§11.11.

· In this section we deduce a specialization formula of the denominator identity (10.9.4). (10.8)

$$\prod_{\alpha\in\Delta_+}(1-e(-\alpha))^{\mathrm{mult}\,\alpha} = \sum_{w\in W}\varepsilon(w)e(w(\rho)-\rho)$$

Let H_0 be a subspace of H, s.t.

(11.11.1) $H_0 \cap \mathrm{Int}\,X_{\mathbb{C}} \neq \phi$.

let $H^* \longrightarrow H_0^*$ via : $\lambda \longmapsto \bar{\lambda}$ denote the restriction map. $\bar{\lambda} = \lambda|_{H_0}$

denote by p the homomorphism of $\mathcal{E}$ to the completed group algebra of H_0^* defined by $p(e(\lambda)) = e(\bar{\lambda})$ $\sum_{\lambda\in H^*} c_\lambda e(\lambda)$ 形式和

· put $\Delta_0 = \{\alpha\in\Delta \mid \bar{\alpha}=0\}$; $\Delta_{0+} = \Delta_0\cap\Delta_+$, $R_0 = \prod_{\alpha\in\Delta_{0+}}(1-e(-\alpha))$.

i.e. $\bar{\alpha}(H_0)=0$ or $\alpha|_{H_0}=0$.

By prop 3.12. the set Δ_0 is finite, hence $\Delta_0 \subset \Delta^{re}$.

⌈ Recall : prop 3.12 c) $\mathrm{Int}\,X = \{h\in H_{\mathbb{R}} \mid \langle\alpha,h\rangle < 0$ only for a finite number of $\alpha\in\Delta_+\}$.

$\mathrm{Int}\,X_{\mathbb{C}} = \{x+iy \mid x\in X,\ y\in H_{\mathbb{R}}\}$. ⌋

· it is clear that Δ_0 satisfies the usual axioms for a finite root system (see Bourbaki [1968]) 拓扑. 序. 代数.

· Denote by W_0 the (finite) subgroup of W generated by reflections r_α ($\alpha\in\Delta_0$) the let ρ_0 (resp. $\rho_0^\vee$) $\rho_0 = \frac{1}{2}\sum_{\alpha\in\Delta_{0+}}\alpha$, $\rho_0^\vee = \frac{1}{2}\sum_{\beta\in\Delta_{0+}^\vee}\beta$.

· Define a polynomial $D(\lambda)$ on H^* by.

$$D(\lambda) = \prod_{\alpha\in\Delta_{0+}}\frac{\langle\lambda,\alpha^\vee\rangle}{\langle\rho_0,\alpha^\vee\rangle}$$

Lemma 11.11 For $\lambda\in H^*$, we have.

$$p\Big(R_0^{-1}\sum_{w\in W_0}\varepsilon(w)e(w(\lambda))\Big) = D(\lambda)e(\bar{\lambda})$$

proof: let $\Pi_0 \subset \Delta_{0+}$ be the set of simple roots of Δ_0. Define homom.

$$F:\ \mathbb{C}[e(-\alpha);\ \alpha\in\Pi_0] \longrightarrow \mathbb{C}[q].$$
$$e(-\alpha)) \longmapsto q. \quad \text{for all } \alpha\in\Pi_0.$$

Then (11.11.2) $p(f) = \lim_{q\to1}F(f)$, $F(e(-\alpha)) = q^{\langle\rho_0^\vee,\alpha\rangle} = q$.

⌈ $f\in\mathcal{E}$, $f=\sum_{\lambda\in H^*}c_\lambda e(\lambda)$, $p(f)=\sum_{\lambda\in H^*}c_\lambda e(\bar{\lambda})$ (if $\lambda\in H_0^* \Rightarrow e(\bar{\lambda})=1$).

