Prop 11.2, Let $\Lambda \in P_+$

(a) $P(\Lambda) \supset W\cdot\{\lambda\in P_+ \mid \lambda$ is nondegenerate with respect to $\Lambda\}$

$\forall u$

Pf: $u = \Lambda - \alpha \in P_+$, where $\alpha = \sum_i k_i\alpha_i$, $k_i \ge 0$, $\sum_i k_i > 0$

$S \cap \{i \mid \langle \Lambda, \alpha_i^\vee\rangle > 0\} \ne \phi$

$\Omega_\alpha = \{\gamma \in Q_+ \mid \gamma \le \alpha$ and $\Lambda - \gamma \in P(\Lambda)\}$ ✓

$\beta = \sum_i m_i \alpha_i$ $\Rightarrow$ $\alpha = \beta$

$\text{supp}\,\beta = \text{supp}\,\alpha$

反证 $\alpha \ne \beta$ $\beta = \sum_i m_i\alpha_i \le \alpha = \sum_i k_i\alpha_i$

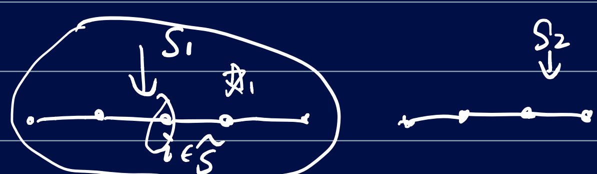
(11.2.4) $\Lambda - \beta - \alpha_i \notin P(\Lambda)$ if $k_i > m_i$

$\tilde{S} = \{j \in S(\Lambda) \mid k_j = m_j\}$, $R = \text{supp}\,\alpha \setminus \tilde{S}$ ($R = \phi$)

☆

$\tilde{S} \ne \phi$

supp α



$S \cap \{i \mid \Lambda(\alpha_i^\vee) > 0\} \ne \phi$

By the property of u (non... with Λ)

$\Rightarrow$ $\exists$ some $i \in S_1$ s.t. $\Lambda(\alpha_i^\vee) > 0$ ☆1

$\exists$ some $j \in S_2$ s.t. $\Lambda(\alpha_j^\vee) > 0$ ☆2

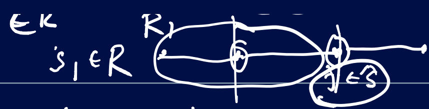
Let $\beta' = \sum_{i\in R} m_i\alpha_i$ $\langle \beta', \alpha_i^\vee\rangle \ge 0$ if $i \in R$

11.2.5

$\langle \beta', \alpha_i^\vee\rangle = \beta(\alpha_i^\vee) - \sum_{j\notin S} m_j \underbrace{\alpha_j(\alpha_i^\vee)}_{a_{ij}} \ge \beta(\alpha_i^\vee) \ge \Lambda(\alpha_i^\vee) \ge 0$

☆2 If $R = S_2$ is a connect component of $\text{supp}\,\alpha$ ($\text{supp}(\Lambda - u)$)

i.e $\exists$ $i \in S_2$ s.t. $\Lambda(\alpha_i^\vee) > 0$ $\langle \beta', \alpha_i^\vee\rangle > 0$



$\hat{S}$ $R = \operatorname{supp}\alpha \setminus S$

(☆$_1$) If R is not connect ~~comp~~ component of $\operatorname{supp}\alpha$

$i \in R$, s_1: $\langle \beta - \beta', \alpha_i^\vee \rangle < 0 \Leftarrow \sum_{j \in S} m_j \alpha_j (\alpha_i^\vee) = \sum_{j \in S} m_j a_{ij} < 0$ ($m_j > 0$)

$\langle \beta', \alpha_i^\vee \rangle = \underbrace{\beta(\alpha_i^\vee)}_{\geq 0} + \underbrace{\langle \beta' - \beta, \alpha_i^\vee \rangle}_{> 0} > 0$

(R_A)

$\Rightarrow$ R is a diagram of finite type

$\langle \alpha', \alpha_i^\vee \rangle \leq 0$ if $i \notin R$ (11.2.7)

