

Then we have $a_{w(\mu)} = a_{\mu}$ for $w \in W$.

This simple formula reduces the computation of the character of $L(\lambda)$ to the computation of the functions a_{λ} . This follows the case is especially simple. Recall: $(12, b, \gamma) : P(\lambda) = \{ \lambda_0 + \frac{1}{2} |\lambda|^2 \delta + \alpha - (\frac{1}{2} |\lambda|^2 + 5) \delta, \text{ where } \alpha \in \bar{\lambda} + \bar{\alpha}, s \in \mathbb{Z}^+ \}$.

Lemma 12.7. Let λ and μ be as in Lemma 12.6. Then $a_{\lambda} = a_{\mu}$. Recall: a_{λ} is (12.7.1)

$$e^{-\frac{1}{2} |\lambda|^2 \delta} \text{ch}_L(\lambda) = a_{\lambda_0} \sum_{\gamma \in \bar{\lambda} + \bar{\alpha}} e^{\lambda_0 + \gamma - \frac{1}{2} |\gamma|^2 \delta}$$

Proof: Use (12.6.3) and the fact that every a_{λ} is W -conjugate to

$$a_{\lambda} = a_{\lambda_0} \text{ due to (12.4.5) and Lemma 12.6.}$$

Try (12.6.3) and (12.7.1) and (12.6.2) $\Rightarrow \text{level}(\lambda) = 1$ iff $\lambda \in \text{Aut}(S(A)) \cdot 0$ (level $(\lambda) = 1$, then $\lambda = \lambda_0 + c\delta$).

$$\text{ch}_L(\lambda) = a_{\lambda} \sum_{\gamma \in \bar{\lambda} + \bar{\alpha}} e^{\lambda_0 + \gamma - \frac{1}{2} |\gamma|^2 \delta}$$

$$\text{i.e. } e^{-\frac{1}{2} |\lambda|^2 \delta} \text{ch}_L(\lambda) = a_{\lambda} \sum_{\gamma \in \bar{\lambda} + \bar{\alpha}} e^{\lambda_0 + \gamma - \frac{1}{2} |\gamma|^2 \delta}$$

then $\lambda = \lambda_0 + c\delta$ and there exists a $\tau \in \text{Aut}(S(A))$

$$u_\lambda = a_{\lambda,0}$$

#

(The function $a_{\lambda,0}$ will be calculated explicitly at the end of this chapter.)

We proceed to rewrite character formulas (10.4.5) and (12.7.1) in terms of theta functions.

$$(10.4.5) \quad \chi(\lambda) = \sum_{w \in W} \varepsilon(w) e(w(\lambda + \rho)) / \sum_{w \in W} \varepsilon(w) e(w(\rho))$$

For $\lambda \in H^+$ such that $\text{level}(\lambda) = k > 0$, set

$$(12.7.2) \quad \theta_\lambda = e^{-\frac{|\lambda|^2}{2k}} \sum_{\alpha \in T} e^{\alpha(\lambda)}, \quad A_\lambda = \sum_{w \in W} \varepsilon(w) \theta_{w(\lambda)}$$

$$\text{Using (6.5.5): } t_\alpha(\lambda) = m\lambda_0 + (\bar{\lambda} + m\alpha) + \frac{1}{2m}(|\lambda|^2 - |\bar{\lambda} + m\alpha|^2)\delta.$$

we obtain where $m := \langle \lambda, k \rangle$

$$(12.7.3) \quad \theta_\lambda = e^{k\lambda_0} \sum_{\gamma \in M + k^{-1}\bar{\lambda}} e^{-\frac{1}{2}k|\gamma|^2\delta + k\gamma}$$

which is a classical theta function (see chap 13 for details)

$$\begin{aligned} \text{[proof of (12.7.3): } \theta_\lambda &= \sum_{\alpha \in M} e^{-\frac{|\lambda|^2}{2k}} e^{t_\alpha(\lambda)} = \sum_{\alpha \in M} e^{k\lambda_0 + (\bar{\lambda} + k\alpha) - \frac{|\lambda|^2}{2k} + \frac{|\lambda|^2}{2k} - \frac{1}{2k}|\bar{\lambda} + k\alpha|^2\delta} \\ &= e^{k\lambda_0} \sum_{\alpha \in M} e^{\bar{\lambda} + k\alpha - \frac{1}{2k}|\bar{\lambda} + k\alpha|^2\delta} = e^{k\lambda_0} \sum_{\gamma \in M + k^{-1}\bar{\lambda}} e^{k\gamma - \frac{1}{2}k|\gamma|^2\delta} \end{aligned}$$

(this series converges absolutely on γ to a holomorphic function.)

