$\ldots = -d_0(x^{(m)}) = \ldots m x$

Lemma 12.8.

a) For $x \in \mathring{g}$ and $n, m \in \mathbb{Z}$ one has:

$$[x^{(m)}, T_n] = 2(K + h^\vee) m x^{(m+n)}$$

Proof: [Recall: P99. $D = \bigoplus_{j \in \mathbb{Z}} \mathbb{C} d_j$ is a Lie algebra with commutation relations: $[d_i, d_j] = (i-j) d_{i+j}$]

· We perform the calculations in the <u>semidirect product $D \ltimes V_c(g')$</u>

First, note

$$(12.8.5) \quad T_0 = -2(K + h^\vee) d + \Omega \quad (\text{by } (12.8.3))$$

Also, it is straightforward to check using (12.8.2) that

$$T_j = \frac{1}{j} [d_j, T_0] \quad \text{if } j \neq 0.$$

[· Given by a Lie action of a Lie algebra g on another Lie algebra A, hence a Lie algebra homomorphism to the derivation on A.

$$\rho : g \to \mathrm{Der}(A)$$

then there is a Lie algebra extension of g by A whose underlying vector space is $\hat{g} = g \oplus A$. and whose Lie bracket is given by the formula. $\quad D \quad V_c(g')$ Note that D is a $\mathbb{Z}$-graded subalg. in $\mathrm{Der}\, L(\mathring{g})$

$$[(x_1, t_1), (x_2, t_2)] = ([x_1, x_2], [t_1, t_2] + \rho(x_1)(t_2) - \rho(x_2)(t_1))$$

then $[d_j, T_0] = [d_j, \sum_i u_i u^i + 2\sum_{n=1}^{\infty} \sum_i u_i^{(-n)} u^{i(n)}]$

$$= \rho(d_j)\left(\sum_i u_i u^i + 2\sum_{n=1}^{\infty} \sum_i u_i^{(-n)} u^{i(n)}\right) \qquad \left(d_j|_{L(\mathring{g})} = -t^{j+1} \frac{d}{dt} \right)$$

$$(*) = d_j \left(2 \sum_{n=1}^{\infty} \sum_i u_i^{(-n)} u^{i(n)} \right) = 2 \sum_{n=1}^{\infty} \sum_i n \left(u_i^{(j-n)} u^{i(n)} - u_i^{(-n)} u^{i(j+n)} \right)$$

since $\sum_i u_i^{(m)} u^{i(n)} = \sum_i u^{i(m)} u_i^{(n)} = \sum_i u_i^{(n)} u^{i(m)} + m\,\delta_{m,-n} (\dim \mathring{g}) K$

(independent of the choice of dual basis) $\qquad$ (12.8.2)

hence $(*) = 2\sum_{n=1}^{\infty} \sum_i \left(-n u_i^{(-n)} u^{i(n+j)} + n u_i^{(n)} u^{i(-n+j)} \right) + \underbrace{(-n+j)\delta \dim \mathring{g} K}_{=0}$

$$= -2 \sum_{n \in \mathbb{Z}} \sum_i n u_i^{(-n)} u^{i(n+j)} + \cancel{(-n+j)\delta_{-n+j,-n} \dim \mathring{g} K} \qquad (\text{since } j \neq 0).$$

右第二项 let $-n = r+j$

$$= -\sum_{n \in \mathbb{Z}} \sum_i n u_i^{(-n)} u^{i(n+j)} - \sum_{n \in \mathbb{Z}} \sum_i n u_i^{(n+j)} u^{i(-n)} - \underbrace{(-n\delta_{-n,-n-j} \dim \mathring{g} K)}_{=0}$$

右第二项 let $r = n$

$$= -\sum_{n \in \mathbb{Z}} \sum_i n u_i^{(-n)} u^{i(n+j)} + \sum_{r \in \mathbb{Z}} \sum_i (r+j) u_i^{(-r)} u^{i(r+j)}$$

$$= j \sum_{n \in \mathbb{Z}} \sum_i u_i^{(-n)} u^{i(n+j)} = j T_j$$

· Now we have, using Th.2.6. and (12.8.5):

$$[x^{(m)}, T_0] = [x^{(m)}, -2(K + h^\vee) d + \Omega] = [x^{(m)}, 2(K + h^\vee) d_0]$$
$$= 2(K + h^\vee) m x^{(m)}$$

recall: ① $d = -d_0$ ② Th.2.6 P23: if V is a restricted $g(A)$-module, then Ω commutes with the action of $g(A)$ on V. ③ $[d, x^{(n)}] = n x^{(n)}$