$\Rightarrow R$ is not of finit type

$R = \phi \Rightarrow \alpha = \beta \Rightarrow \mu \in P(\Lambda)$ #

§11.3 (Geometric properties of the set of weights $P(\Lambda)$, $\Lambda \in P_+$)

prop 11.3

(a) $P(L(\Lambda)) = (\Lambda + Q) \cap$ the convex hull of $W.\Lambda$ (c.h.$(W.\Lambda)$), where $P(L(\Lambda)) = P(\Lambda)$

(2) If $\lambda, \mu \in \mathfrak{h}^*$ s.t $\mu \in \lambda + Q \cap$ c.h.$(W.\lambda)$ ($\mu \in P(\lambda)$)

$\Rightarrow$ then $\operatorname{mult}_{L(\Lambda)}(\mu) \geq \operatorname{mult}_{L(\Lambda)}(\lambda)$

"$\subseteq$"

Pf: $P(\Lambda) \subset \Lambda + Q$ ✓

$P(\Lambda) \subset$ c.h.$(W.\Lambda)$ ☆

By induction on $\operatorname{ht}(\Lambda - \lambda)$

$\Lambda = \lambda$ ✓ If $\lambda < \Lambda$, there exists i, s.t

$\lambda + \alpha_i \in P(\Lambda)$, take the maximal s s.t $\mu = \lambda + s\alpha_i \in P(\Lambda)$

Since $ht(\Lambda - (\lambda + s\alpha_i)) < ht(\Lambda - \lambda)$

~~By~~ prop 11.1 $u \in ch(W.\Lambda)$

$\Rightarrow$ $\lambda + t\alpha_i \in P(\Lambda) \iff t \in [-p, q]$, $p - q = \lambda(\alpha_i^\vee)$

$\Rightarrow \lambda \in [u, r_i(u)]$ interval

$$\left[\begin{array}{l} r_i(u) = u - (\lambda(\alpha_i^\vee) + 2s)\alpha_i \\ u = \lambda + s\alpha_i \end{array} \right] \quad p - q = \lambda(\alpha_i^\vee)$$

$\Rightarrow \lambda \in ch(W.\Lambda)$

$P(\Lambda) \subset (\Lambda + Q) \cap ch(W.\Lambda)$

"$\supseteq$" $\forall \lambda = \sum_w c_w W(\Lambda) \in \Lambda + Q$, where $c_w \geq 0$ and $\sum_w c_w = 1$

$\Lambda + Q$ $ch\, W(\Lambda)$ $\lambda \in P(V)$ $w(\lambda) \in P(V)$

(11.3.1) $\Lambda - \lambda = \sum_w c_w (\Lambda - w(\Lambda)) \in Q_+$

$r_i(\lambda) = \lambda - \lambda(\alpha_i^\vee)\alpha_i$ ht $\lambda \in ch\, W(\Lambda)$

replacing λ by $w(\lambda)$ — with minimal $ht(\Lambda - \lambda)$, $w(\lambda)$

$\lambda + t\alpha_i$ $\lambda(\alpha_i^\vee) < 0$ $\exists$ $\lambda + \alpha_i$

We may assume that $\lambda \in P_+$; ≥ 0 (11.3.1) $\Rightarrow$ λ is nondegenerat with respect Λ $\overset{\text{prop 11.2(a)}}{\Longrightarrow}$ $\lambda \in P(\Lambda)$

$P(\Lambda) = w(\lambda| \cdots \cdots)$

$\checkmark$ $\text{Supp}(\Lambda - \lambda) = \bigcup_w \text{Supp}(\Lambda - w(\lambda)) \supset S$ $\checkmark$

$w(\lambda) \in P(\Lambda)$ $w(\lambda)$ is non.... with Λ

$S \cap \{i \mid \lambda(\alpha_i^\vee) > 0\} \neq \emptyset$

(b). We may assume that $\lambda \in P_+$

$\lambda \notin P(\Lambda)$ $\quad \text{mult}(\lambda) = 0 \quad \text{mult}_{L(\Lambda)}(\mu) \geq \text{mult}(\lambda)(\mu)$

$\Rightarrow \lambda \in P(\Lambda)$

$\lambda \in P(\Lambda)$ ⟲ W 权重集在 Weyl 群作用不变.