Note that θ_λ depends only on $\lambda \bmod kM + \phi\delta$? $\theta_\lambda = \theta_{\lambda_1}$ if $\lambda_1 = \lambda + kM + \phi\delta$
 $\lambda_1 = \lambda + kM + \phi\delta \Rightarrow |\lambda_1|^2 = |\lambda|^2$

Using theta functions, we can rewrite the denominator identity (10.4.4) in yet another form.)

Recall: (10.4.4) $\prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{\text{mult } \alpha} = \sum_{w \in W} e(w) e(w(\rho) - \rho)$

(the dual Coxeter number) $\langle \rho, K \rangle = h^\vee$

(6.2.8): $\rho = \bar{\rho} + h^\vee \lambda_0$

Using the decomposition $W = \tilde{W} \ltimes T$, we get:

$$\sum_{w \in W} e(w) e(w(\rho) - \rho) = e^{-\rho} \sum_{w \in \tilde{W}} e(w) \sum_{\alpha \in M} e^{w(\alpha)\rho} = e^{-\rho + \frac{|\rho|^2}{2h^\vee} \delta} A_\rho.$$

Hence, (10.4.4) can be rewritten as follows:

(12.7.4) $A_\rho = \sum_{w \in \tilde{W}} e(w) \underbrace{e(w(\lambda_0 + \rho))}_{w(\lambda_0) = \lambda_0} = \underbrace{e^{\lambda_0 + \bar{\rho} - \frac{|\rho|^2}{2h^\vee} \delta}}_P \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\text{mult } \alpha}.$
by (10.4.4) & equality above

Introduce the following number for $\lambda \in \rho_+^\vee = \{ \lambda \in \rho_+ \mid \langle \lambda, K \rangle = k \}$.

(12.7.5) $m_\lambda = \frac{|\lambda + \rho|^2}{2(k + h^\vee)} - \frac{|\rho|^2}{2h^\vee}$

this number is called the molecular anomaly of λ (分子異常)

and the normalized character:

$$\chi_\lambda = e^{-m_\lambda \delta} \text{Ch}(\chi_\lambda)$$

For a weight $\lambda \in \rho(\Lambda)$, introduce the number

(12.7.6) $m_{\lambda, \Lambda} = m_\lambda - \frac{|\lambda|^2}{2k}$

(it is clear that $m_{\lambda, \Lambda}$ is a rational number.)

For $\lambda \in \mathfrak{H}^*$, set

(12.7.7) $C_\lambda^\wedge = e^{-m_{\lambda, \Lambda} \delta} \sum_{n \in \mathbb{Z}} \text{mult}_{L(\Lambda)}(\lambda - n\delta) e^{-n\delta}$

The function $C_\lambda^\wedge$ is called the string function of $\lambda \in \mathfrak{H}^*$.

(Just as the series $a_\lambda^\wedge$, the series converges absolutely to a holomorphic function on $\mathfrak{Y}$.)

Remark: (1) $C_\lambda^\wedge = e^{-m_{\lambda, \Lambda} \delta} a_\lambda^\wedge$ if $\lambda \in \text{max}(\Lambda)$.