Finally, for $j \neq 0$, we have: $\left[[x^{(m)}, d_j] = -d_j(x^{(m)}) = \ldots m x^{(m+j)} = m x^{(m+j)} \right]$

$$[x^{(m)}, T_j] = \frac{1}{j}[x^{(m)}, [d_j, T_0]] = \frac{1}{j}[[x^{(m)}, d_j], T_0] + \frac{1}{j}[d_j, [x^{(m)}, T_0]]$$

$$= \frac{1}{j}[m x^{(m+j)}, T_0] + \frac{1}{j} 2(K+h^\vee)[d_j, m x^{(m)}]$$

$$= \frac{1}{j} m\, 2(K+h^\vee)(m+j) x^{(m+j)} + \frac{1}{j} 2(K+h^\vee)(-m^2 x^{(m+j)})$$

$$= 2(K+h^\vee) m x^{(m+j)}$$

#

2) Let V be a restricted $\mathfrak{g}$-module and let $v \in V$ be such that $n_+(v) = 0$ and $h(v) = \langle \lambda, h \rangle v$ for some $\lambda \in H^*$. Then

$$T_0(v) = (\bar\lambda \mid \bar\lambda + 2\bar\rho) v$$

proof. follows immediately from (12.8.5) and (2.6.2).

[Recall: P24. (2.6.2) $\Omega(v) = (\lambda + 2\rho \mid \lambda) v$.

Then $T_0(v) = \Big(\sum_i u_i u^i + 2\underbrace{\sum_{n=1}^{\infty} \sum_i u_i^{(-n)} u^{i(n)}}_{=0} \Big)(v)$ $\quad (L(\mathfrak{g})_{j\delta} = t^j \otimes \mathfrak{H})$

? $= \mathring{\Omega}(v) = (\bar\lambda \mid \bar\lambda + 2\bar\rho) v.$ (by P27 ex. 28).

$T_0(v) = -2\lambda(K)\lambda(d)v - 2h^\vee \lambda(d) v + (\lambda \mid \lambda + 2\rho) v.$ | $\lambda = \bar\lambda + \langle \lambda, K \rangle \Lambda_0 + (\lambda \mid \Lambda_0)\delta$

$=$ | $\rho = \bar\rho + h^\vee \Lambda_0$]

#

- Now we can calculate $[T_m, T_n]$.

(One should be careful about staying within the algebra $U_c(\mathfrak{g}')$.)

(1) We may assume that $m > n$. Let first $m+n \neq 0$, $m \neq 0$, $n \neq 0$.

Then we have:

$$[T_m, T_n] = \sum_{j \in \mathbb{Z}} \sum_i [u_i^{(-j)} u^{i(m+j)}, T_n]$$

$$= \sum_{j \in \mathbb{Z}} \sum_i [u_i^{(-j)} u^{i(m+j)}, T_n] - u_i^{(-j)} T_n u^{i(m+j)} + u_i^{(-j)} T_n u^{i(m+j)}$$

$$= \sum_{j \in \mathbb{Z}} \sum_i \left(u_i^{(-j)} [u^{i(m+j)}, T_n] + [u_i^{(-j)}, T_n] u^{i(m+j)} \right)$$

(by Lem. 12.8) $= 2(K+h^\vee) \sum_{j \in \mathbb{Z}} \sum_i \left((m+j) u_i^{(-j)} u^{i(m+j+n)} \underline{- j u_i^{(-j+n)} u^{i(m+j)}} \right)$ (*)

Replacing j by $j+n$ in (*) we obtain:

(12.8.6) $[T_m, T_n] = 2(K+h^\vee) \sum_{j \in \mathbb{Z}} \sum_i \left((m+j) u_i^{(-j)} u^{i(m+j+n)} - (j+n) u_i^{(-j)} u^{i(m+j+n)} \right)$

$$= 2(K+h^\vee) \sum_{j \in \mathbb{Z}} \sum_i (m-n) u_i^{(-j)} u^{i(m+n+j)}$$

Thus, we have, provided that $m+n \neq 0$, $m \neq 0$, $n \neq 0$.

(12.8.7) $[T_m, T_n] = 2(K+h^\vee)(m-n) T_{m+n}$

(2) A similar calculation shows that (12.8.7) holds when $m+n \neq 0$ but m or $n = 0$.

(3) Let now $m+n=0$, $m>0$. Then the right-hand side of (12.8.6) does not lie in $U_c(g')$. ?