By (10.9.3) : $F_1(N_\lambda) = \prod_{\alpha\in\Delta_+^\vee}(1-q^{\langle\lambda,\alpha\rangle})^{\mathrm{mult}\,\alpha}$ where $\lambda\in P_{++}$,

$$N_\lambda = \sum_{w\in W}\varepsilon(w)e(w(\lambda)-\lambda)$$ ⌋

we have (11.11.3) $F\Big(\sum_{w\in W_0}\varepsilon(w)e(w(\lambda)-\lambda)\Big) = \prod_{\alpha\in\Delta_{0+}}(1-q^{\langle\lambda,\alpha^\vee\rangle})$.

(11.11.2) and (11.11.3) together with (10.8.5) prove the lemma.

⌈ (10.8.5) $\prod_{\alpha\in\Delta_+}(1-q^{\langle\rho^\vee,\alpha\rangle})^{\mathrm{mult}\,\alpha} = \prod_{\alpha\in\Delta_+^\vee}(1-q^{\langle\rho,\alpha\rangle})^{\mathrm{mult}\,\alpha}$.

Then we have $\prod_{\alpha\in\Delta_{0+}}(1-q^{\langle\rho_0^\vee,\alpha\rangle}) = \prod_{\alpha\in\Delta_{0+}^\vee}(1-q^{\langle\rho_0,\alpha\rangle})$.

by (11.11.2) we only need to calulate.

$$\frac{\lim_{q\to 1} F\left(\dfrac{\sum_{w\in W_0}\varepsilon(w)e(w(\lambda)-\lambda)}{\prod_{\alpha\in\Delta_+^0}(1-e(-\alpha))}\right)}{} = \lim_{q\to 1}\frac{\prod_{\alpha\in\Delta_+^0}(1-q^{\langle\lambda,\alpha^\vee\rangle})}{\prod_{\alpha\in\Delta_+^0}(1-q^{\langle\rho_0^\vee,\alpha\rangle})}$$

$$= \lim_{q\to 1}\prod_{\alpha\in\Delta_+^0}\frac{1-q^{\langle\lambda,\alpha^\vee\rangle}}{1-q^{\langle\rho_0,\alpha\rangle}} \overset{\text{L'Hospitals}}{=\!=} \prod_{\alpha\in\Delta_+^0}\frac{\langle\lambda,\alpha^\vee\rangle}{\langle\rho_0,\alpha^\vee\rangle} = D(\lambda).$$

Hence. $p_0(R_0^{-1}\sum_{w\in W_0}\varepsilon(w)e(w(\lambda))) = p(R_0^{-1}\sum_{w\in W_0}\varepsilon(w)e(w(\lambda)-\lambda)e(\lambda))$

$= p(R_0^{-1}(\quad))\,p(e(\lambda)) = D(\lambda)e(\bar\lambda)$.

#.

· Now we can deduce the specialization formula :

$$(11.11.4).\quad \prod_{\alpha\in\Delta_+\setminus\Delta_0}(1-e(-\bar\alpha))^{\operatorname{mult}\alpha} = \sum_{w\in W_0\backslash W}\varepsilon(w)D(w(\rho))e(\overline{w(\rho)}-\bar\rho).$$

$W_0\backslash W$ denotes a set of representatives of left coset of W_0 in W.

Proof of (11.11.4).

indeed,
$$\frac{\prod_{\alpha\in\Delta_+}(1-e(-\alpha))^{\operatorname{mult}\alpha}}{\prod_{\alpha\in\Delta_+^0}(1-e(-\alpha))} = \frac{\sum_{w\in W}\varepsilon(w)e(w(\rho)-\rho)}{\prod_{\alpha\in\Delta_+^0}(1-e(-\alpha))}.$$

$$\|$$

$$\prod_{\alpha\in\Delta_+\setminus\Delta_0}(1-e(-\alpha))^{\operatorname{mult}\alpha} \xrightarrow{p} \prod_{\alpha\in\Delta_+\setminus\Delta_0}(1-e(-\bar\alpha))^{\operatorname{mult}\alpha}.$$