(2) $\left(\text{if } a_\lambda^\wedge = \sum_{n=0}^{\infty} \text{mult}_{L(\Lambda)}(\lambda - n\delta) e^{-n\delta}, \text{ if } \lambda \in \text{max}(\Lambda), \text{ then } n \geq 0. \right)$

(3) (12.7.8) $C_{w(\lambda)}^\wedge = C_\lambda^\wedge$ for $w \in W$

(4) $\left(\text{if } \text{mult}_{L(\Lambda)}(\lambda - n\delta) = \text{mult}_{L(\Lambda)}(w(\lambda) - n\delta) = \text{mult}_{L(\Lambda)}(w(\lambda - n\delta)) \text{ if } \lambda \in \rho_+ \right)$

(5) (12.7.9) $C_{w(\alpha) + k\rho + a\delta}^\wedge = C_\alpha^\wedge$ for $\lambda \in \mathfrak{H}^*$, $w \in \tilde{W}$, $r \in M$, $a \in \mathbb{C}$.

Since $W = \tilde{W} \ltimes T$ and (6.2.8): $t_\alpha(\lambda) = \lambda + \langle \lambda, K \rangle \alpha - (|\lambda| \alpha) + \frac{|\lambda|^2}{2} \langle \lambda, K \rangle \delta$

Then $C_\lambda^\wedge = C_{w(\lambda)}^\wedge = C_{w(\lambda) + k w(\alpha) + a\delta}^\wedge = C_{w(\alpha) + k\rho + a\delta}^\wedge$ where $w \in \tilde{W}$, $\alpha \in M$, $r = w(\alpha) \in M$, $a \in \mathbb{C}$

(4) (12.7.10) $C_\lambda^\wedge = C_\lambda^\wedge + a\delta$ for $a \in \mathbb{C}$.

Γ by (12.7.5) $m_\lambda = \frac{1 + |P|^2}{2(k+h)} - \frac{|P|^2}{2h}$, $m_{\lambda, \lambda} = m_\lambda - \frac{|P|^2}{2h}$.

(12.7.7) $\text{mult}_{L(\lambda)}(\lambda - n\delta) = \text{mult}_{L(\lambda + a\delta)}(\lambda - n\delta)$?

$\langle \lambda + a\delta, h \rangle = \langle \lambda, h \rangle$

• Using $W = \tilde{W} \rtimes T$ and (12.7.1), we can rewrite the character formula (10.4.5) as follows:

(12.7.11) $\chi_\lambda = A_{\lambda+p} / A_p$ (by (12.7.4) $\Rightarrow$ via N)

Γ $A_p = e^{-\frac{|P|^2}{2h}} \sum_{w \in W} \epsilon(w) e^{w(p)}$, $A_{\lambda+p} = e^{-\frac{1 + |P|^2}{2(k+h)}} \sum_{w \in W} \epsilon(w) e^{w(\lambda+p)}$

then $A_{\lambda+p} / A_p = e^{-m_\lambda \delta} \text{ch}_{L(\lambda)} = \chi_\lambda$ by (12.7.5)

• On the other hand, we have by (12.7.1) and the definitions of θ_λ and $C_\lambda^\wedge$:

$$\chi_\lambda = \sum_{\substack{\lambda \in p^k \text{ mod } \mathbb{C}\delta \\ \lambda \text{ mod } T}} C_\lambda^\wedge \theta_\lambda.$$

Γ (12.7.11) $\text{ch}_{L(\lambda)} = \sum_{\lambda \in \text{max}(\lambda)} e^{\lambda} a_\lambda^\wedge = \sum_{\substack{\lambda \in \text{max}(\lambda) \\ \lambda \text{ mod } T}} \left(\sum_{\tau \in T} e^{\tau \alpha} \right) a_\lambda^\wedge$

(12.7.12) $\theta_\lambda = e^{-\frac{|P|^2}{2h}} \sum_{\tau \in T} e^{\tau \alpha}$,

(12.7.7) $C_\lambda^\wedge = e^{-m_\lambda \delta} \sum_{n \in \mathbb{Z}} \text{mult}_{L(\lambda)}(\lambda - n\delta) e^{-n\delta}$.

• $\chi_\lambda = e^{-m_\lambda \delta} \text{ch}_{L(\lambda)} = e^{-m_\lambda \delta} \sum_{\substack{\lambda \in \text{max}(\lambda) \\ \lambda \text{ mod } T}} \left(e^{-\frac{|P|^2}{2h}} \theta_\lambda \right) e^{m_\lambda \delta} C_\lambda^\wedge$ by Rem (1).

