Recall: $\Omega_1(x) = \sum_{\alpha\in\Delta_+}\sum_i [e_{-\alpha}^{(i)}[e_\alpha^{(i)}, x]_-]$ $(x\in n_-)$

Lem 11.6: If $\alpha\in\Delta_+$ and $x\in g_{-\alpha}$, then

$$\Omega_1(x) = (2(\rho|\alpha) - (\alpha|\alpha))x.$$

§11.7

Now we are in position to prove the following fundamental result.

Th 11.7 let $g(A)$ be a symmetrizable Kac-Moody algebra, Then

a) The restriction of Hermitian form $(\cdot|\cdot)_0$ to every root space g_α $(\alpha\in\Delta)$ is positive-definite, i.e. $(\cdot|\cdot)_0$ is p-d. on $n_-\oplus n_+$

proof: Using w_0, it suffices to show that $(\cdot|\cdot)_0$ is positive definite on $g_{-\alpha}$.

We do it by induction on ht α. $A=DB$.

The case ht$\alpha=1$ is clear by (2.2.1) $(e_i|f_j)=\delta_{ij}\varepsilon_i$

[$(f_j|f_j)_0 = -(w_0(f_j)|f_j) = (e_j|f_j) = \varepsilon_j$]

Otherwise put $S=\{\beta\in\Delta_+ \mid \beta<\alpha\}$ and use the inductive assumption to choose, for every $\beta\in S$, an orthonormal basis $\{e_{-\beta}^{(i)}\}$ of $g_{-\beta}$ w.r.t. $(\cdot|\cdot)_0$, Then, setting $e_\beta^{(i)} = -w_0(e_{-\beta}^{(i)})$, we have $(e_\beta^{(i)}|e_{-\beta}^{(j)}) = (e_{-\beta}^{(i)}|e_{-\beta}^{(j)})_0 = \delta_{ij}$.

- Now we apply lem 11.6 with this choice of $e_\beta^{(i)}$ and $e_{-\beta}^{(i)}$ (the choice for the $\beta\in\Delta_+\setminus S$ is arbitary).

For $x\in g_{-\alpha}$ we have:

$$\underline{(2(\rho|\alpha)-(\alpha|\alpha))(x|x)_0} = (\Omega_1(x)|x)_0$$

$$= \sum_{\beta\in S}\sum_i([e_{-\beta}^{(i)}, \underbrace{[e_\beta^{(i)},x]}]|x)_0 = \sum_{\beta\in S}\sum_i(\underbrace{[e_\beta^{(i)},x]}|[e_\beta^{(i)},x])_0.$$

$[e_\alpha^{(i)},x]_- \neq 0 \Rightarrow \beta\in S.$

By the inductive assumption, the last sum is nonnegative.

using (11.6.1) $2(\rho|\alpha) > (\alpha|\alpha)$ if $\alpha \in \Delta_+ \setminus \Pi$.

we get $(x|x)_0 \geq 0$.

Since $(\cdot|\cdot)_0$ is nondegenerate on $\mathfrak{g}_{-\alpha}$, we deduce that it is positive - definite (i.e. $(x|x)_0 = 0 \Leftrightarrow x = 0$).

[there is i s.t. $[e_i, x] \neq 0$, $\alpha_i \in S$, since $x \in \mathfrak{g}_{-\alpha}$]

#

b) Every integrable highest - weight module $L(\Lambda)$ over $\mathfrak{g}(A)$ is unitarizable. Conversely, if $L(\Lambda)$ is unitarizable, then $\Lambda \in P_+$.

integrable $\Leftrightarrow$ unitarizable.

proof: "$\Rightarrow$"

Using Lem 11.5, one has to show for b) that the restriction of H to $L(\Lambda)_\lambda$ is positive - definite.

[by Lem 11.5: w.r.t. Hermitian form, $L(\Lambda)$ decomposes into an orthogonal direct sum of weight space]

We prove this by induction on $ht(\Lambda - \lambda)$.

· If $ht(\Lambda - \lambda) = 0$. then $u = c v_\Lambda$ for some $c \neq 0 \in \mathbb{C}$ and $H(u, u) = H(c v_\Lambda, c v_\Lambda) = c\bar{c} H(v_\Lambda, v_\Lambda) = c\bar{c} > 0$.