$\rightsquigarrow \exists$ $\quad \lambda \xrightarrow{w} \in P_+ \quad \lambda(\alpha_i^\vee) < 0$

$\exists\, w.$ s.t $w(\lambda) \in P_+$ $\quad w\lambda \in P(\Lambda)$

$\Rightarrow$ We can assume that $\lambda \in P_+ \cap P(\Lambda)$

$\mu \in (\lambda + Q) \cap \text{c.h.}(W.\lambda) \Rightarrow \mu \in P(\lambda)$

$\quad\quad \| $ by (a) $P(\lambda)$

$\mu \in (\lambda + Q) \cap \text{ch.}(W.\lambda) = P(\lambda)$

$\mu \in P(\lambda)$; If $\lambda = \mu$ ✓ $\quad P(\lambda) \quad \lambda - \mu \in Q$

otherwise $\mu + \alpha_i \in P(\lambda)$ $(\mu \in P(\lambda) \quad \mu < \lambda)$

for some i, let $s > 0$. s.t $(\mu + s\alpha_i) \in P(\lambda)$ $\quad \mu + (s+1)\alpha_i \notin P(\lambda)$

By induction $ht(\lambda - \mu)$

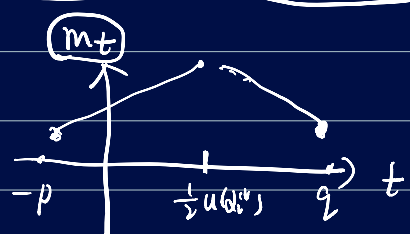
$ht(\lambda - (\mu + s\alpha_i)) < ht(\lambda - \mu)$

$\text{mult}(\mu) \geq$ ☆ $\quad \text{mult}_{L(\lambda)}(\mu + s\alpha_i) \geq \text{mult}_{L(\lambda)}(\lambda)$ ✓

⑨

$\Rightarrow \mu$ lies in $[\mu + s\alpha_i, r_i(\mu + s\alpha_i)]$

By prop 11.1 (b)(c)



$\text{mult}_{L(\lambda)}(\mu) \geq \min\{\text{mult}_{L(\lambda)}(\mu + s\alpha_i), \text{mult}(r_i(\mu + s\alpha_i))\}$

$= \mathrm{mult}_{L(\Lambda)}(\mu + s\alpha_i) \geq \mathrm{mult}_{L(\Lambda)}(\lambda)$

$\mu \in (\lambda + Q) \cap \mathrm{ch.}(W\cdot\lambda) \Rightarrow$ #

§11.4 In the rest of the chapter, $A \to$ sym. GCM
$(\cdot|\cdot) \to g(A)$

Prop 11.4, Let $\Lambda \in P_+$ and $\lambda, \mu \in P(\Lambda)$, Then

(a) $(\Lambda|\Lambda) - (\lambda|\mu) \geq 0$ and equality holds $\Leftrightarrow$
$\lambda = \mu \in W \cdot \Lambda$

(b) $|\Lambda + \rho|^2 - |\lambda + \rho|^2 \geq 0$ and eq... $\Leftrightarrow$
$\lambda = \Lambda$

Pf: Since $(\cdot|\cdot)$ and $P(\Lambda)$ are W-invariant.

We can assume that in the proof of (a) that $\lambda \in P_+$
then since $\beta = \Lambda - \lambda \in Q_+$
$\beta_1 = \Lambda - \mu \in Q_+$

we have
$(\Lambda|\Lambda) - (\lambda|\mu) = (\Lambda|\beta) + (\lambda|\beta_1) \geq 0$
$\lambda = \Lambda - \beta$, $\mu = \Lambda - \beta_1$

$(\Lambda|\beta) = \frac{1}{2} k_i (\alpha_i|\alpha_i) \Lambda(\alpha_i^\vee) \geq 0$

$\beta = \sum k_i \alpha_i$, $k_i > 0$, $(\alpha_i|\alpha_i) > 0$, $\Lambda(\alpha_i^\vee) \geq 0$ since $\Lambda \in P_+$

$(\lambda \mid \beta_1) = \cdots \lambda(\alpha_i^\vee) \geq 0$ ($\lambda \in P_+$)

In the case of equality, we have

$(\Lambda|\beta) = (\lambda|\beta_1) = 0 \Longrightarrow \begin{cases} \Lambda = \lambda \\ \lambda = \alpha u \end{cases}$