$= \sum_{\substack{\lambda \in \text{max}(\lambda) \\ \lambda \text{ mod } T}} \theta_\lambda C_\lambda^\wedge$ since $m_{\lambda, \lambda} = m_\lambda - \frac{|P|^2}{2h}$

$= \sum_{\substack{\lambda \in p^k \text{ mod } \mathbb{C}\delta \\ \lambda \text{ mod } T}} C_\lambda^\wedge \theta_\lambda$ by Rem (4) and a maximal weight is W -equivalent to a unique dominant maximal weight.

• Using (12.7.9) we can rewrite this as follows:

(12.7.12) $\chi_\lambda = \sum_{\lambda \in p^k \text{ mod } (kM + \mathbb{C}\delta)} C_\lambda^\wedge \theta_\lambda$

Γ and the argument before (12.7.3) i.e. θ_λ depends only on $\lambda \text{ mod } kM + \mathbb{C}\delta$

Comparing (12.7.11) and (12.7.12) gives the theta function identity

(12.7.13) $A_{\lambda+p} = A_p \sum_{\lambda \in p^k \text{ mod } (kM + \mathbb{C}\delta)} C_\lambda^\wedge \theta_\lambda$

§ 12.8. Sugawara construction

(For the sake of simplicity, we consider only the case of a non-degenerated affine algebra $g' = g'(X_i^{(n)})$, leaving the general case as an exercise (12.10).)

Recall that $g' = \tilde{L}(g) = \mathbb{C}[t, t^{-1}] \otimes g + \mathbb{C}K$ with commutation relations

$$(12.8.1) \quad [x^{(m)}, y^{(n)}] = [x, y]^{(m+n)} + m\delta_{m,n}(x|y)K$$

(Here g is a simple $\mathfrak{f}$ -d Lie algebra of type X_c , $(x|y)$ is the normalized (by (6.4.5): $(\alpha|\alpha) = 2$ if $\alpha \in \Delta_c$) invariant bilinear form on g , and $x^{(n)}$ stands for $t^n \otimes x$ ($n \in \mathbb{Z}$, $x \in g$))

• let $\{u_i\}$ and $\{u^i\}$ be dual bases of g , i.e. $(u_i|u^j) = \delta_{ij}$.

Recall that $\tilde{\Omega} = \sum_i u_i u^i$ (the Casimir operator of g is independent of the choice of these dual bases. (see ex. 8.17)) In particular, $\sum_i u_i u^i = \sum_i u^i u_i$

(This proves the following useful formula:)

Rem:

$$(1) \quad (12.8.2) \quad \sum_i [u_i^{(m)}, u^{i(n)}] = m\delta_{m,n}(\dim g)K.$$

$$\text{Proof: } \sum_i [u_i^{(m)}, u^{i(n)}] = \sum_i [u_i u^i]^{(m+n)} + m\delta_{m,n}(u_i|u^i)K$$

$$= \left(\sum_i u_i u^i - \sum_i u^i u_i \right)^{(m+n)} + \sum_i m\delta_{m,n}(u_i|u^i)K$$

$$= m\delta_{m,n}(\dim g)K. \quad (\text{易证 } \uparrow \text{ 数也是 } g \text{ 的秩})$$

(2) (12.8.3) $\Omega = 2(K + h^\vee)d + \tilde{\Omega} + 2 \sum_{n=1}^{\infty} \sum_i u_i^{(-n)} u^{i(n)}$ is the Casimir operator of the affine algebra $g = g(X_i^{(n)}) = g' + \mathbb{C}d$, with $[d, x^{(n)}] = n x^{(n)}$

Recall: $P_{\text{irr.}} \Omega = 2\nu^{-1}(p) + \sum_i u_i u^i + \Omega_0$ where $\Omega_0 = 2 \sum_{\alpha \in \Delta_+} \sum_i e_{-\alpha}^{(1)} e_{\alpha}^{(1)}$

~~need to prove $\nu^{-1}(p) = (K + h^\vee)d$?~~

• Since Ω is independent of the choice of the dual bases $\{u_i\}$ and $\{u^i\}$, we can check this for a special choice of these bases

consider the root space decomposition $g = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathbb{C}e_{\alpha}$, s.t.