$$R_0^{-1}\sum_{w\in W}\varepsilon(w)e(w(\rho)-\rho) = \sum_{v\in W_0\backslash W}\varepsilon(v)\,R_0^{-1}\sum_{u\in W_0}\varepsilon(u)e(u(v(\rho))-\rho)$$

$$\left[\quad W=\bigcup_{v\in R}vW_0\ ,\quad \varepsilon(w)=\det{}_H\cdot w\ \ \&\ \ \varepsilon(w)\varepsilon(v)=\varepsilon(wv)\quad\right].$$

$$p\Big(\sum_{v\in W_0\backslash W}\varepsilon(v)\,(R_0^{-1}\sum_{u\in W_0}\varepsilon(u)e(u(v(\rho)))-\rho)\Big)$$

$$=\sum_{v\in W_0\backslash W}\varepsilon(v)\,p\,(R_0^{-1}\sum_{u\in W_0}\varepsilon(u)e(u(v(\rho)))\cdot e(-\rho)).$$

$$=\sum_{v\in W_0\backslash W}\varepsilon(v)D(v(\rho))e(\overline{v(\rho)})e(-\bar\rho)$$

· Now we consider a very special case of identity (11.11.4).
Fix a sequence of nonnegative integers $s=(s_1,\cdots,s_n)$ s.t. the subdiagram $\{i\in S(A)\mid s_i=0\}\subset S(A)$ is a union of diagrams of finite type. Fix an element $h^s\in H$ such that :

$$\langle\alpha_i, h^s\rangle = s_i\quad (i=1,\cdots,n).$$

The subspace $\mathbb{C}h^s$ of H satisfies (11.11.1) by prop 3.12.

[i.e. $\mathbb{C}h^s\cap \operatorname{Int}X_{\mathbb{C}}\neq\phi$.

If. $\mathbb{C}h^s\cap\operatorname{Int}X_{\mathbb{C}}=\phi$, $\underset{\text{prop 3.12(f).}}{\Rightarrow}$ $|W_{h^s}|=\infty$ where $W_{h^s}=\{w\in W\mid w(h_s)=h_s\}$.
: $|W_h|<\infty$ iff $h\in\operatorname{Int}X$

if $\langle\alpha_i,h^s\rangle=0$ $\Rightarrow$ $r_i\in W_{h^s}$ where $r_i=r_{\alpha_i^\vee}$.
since by prop 3.12 (a) W_{h^s} is generated by r_i.
then Contradicting to the condition $\{i\in S(A)\mid s_i=0\}$ is a union of finite type]

Define $\lambda\in(\mathbb{C}h^s)^*$ by $\langle\lambda,h^s\rangle=1$ and set $q=e(-\lambda)$

Then $p(e(-\alpha_i)) = q^{s_i}$ $(\star)$ i.e. p is the specialization of type s. (p180).

[$p(e(-\alpha_i)) = e(-\bar{\alpha}_i)$. $H^* \to (\mathbb{C}h^s)^*$
$\lambda \longmapsto \bar{\lambda}$.

$(e(-\lambda))^{s_i} = e(-s_i\lambda)$ & $\langle s_i\lambda, h^s\rangle = s_i$.
Then $p(e(-\alpha_i)) = (e(-\lambda))^{s_i} = q^{s_i}$]

Now (11.11.4) can be written as follows:

(11.11.5) $$\prod_{j\geq 1}(1-q^j)^{\dim g_j(s)} = \sum_{w\in W^s} \varepsilon(w) D_s(w(\rho)) q^{\langle \rho - w(\rho), h^s\rangle}$$

Here $D_s(\lambda) = \prod_{\alpha\in\Delta_{s+}} \frac{\langle\lambda,\alpha^\vee\rangle}{\langle\rho_s,\alpha^\vee\rangle}$, where $\Delta_{s+} = \{\alpha\in\Delta_+ \mid \langle\alpha, h^s\rangle = 0\}$.

and ρ_s. W^s: representatives of W_s. (is generated by r_α, $\alpha\in\Delta_{s+}$).