· suppose $H(u, u) > 0$ when $ht(\Lambda - \lambda) < k$ for some integer $k \geq 1$. let $\lambda \in P(\Lambda) \setminus \{\Lambda\}$ and $v \in L(\Lambda)_\lambda$. Thanks to a), we can choose a basis $\{e_\alpha^{(i)}\}$ of $\mathfrak{g}_\alpha$ such that $\{-\omega_0(e_\alpha^{(i)})\}$ is dual (w.r.t. $(\cdot|\cdot)$) basis of $\mathfrak{g}_{-\alpha}$.

Then we have:

$$\Omega = 2\nu^{-1}(\rho) + \sum_i u_i u^i - 2\sum_{\alpha \in \Delta_+} \sum_i \omega_0(e_\alpha^{(i)}) e_\alpha^{(i)}$$

[$\Omega = 2\nu^{-1}(\rho) + \sum_i u_i u^i + \Omega_0$, where $u_1, u_2, \cdots$ and $u^1, u^2, \cdots$ are dual bases of H, & $\Omega_0 = 2\sum_{\alpha\in\Delta_+}\sum_i e_{-\alpha}^{(i)} e_{\alpha}^{(i)}$]

and hence (11.7.1) $\Omega(v) = (\lambda+2\rho|\lambda)v - 2\sum_{\alpha\in\Delta_+}\sum_i \omega_0(e_\alpha^{(i)}) e_\alpha^{(i)}(v)$.

[$2\nu^{-1}(\rho).v = \langle 2\nu^{-1}(\rho), \lambda\rangle v = (2\rho|\lambda)v$

Pro (2.5.2) $\sum_i \langle\lambda, u^i\rangle\langle\mu, u_i\rangle = (\lambda|\mu)$.

$\sum_i u_i u^i . v = \sum_i u_i . \langle\lambda, u^i\rangle v = \sum_i \langle\lambda, u^i\rangle\langle\lambda, u_i\rangle v = (\lambda|\lambda)v$.]

· computing $H(\Omega(v), v)$ in two different ways by making use of Cor. 2.6 and (11.7.1).

[Cor. 2.6: if v is a highest-weight of $L(\Lambda)$, then

$\Omega(v) = (\Lambda+2\rho|\Lambda)v$ and $\Omega = (\Lambda+2\rho|\Lambda)1_{L(\Lambda)}$]

and equating the results we obtain:

$(|\Lambda+\rho|^2 - |\lambda+\rho|^2)H(v,v) = ((\Lambda+2\rho|\Lambda) - 2(\rho|\lambda) - (\lambda|\lambda))H(v,v)$

$= (\Omega - (\lambda+2\rho|\lambda))H(v,v)$ (by cor. 2.6).

$= 2\sum_{\alpha\in\Delta_+}\sum_i -\omega_0(e_\alpha^{(i)}) e_\alpha^{(i)} H(v,v)$ (by (11.7.1))

$= 2\sum_{\alpha\in\Delta_+}\sum_i H(e_\alpha^{(i)}(v), e_\alpha^{(i)}(v))$ (by contravariant).

by the inductive assumption, the last sum is nonnegative.

Using prop 11.4 b): ($|\Lambda+\rho|^2 - |\lambda+\rho|^2 \geq 0$, "=" iff $\lambda = \Lambda$).

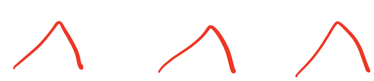
we deduce that $H(v,v) \geq 0$.

since H is nondegenerate on $L(\Lambda)_\lambda$. ⇒ p-d.

$L(\Lambda) = M(\Lambda)/M'(\Lambda)$ $\quad\quad$ $U(n_-)(v_\Lambda)$

sl_2 $\quad -1 \quad -2 \quad -3$

可积, 相当于可提升到群上. $\quad\quad$ 从正的 当然 可约.