$\beta = 0$, $\beta_1 = 0$

$\beta \neq 0$, $\exists\, i \in \operatorname{Supp}\beta$, $\Lambda(\alpha_i^\vee) > 0 \Longrightarrow (\Lambda|\beta) > 0$ ↯

$\exists\, k_i > 0$

$\begin{cases} \lambda \in P(\Lambda) \\ \lambda \in P_+ \end{cases} \Longrightarrow \lambda$ is nondegenerat.. with respect to Λ

$\forall S \subset \operatorname{supp}(\Lambda - \lambda)$ $\quad S \cap \{i \mid \Lambda(\alpha_i^\vee) > 0\} \neq \emptyset$

$\operatorname{supp}(\Lambda - \lambda) = \operatorname{supp}(\beta)$

$\Rightarrow \beta = 0 \Rightarrow \lambda = \Lambda$

$\Rightarrow (\Lambda|\beta_1) = 0 \qquad \Rightarrow \lambda = \Lambda = u$

$(\Lambda|\beta_1) = 0 \Rightarrow \beta_1 = 0$

(b) $(\Lambda+\rho \mid \Lambda+\rho) - (\lambda+\rho \mid \lambda+\rho)$

$= \underbrace{((\Lambda|\Lambda) - (\lambda|\lambda))}_{\geq 0} + 2(\Lambda - \lambda \mid \rho) \geq 0$

($\Lambda - \lambda \in Q_+$, ≥ 0)

by (a) $\lambda, u \in P(\Lambda)$ $\quad \lambda = u$

$(\Lambda|\Lambda) - (\lambda|u) \geq 0$

Clearly equality occurs $\Longleftrightarrow \Lambda = \lambda$

#

§11.5

Def: V be a $\mathfrak{g}(A)$-module. A Hermitian form on V is contravariant if $H(g(x) \mid y) = -H(x \mid \omega_0(g)y)$ for all $g \in \mathfrak{g}(A)$, $x, y \in V$

example $(\cdot|\cdot)_0$ on $\mathfrak{g}(A)$ in §2.7

$(x|y)_0 := -(\omega_0(x) | y)$

$\omega_0 \to$ compact involution

Since $(\mathrm{ad}(g)(x) \mid y)_0 = -(x \mid \mathrm{ad}(\omega_0(g))(y))_0$

Lemma 11.5

$\Lambda \in \mathfrak{h}_{\mathbb{R}}^*$, $\mathfrak{g}(A)$-module $L(\Lambda)$ carries a unique up to constant factor, non.. contravariant. Hermitial form.

$L(\Lambda) \longrightarrow$ decompose into an orthogonal direct sum of weight space.

~~$L(\Lambda)$~~

Pf: Denote by $\mathfrak{g}(A)_{\mathbb{R}}$ the real subalgebra generated by e_i, f_i $(i=1..n)$ $\mathfrak{h}_{\mathbb{R}}$, $L(\Lambda)_{\mathbb{R}} = U(\mathfrak{g}(A)_{\mathbb{R}})(v_\Lambda)$

Prop 9.4: Every $\mathfrak{g}(A)$-module $L(\Lambda)$ carries a unique up to constant fact nonde... Contra...v bilinear form B, this form is symm.. and $L(\Lambda)$ decompose into orthogonal direct sum of weight space with respect to this form)

[$f: L(\Lambda) \longrightarrow L(\Lambda)^*$

$B(x, y) = f(x)(y)$, where $x, y \in L(\Lambda)$

$H(\cdot|\cdot): L(\Lambda) \times L(\Lambda) \longrightarrow \mathbb{C}$

$H(c_1 u | c_2 v) = c_1 \bar{c_2} B(u|v)$ for $c_1, c_2 \in \mathbb{C}$, $u, v \in L(\Lambda)$]

$g(A) \longrightarrow g(A)_\mathbb{R} \longrightarrow \mathrm{End}\, L(\Lambda)_\mathbb{R}$

$B(\cdot|\cdot)$ on $L(\Lambda)_\mathbb{R}$ $\rightsquigarrow$ Hermitian extension of B from $L(\Lambda)_\mathbb{R}$ to $L(\Lambda)$