So. $W = W_s W^s$, $g(A) = \bigoplus_j g_j(s)$ is the $\mathbb{Z}$-gradation of $g(A)$ of type s.

[by $(\star)$ we have $e(-\bar{\alpha}_i) = q^{s_i} = q^{\langle\alpha_i, h^s\rangle}$.

(11.11.4) $\prod_{\alpha\in\Delta_+\setminus\Delta_0}(1-e(-\bar{\alpha}))^{\mathrm{mult}\,\alpha} = \sum_{w\in W_0\setminus W}\varepsilon(w) D(w(\rho)) e(\overline{w(\rho)} - \bar{\rho})$.

除去 $\langle\alpha, h^s\rangle = 0$ $\Downarrow$ $e(-\bar{\alpha}) = q^{\langle\alpha, h^s\rangle}$ $\Downarrow$ by $(\star)$.

$\prod_{j\geq 1}(1-q^j)^{\dim g_j(s)}$

$g_j(s) = \bigoplus_\alpha g_\alpha$

$\alpha = \sum k_i\alpha_i \in Q$ s.t. $\sum k_i s_i = j$.

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§ 11.12.

- discussing the unitarizability problem for the oscillator and the Virasoro algebras (c.f. § 9.13 and § 9.14).

Recall: Lie algebra $\mathcal{A} = (s + \mathbb{C}d_0) \oplus \mathcal{A}_0$. is called an oscillator algebra. where s is infinit. d. Heisenberg Lie alg.
d_0 is a derivation. $\mathcal{A}_0$ is a f. d. central ideal.

- $Vir = \bigoplus_{j\in\mathbb{Z}}\mathbb{C}d_j \oplus \mathbb{C}c$, where $\bigoplus_{j\in\mathbb{Z}}\mathbb{C}d_j$ is Lie algebra. (where d_j is a derivation of $\mathcal{L}(g)$), c is 1-dimensional central.

- A contravariant Hermitian form H on $\mathcal{A}$ (resp. Vir)-module V. is a Hermitian form w. r. t. whi

- $\mathcal{A}$-module V_1: $H(p_n x, y) = H(x, q_n y)$. $H(cx, y) = H(x, cy)$.
$H(ax, y) = H(x, ay)$ $\forall x, y\in V_1$

- Vir-module V_2: $H(d_n x, y) = H(x, d_{-n} y)$, $H(cx, y) = H(x, cy)$
$\forall x, y\in V_2$.

- $\mathcal{A}$-mod $R_{a,b,\lambda}$ (resp. the Vir-mod $L(c,h)$) admits a contravariant Hermitian form iff $a, b\in\mathbb{R}$, and $\lambda\in\mathcal{A}^*_{0\mathbb{R}}$ (resp. $c, h\in\mathbb{R}$). (c.f. P163 P164.

[where $L(c,h)$ is a irreducible Vir-module with high weigh c, h.
$R_{a,b,\lambda}$ is ......

· Lem 11.5. Let $\Lambda \in H_R^*$, the $g(A)$-module $L(\Lambda)$ carries a unique up to constant factor, nondegenerate contravariant Hermitian form.

and this form is uniquely determined by condition

$$H(1,1)=1 \quad (\text{resp } H(v_{c,h}, v_{c,h})=1$$

· A-module $R_{a,b,\lambda}$ is unitarizable iff $a>0$.