"$\Leftarrow$" To prove the converse, note that by

$$(3.2.4) \qquad e(v_j) = (\lambda - j + 1) v_{j-1}$$

where $v_j = (j!)^{-1} f^j(v)$.

then $e_i(f_i^k(v_\lambda)) = k(\langle \Lambda, \alpha_i^\vee \rangle + 1 - k) f_i^{k-1}(v_\lambda)$.

we have: $0 \le H(f_i^k v_\lambda, f_i^k v_\lambda) = H(f_i^{k-1} v_\lambda, e_i f_i^k v_\lambda)$

$= k(\langle \Lambda, \alpha_i^\vee \rangle + 1 - k) H(f_i^{k-1} v_\lambda, f_i^{k-1} v_\lambda)$

$= \prod_{j=1}^{k} j \underline{(\langle \Lambda, \alpha_i^\vee \rangle + 1 - j)}$

since $j \ge 1 \Rightarrow \langle \Lambda, \alpha_i^\vee \rangle$ is nonnegative real number.

if $\langle \Lambda, \alpha_i^\vee \rangle$ is not an integer, we take $k = [|\langle \Lambda, \alpha_i^\vee \rangle|] + 2$

then $\prod_{j=1}^{k} j(\langle \Lambda, \alpha_i^\vee \rangle + 1 - j) < 0$. since $\langle \Lambda, \alpha_i^\vee \rangle + 1 - k < 0$.

which contradicts to condition. Thus $\Lambda \in P_+$.

#

· warning: The restriction of $(\cdot|\cdot)_0$ to H and even H' is in general an idefinite Hermitian form:

$(h_1 | h_2)_0 = -(w_0(h_1) | h_2) = (h_1 | h_2)$

the matrix $((\alpha_i^\vee | \alpha_j^\vee)_0)$ is a symmetrization of matrix A^T.

$A = DB$. $\quad D = \mathrm{diag}(\varepsilon_1, \cdots, \varepsilon_n)$ $\quad \varepsilon_i \ne 0$. $\quad B = (b_{ij})$.

Then B is called a symmetrization of A.

$B' = (\alpha_i^\vee | \alpha_j^\vee)_0 = (\alpha_i^\vee | \alpha_j^\vee) = b_{ij} \varepsilon_i \varepsilon_j$.

$A^T = D' B'$ $\qquad A = (a_{ij}) \quad a_{ij} = b_{ij} \varepsilon_i$.

$$\begin{pmatrix} \frac{1}{\varepsilon_1} & & & \\ & \frac{1}{\varepsilon_2} & & \\ & & \ddots & \\ & & & \frac{1}{\varepsilon_n} \end{pmatrix} \begin{pmatrix} b_{11}\varepsilon_1\varepsilon_1 & \cdots & b_{1n}\varepsilon_1\varepsilon_n \\ \vdots & & \\ b_{n1}\varepsilon_n\varepsilon_1 & \cdots & b_{nn}\varepsilon_n\varepsilon_n \end{pmatrix} = \begin{pmatrix} A^T \end{pmatrix}$$

· if fact, $(\cdot|\cdot)_0$ is positive-definite (resp. positive-semidefinite) on $\mathfrak{h}'$ iff A is of finite (resp. affine) type.

- hence, using Th 11.7 a), the $(\cdot|\cdot)_0$ is positive-definite on $\mathfrak{g}(A)$. so that $\mathfrak{g}(A)$ carries a positive-definite $\mathfrak{k}(A)$-invariant Hermitian form.

$\mathfrak{k}(A) = \{ x \in \mathfrak{g}(A) \mid \omega_0(x) = x \}$.

Recall $(\cdot|\cdot)$ is invariant i.e. $([x,y]|z) = (x|[y,z])$.

since $([u,x]|y)_0 = -(x|[\omega_0(u),y])_0$. where $x, y, u \in \mathfrak{k}(A)$.

$= -(x|[u,y])_0$.

i.e. $([x,u]|y)_0 = (x|[u,y])_0$.

§ 11.8.

we deduce from Th 11.7 b) another complete reducibility result.