#

[Recall in §9.4 $(L(\Lambda), \pi_\Lambda^*)$ by

$\pi_\Lambda^*(g)(u) = \pi_\Lambda(\omega(g)u)$

$\pi: g(A) \to \mathrm{End}(L_\Lambda)$

$B(g(x)|y) = -B(x|\omega(g)y)$

expectation value $\langle v \rangle \in \mathbb{C}$ $\forall v \in V$

$v = \langle v \rangle v_\Lambda + \sum_{\alpha \in Q_+ \setminus \{0\}} v_{\Lambda-\alpha} \in V_{\Lambda-\alpha}$

$\omega \rightsquigarrow -\hat{\omega}$

$\langle \hat{\omega}(a) v_\Lambda \rangle = \langle a v_\Lambda \rangle$ $a \in U(g_A)$

$B(a v_\Lambda, a' v_\Lambda) = \langle \hat{\omega}(a) a' v_\Lambda \rangle$

and normalized by $B(v_\Lambda, v_\Lambda) = 1$]

As in §9.4 $H(g v_\Lambda, g' v_\Lambda) = \langle \hat{\omega}_0(g) g' v_\Lambda \rangle$

where $g, g' \in U(g(A))$, $\hat{\omega}_0$ is the extension ω_0

to be an antilinea anti-involution of $U(\mathfrak{g}(A))$

by $H(v_\lambda, v_\lambda) = 1$

Rmk: ① g and $-\omega_0(g)$ are adjoint operator on $L(\lambda)$ with respect to H

$H(g(x), y) = -H(x, \omega_0(g)y)$

☆ ② Thus, the compact form $\mathfrak{g}(A)$ is represented on $L(\lambda)$ by skew-adjoint operators

$\omega_0 / \mathfrak{k}(A) = \mathfrak{k}(A)$

The $\mathfrak{g}(A)$-module $L(\lambda)$ is called unitarizable if the Hermitian form H defined in (11.5.1) is positive definite

§11.6 $\sqrt{2}_1$

claim: $2(\rho|\alpha) > (\alpha|\alpha)$ if $\alpha \in \Delta_+ \setminus \Pi$

pf: If $\alpha \in \Delta_+^{im}$, this is clear, since then $(\alpha|\alpha) \le 0$

But $(\rho|\alpha) > 0$ for all $\alpha \in \Delta_+$

If $\alpha \in \Delta_+^{re} \setminus \Pi$, $\alpha^\vee \in (\Delta_+^\vee)^{re} \setminus \Pi^\vee$

$$\frac{2(\rho|\alpha)}{(\alpha|\alpha)} = \rho(\alpha^\vee) = (\sum k_i)\rho(\alpha_i^\vee) = ht(\alpha^\vee) > 1$$

($\rho(\alpha_i^\vee) = \frac{1}{2}a_{ii} = 1$)

$\alpha^\vee = \frac{2}{(\alpha|\alpha)} \nu^{-1}(\alpha)$ $\qquad (\alpha|\alpha) > 0$

$\alpha^\vee = \sum_i^{\ell} k_i \alpha_i^\vee$

$\sum k_i > 1$

$$2(\rho|\alpha) > (\alpha|\alpha)$$ #

By analogy with the "partial" Casimir operator Ω_0 we define an operator Ω_1 on n_- as follows:

$x \in n_-$

$$\Omega_1(x) = \sum_{\alpha\in\Delta_+} \sum_{i}^{m_\alpha = \text{mult}(\alpha)} [e_{-\alpha}^{(i)}\ [e_\alpha^{(i)}, x]_-\ ]$$

$e_{-\alpha}^{(i)} \in g_{-\alpha}$

Here, as before $\{e_\alpha^{(i)}\}$, $\{e_{-\alpha}^{(i)}\}$ are dual bases of g_α and $g_{-\alpha}$ with the bilinear form $(\cdot|\cdot)$, "$-$": the projection on n_-