$(e_{\alpha}|e_{-\alpha}) = 1$, and choose dual bases $h_1, \dots, h_r$ and $h^1, \dots, h^r$ of $\mathfrak{h}$. Then we have: $\tilde{\Omega} = \sum_i h_i h^i + \sum_{\alpha \in \Delta} e_{-\alpha} e_{\alpha}$

Using that $[e_{\alpha}, e_{-\alpha}] = \nu^{-1}(\alpha)$ (by Th. 7.3.6) and that

$\tilde{p} = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$, we obtain the form of $\tilde{\Omega}$ given in § 7.5:

$$\tilde{\Omega} = \sum_{i=1}^r h_i h^i + 2\nu^{-1}(\tilde{p}) + 2 \sum_{\alpha \in \Delta_+} e_{-\alpha} e_{\alpha} \quad \sum_{\alpha \in \Delta} \sum_i e_{\alpha}^{(1)} e_{\alpha}^{(1)} + \sum_i h_i h^i$$

(Hence (12.8.3) can be rewritten in the same form as Ω def. in § 7.5, due to (6.3.5), (6.3.8) and § 7.4 (see ex. 7.16) Piv.)

By (6.3.5) $(K|\alpha_i^\vee) = 0$ ($i = 0, \dots, l$), $(K|K) = 0$, $(K|d) = a_0 = 1$

$$(6.3.8) \quad p = \tilde{p} + h^\vee \Lambda_0$$

need to prove $\nu^{-1}(\tilde{p}) + h^\vee d = \nu^{-1}(p) = \nu^{-1}(\tilde{p} + h^\vee \Lambda_0) = \nu^{-1}(\tilde{p}) + h^\vee d$

$2(K|d)$ is part of dual basis of $\mathfrak{h}$, since $d^\vee, \dots, d_l^\vee, d$ form a basis of $\mathfrak{h}$: $\mathfrak{h} = \mathfrak{h} + \mathbb{C}K + \mathbb{C}d$.

In order to perform calculations, it is convenient to introduce the restricted completion $\bar{U}(g')$ of the universal enveloping algebra $U(g')$.

Consider the category of all restricted g' -modules, i.e. modules V such that for any $v \in V$, $x^{(j)}(v) = 0$ for all $x \in g$ and all $j \gg 0$.
 [Result: For a $g(A)$ -module (resp. $g'(A)$ -module) V is called restricted if for every $v \in V$, we have $g_\alpha(v) = 0$ for all but a finite number of positive roots α .]

Consider all series $\sum_{j=1}^{\infty} u_j$ with $u_j \in U(g')$ such that for any restricted g' -module V and any $v \in V$, $u_j(v) = 0$ for all but finitely many u_j .

(we identify two such series if they represent the same operator in every restricted g' -module.)

We thus obtain an algebra $\bar{U}(g')$ which contains $U(g')$ (since, by ex. 11.22, $\text{Pr} 11$: $\bigoplus L(\lambda)$ is a faithful $U(g')$ -module) and acts on every restricted g' -module.
 (可积的不同模块的直和)

[if M is an R -module and $\text{Ann}_R(M) = 0$, then M is called a faithful module.]

c.f. P.S. prop 9.10. b)

$$A \hookrightarrow \text{End}(M)$$

faithful $\Rightarrow$ A 中元素

$$U(g') \hookrightarrow \bigoplus L(\lambda) \text{ but } \bar{U}(g')$$

(Note that the derivations d_n (see §7.5) extend from $U(g')$ to derivations of $\bar{U}(g')$.)

Now we introduce the Sugawara operators T_n ($n \in \mathbb{Z}$).

$$(17.8.4) \quad T_0 = \sum_i u_i u^i + 2 \sum_{n=1}^{\infty} \sum_i u_i^{(-n)} u^{i(n)}$$

$$T_n = \sum_{m \in \mathbb{Z}} \sum_i u_i^{(-m)} u^{i(m+n)} \text{ if } n \neq 0.$$

Rem:

(1) Due to (17.8.2) $T_0, T_n \in \bar{U}(g')$

(2) T_0, T_n are independent of the choice of dual bases of g .