Lem 11.8. Let $h \in \operatorname{Int} X_{\mathbb{C}}$, Then for every $r \in \mathbb{R}$, the number of eigenvalues (counting multiplicities) λ of h in $L(\Lambda)$ $(\Lambda \in P_+)$, such that $\operatorname{Re}\lambda > r$ is finite.

proof: This follows from prop 10.6 d): The series $\sum_{w \in W} \varepsilon(w) e^{w(\Lambda+\rho)}$ converges absolutely on $\operatorname{Int} X_{\mathbb{C}}$ to holomorphic function. and diverges absolutely on $\mathfrak{h} \setminus \operatorname{Int} X_{\mathbb{C}}$. #

prop. 11.8. Let $\mathcal{A} \subset \mathfrak{g}(A)$ an ω_0-invariant subalgebra which is normalized by an element $h \in \operatorname{Int} X_{\mathbb{C}}$, (i.e. $[h, \mathcal{A}] \subset \mathcal{A}$). Then w.r.t. $\mathcal{A}$, the module $L(\Lambda)$ $(\Lambda \in P_+)$ decomposes into an orthogonal (w.r.t. H) direct sum of irreducible $\mathfrak{h}$-invariant submodules.

proof: put $A_1 = A + \mathbb{C}h$.

By Th. 11.7 b) and Lem. 11.8. $\Rightarrow$ $L(\Lambda)$ decomposes into an orthogonal direct sum of f-d eigenspaces of h.

$x \in V_{\lambda_1}$, $y \in V_{\lambda_2}$. $\lambda_1 H(x,y) = \bar{\lambda}_2 H(x,y) \Rightarrow H(x,y) = 0$.

It follows, using Th. 11.7 b). and w_0 - invariant of A_1, that for every A_1 - submodule $V \subset L(\Lambda)$, the subspace $V^\perp$ is also an A_1 - submodule and $L(\Lambda) = V \oplus V^\perp$.

[$\because \forall\, y \in V^\perp$, and $a + kh \in A_1$, $x \in V$,

$$H((a+kh).y, x) = -H(y, w_0(a+kh)(x))$$
$$= -H(y, \underbrace{w_0(a).x - k h.x}_{\in V}). = 0.$$

$\Rightarrow V^\perp$ is an A_1 - submodule.]

. Hence $L(\Lambda)$ decomposes into an orthogonal direct sum of irreducible A_1 - modules.

$$L(\Lambda) = \underbrace{V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots}_{\text{irred } A_1\text{-mod}} \oplus V_{\lambda_n} \oplus \cdots\cdots$$

. Let $V \subset L(\Lambda)$ be an irreducible A_1 - submodule.

It remains to show to V remains irreducible when restricted to A.

Let V_λ denote the λ - eigenspace of h in V, let λ_0 be the eigenvalue of h with maximal real part.

Let A^λ denote the eigenspace of ad$h$ in A, ($[h, a] = \lambda a$).

We denote by A_0, (resp A_+ or A_-). the sum of all A^λ with Re$\lambda = 0$ (resp > 0, or < 0). Then $A = A_0 \oplus A_+ \oplus A_-$.

and it is clear that V_{λ_0} is an irreducible A_0 - module.

and $\{x \in V_\lambda \mid A_+(x) = 0\} = 0$ if Reλ < Reλ_0 (*)

Γ. V_{λ_0} is an A_0-module: let $x_0 \in A_0$ and $\mathrm{Re}\,[h, x_0] = 0$.

& let $u \in V_{\lambda_0}$, $h.u = \lambda_0 u$.

then $h.x_0.u = \underbrace{[h\ x_0].u} + x_0.h.x = \overset{\text{imaginary}}{(\ )}\, x_0 u + \lambda_0 x_0 u$.

$\Rightarrow x_0.u \in V_{\lambda_0}$.

· irreducible: if $X \subset V_{\lambda_0}$. $\quad h.x_0.u = (\quad) + \lambda_0 x_0 u$.

· (*) $h.a_+.x = 0 = [h, a_+]x + a_+.h.x = [h\ a_+]x + \lambda a_+.x$.

$\Rightarrow x = 0$. obviously $\mathrm{Re}\lambda = \mathrm{Re}\lambda_0$, $\Rightarrow a_+.x = 0$, $\forall\ x \in V_{\lambda_0}$

thus V is a irreducible A-module.

对 $\exists$ A submodule K, choose $V_{\lambda_0} \not\subset K$

Then $\exists$ max $\mathrm{Re}(\lambda) = k$. $\quad k < \mathrm{Re}\lambda_0$.

but $A_+(V_\lambda)$ 全部 加.

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§11.9. Action of Imaginary root vectors.