$M(0) = U(g(A)) / \langle n_+, H\rangle$

Lemma 11.6 If $\alpha\in\Delta_+$, $x\in g_{-\alpha}$, then

$$\Omega_1(x) = (\underbrace{2(\rho|\alpha) - (\alpha|\alpha)}_{>0})x$$

Pf: We calculate in $M(0)$ the expression $\Omega_0 x(v)$

$v \to$ h.w vector. $\Omega_0 = 2\sum_{\gamma\in\Delta_+}\sum_{i=1}^{m_\gamma} e_{-\gamma}^{(i)} e_\gamma^{(i)}$

$u \in U_\lambda$

By (2.6.1) $\to$ $[\Omega_0, u] = -u\,(2(\rho|\lambda) + (\lambda|\lambda) + 2\nu^{-1}(\lambda))$, we have

(11.6.2) $\Omega_0\, x(v) = (2(\rho|\alpha) - (\alpha|\alpha))\, x(v)$

$x\in g_{-\alpha}$ $[\Omega_0, x](v) = -x\,(2(\rho|-\alpha) + (-\alpha|-\alpha) + 2\nu^{-1}(-\alpha))(v)$

$= (2(\rho|\alpha) - (\alpha|\alpha))\,x(v)$ $2\nu^{-1}(-\alpha)$ ↓ 0

On the other hand, by the definition of Ω_0,

$[uu_1, x] = [u, x]u_1 + u[u_1, x]$

$$\Omega_0\, x(v) = 2\sum_{\gamma\in\Delta_+}\sum_{i=1} e_{-\beta}^{(i)} e_{\beta}^{(i)} x(v)$$

$$= 2\sum_{\gamma\in\Delta_+}\sum_{i=1} e_{-\beta}^{(i)}[e_{\beta}^{(i)}, x](v) + 2\sum_{\gamma\in\Delta_+}\sum_i [e_{-\beta}^{(i)}, x] e_{\beta}^{(i)}(v)$$

0

$x \to -\alpha$

$e_\beta^{(i)} \to \beta$

$\beta > \alpha$

$[e_\beta^{(i)}, x](v) = 0$

$\beta < \alpha$

Putting $S = \{\beta\in\Delta_+ \mid \beta < \alpha\}$

$$\Omega_0\, x(v) = 2\sum_{\beta\in S}\sum_i e_{-\beta}^{(i)}[e_{\beta}^{(i)}, x](v)$$

$$= \sum_{\beta\in S}\sum_i \left([e_{-\beta}^{(i)}, [e_\beta^{(i)}, x]] + [e_\beta^{(i)}, x]e_{-\beta}^{(i)} + e_{-\beta}^{(i)}[e_\beta^{(i)}, x]\right)(v)$$

0

$$\left([e_{-\beta}^{(i)}, [e_\beta^{(i)}, x]] = e_{-\beta}^{(i)}[e_\beta^{(i)}, x] - [e_\beta^{(i)}, x]e_{-\beta}^{(i)}\right.$$

Using (2.4.4) $\Rightarrow [e_\beta^{(i)}, x]e_{-\beta}^{(i)} + e_{-\beta}^{(i)}[e_\beta^{(i)}, x] = 0$

[☆] $$\Omega_0\, x(v) = \sum_{\beta\in S}\sum_i [e_{-\beta}^{(i)}[e_\beta^{(i)}, x]](v)$$

(2.4.4) $$\sum_s e_{\alpha}^{(s)}[z, e_\alpha^{(s)}] = -\sum_s [z, e_{-\beta}^{s}]e_\beta^{s}$$ in $U(g)$

11.6.2 $\Omega_0\, x(v) = 2(\rho|\alpha) - (\alpha|\alpha)\, x(v)$

(☆) $\Omega_0 \cdot x(v) = \sum \sum [e_{-\beta}^{(i)}[e^{(i)}, x](v)$

$$\Omega_1(x) = \sum_{\beta \in \Delta_+} \sum_i \left[e_{-\beta}^{(i)} \left[e_\beta^{(i)}, x \right]_- \right](u)$$

$\beta' \in \Delta_+ \setminus S$ $\quad [e_\beta^{(i)}, x] \in g_{\beta-\alpha}$

$\beta > \alpha$ $\quad [e_\beta^{(i)}, x]_- = 0$

$\Rightarrow \Omega_1(x) = (2(\rho|\alpha) - (\alpha|\alpha))\, x$

As $M(0)$ is free $U(n_-)$ module (prop 9.2 b)

#