(9.13.3) $\quad B(x_1^{k_1}\cdots x_n^{k_n},\ x_1^{k_1}\cdots x_n^{k_n}) = a^{\sum k_i}\prod k_j!$

prop 11.2.

a) If the Vir-module $L(c,h)$ is unitarizable, then $h\geq 0$ and $c\geq 0$

proof: Let v be the highest-weight vector of a unit. Vir-mod $L(c,h)$. Then $H(d_{-1}(v), d_{-1}(v)) = H(v, d_1 d_{-1}(v))$

$= H(v, 2d_0 v) + H(v, d_{-1}d_1(v)) = 2h.$

[$[d_1, d_{-1}] = d_1 d_{-1} - d_{-1}d_1 = 2d_0$ & $d_0(v) = hv$ P163]

Hence unitarizability $\Rightarrow h\geq 0$.

looking at $H(d_{-j}(v), d_{-j}(v))$

$= H(v, d_j d_{-j}(v)) = 2jh + \frac{1}{12}(j^3-j)c.$

[$[d_i\ d_j] = (i-j)d_{i+j} + \frac{1}{12}(i^3-i)\delta_{i,-j}c$ P99.]

as $j\to\infty \Rightarrow c\geq 0$.

b). If $V = L(0,h)$ is unitarizable, then $h=0$, and hence V is the trivial 1-dimen Vir-module.

proof: Consider the subspace V_n of $L(0,h)$ spanned by $d_{-2n}(v)$ and $d_{-n}^2(v)$. Since $H(\ )$ is p.d. V_n, we have $\det_{V_n} H\geq 0$. which, after a simple calculation gives

$4n^3h^2(8h-5n)\geq 0$ for all integral $n>0$

$4n^3h^2(8h-9n)\geq 0.$

$\Rightarrow h=0.$

$$\left[\quad \begin{vmatrix} H(d_{-2n}(v), d_{-2n}(v)) & H(d_{-2n}(v), d_{-n}^2(v)) \\ H(d_{-n}^2(v), d_{-2n}(v)) & H(d_{-n}^2(v), d_{-n}^2(v)) \end{vmatrix} = \begin{vmatrix} 4nh & 6n^2h \\ 6n^2h & 8n^2h^2 \end{vmatrix}\right.$$

using $[d_j\ d_{-j}] = 2jd_0 + \frac{1}{12}(j^3-j)c = 2jd_0.$

$\cdot H(d_{-n}^2(v), d_{-n}^2(v)) = H(d_{-n}(v), (d_n d_{-n})d_{-n}(v))$

$= H(d_n(v), d_{-n}(d_n d_{-n})(v)) + H(d_{-n}(v), 2nh\, d_{-n}(v)).$

$= H(d_{-n}(v), 2nh\, d_{-n}(v)) + H(d_{-n}(v), 2nh\, d_{-n}(v))$

$= 4nh\, H(d_{-n}(v), d_{-n}(v)) = 8n^2h^2.$

$\cdot H(d_{-n}^2(v), d_{-2n}(v)) = H(d_{-n}(v), d_n d_{-2n}(v)) = 6n^2h.$]

· trivial since $d_0(v) = h(v) = 0$, $c(v) = 0$.

· 1-dim: since $h\geq 0$ and $c\geq 0$ by a), then $\forall v'\in V$, we have $d_0(v') = c(v') = 0$ since v is highest weight vector.

i.e $\forall v'\in V$, $v' = kv$, $k\in\mathbb{C}$.

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c) Let V be a unitarizable Vir-module s.t. d_0 is diagonalizable with f.d. eigenspaces and with spectrum bounded below.
Then V decomposes into an orthogonal direct sum of unitarizable Vir-module $L(c,h)$, and the spectrum of d_0 is non-negative.

proof: Note that if U is a Vir-submodule of V, it is graded w.r.t. the eigenspace decomposition of d_0. (by prop 1.5) p8.
hence

$$V = U \oplus U^{\perp}, \quad \text{since } U^{\perp} \text{ is also Vir-submodule.}$$

we deduce that V is an orthogonal direct sum of unitarizable irreducible of unitariable irreducible Vir-module V_i s.t. d_0 is diagonalizable on V_i with spectrum bounded below.
Thus the V_i are highest-weight modules $L(c,h)$.
(p164: irreducible Vir-module $L(c,h)$ with hight weight (c,h) is unique)
and $d_0 \geq 0$ by a)

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