We proceed as follows: since $\mathring{\Omega}$ is independent of the choice of dual bases, we have:

$$\sum_i u_i^{(-j)} u^{i(j+m)} = \sum_i u_i^{(j+m)} u^{i(-j)} \quad \text{when } m \neq 0.$$

Hence we can write (here and further we drop the sign of summation over i, but assume that it is present)

$$T_m = \sum_{j\geq 0} u_i^{(-j)} u^{i(j+m)} + \sum_{j>0} u_i^{(-j+m)} u^{i(j)}$$

$$\left[\ T_n = \sum_{m\in\mathbb{Z}} \sum_i u_i^{(-m)} u^{i(m+n)} \quad \text{if } n\neq 0 \ \right]$$

We have:

$$\cancel{[T_m, T_{-m}] = \Big[\sum_{j\geq 0} u_i^{(-j)} u^{i(j+m)} + \sum_{j>0} u_i^{(-j+m)} u^{i(j)},\ \sum_{j\geq 0} u_i^{(-j)} u^{i(j-m)} + \sum_{j>0} u_i^{(-j-m)} u^{i(j)}\Big]}$$

(Recall: $[T_m, T_n] = 2(K+h^\vee) \sum_{j\in\mathbb{Z}} \sum_i \left((m+j) u_i^{(-j)} u^{i(m+j+n)} - j u_i^{(-j+n)} u^{i(m+j)}\right)$)

$$[T_m, T_{-m}] = \underline{2(K+h^\vee)\, m\mathring{\Omega}} \quad (j=0)$$

$$+\ 2(K+h^\vee) \sum_{j>0} \left((j+m) u_i^{(-j)} u^{i(j)} - j u_i^{(-j-m)} u^{i(j+m)}\right) \quad (j>0)$$

$$+\ 2(K+h^\vee) \sum_{j>0} \left((-j+m) u_i^{(j)} u^{i(-j)} + j u_i^{(j-m)} u^{i(-j+m)}\right) \quad (j<0)$$

[换易两交换 $= 2(K+h^\vee) \sum_{j>0} \left((-j+m) u^{i(j)} u_i^{(-j)} + j u^{i(j-m)} u_i^{(-j+m)}\right)$

$$= 2(K+h^\vee) \sum_{j>0} (-j+m) u_i^{(-j)} u^{i(j)} + (-j+m) j \dim \mathring{g} K + j u_i^{(-j+m)} u^{i(j-m)} + j(j-m) \dim \mathring{g} K$$

$$= 2(K+h^\vee) \sum_{j>0} (-j+m) u_i^{(-j)} u^{i(j)} + j u_i^{(-j+m)} u^{i(j-m)}$$]

Continue above:

$$[T_m, T_{-m}] = 2(K+h^\vee) m\mathring{\Omega} + 2(K+h^\vee) \sum_{j>0} \Big(\underline{(j+m) u_i^{(-j)} u^{i(j)}} - j u_i^{(-j-m)} u^{i(j+m)}$$
$$+\ j u_i^{(-j+m)} u^{i(j-m)} + \underline{(m-j) u_i^{(-j)} u^{i(j)}}\Big)$$

$$= 2(K+h^\vee)\Big(mT_0 + \sum_{j>0} j u_i^{(-j+m)} u^{i(j-m)} - j u_i^{(-j-m)} u^{i(j+m)}\Big)$$

[Recall $T_0 = \mathring{\Omega} + 2\sum_{n\geq 1} \sum_i u_i^{(-n)} u^{i(n)}$]

Replacing j by $j+m$ in the first summation and j by $j-m$ in the second summation, we obtain:

$$[T_m, T_{-m}] = 2(K+h^\vee)\Big(mT_0 + \sum_{j>-m} (j+m) u_i^{(-j)} u^{i(j)} + \sum_{\boxed{j>m}} (m-j) u_i^{(-j)} u^{i(j)}\Big)$$

$$= 2(K+h^\vee)\Big(2mT_0 + \underline{\sum_{j=-m+1}^{-1} (j+m) u_i^{(-j)} u^{i(j)}} - \underline{\sum_{j=1}^{m-1} (m-j) u_i^{(-j)} u^{i(j)}}\Big) \quad (T_0 \text{部分多加了})$$

对中 $-j\to j$

$$= 2(K+h^\vee)\Big(2mT_0 + \sum_{j=1}^{m-1} (-j+m)\big(u_i^{(j)} u^{i(-j)} - u_i^{(-j)} u^{i(j)}\big)\Big) \quad (*)$$

交易

$$= 2(K+h^\vee)\Big(2mT_0 + \sum_{j=1}^{m-1} (m-j)\,[u^{i(j)}, u_i^{(-j)}]\Big) \overset{(12.8.2)}{=} 2(K+h^\vee)\Big(2mT_0 + \sum_{j=0}^{m-1} j(m-j)(\dim \mathring{g})K$$

(为0也无多项)

since $\sum_{j=0}^{m-1} j(m-j) = \frac{m^3-m}{6}$

$\ulcorner$ $\sum_{j=0}^{m-1} j(m-j) = m-1 + 2(m-2) + 3(m-3) + \cdots + (m-1)(m-m+1)$

$= m + 2m + \cdots + (m-1)m - (1+4+9+\cdots+(m-1)^2)$

$= \frac{m(m-1)m}{2} - \frac{(2m-1)m(m-1)}{6} = \frac{m^3-m}{6}$ $\lrcorner$

combining with (12.8.7), we obtain the final formula:

(12.8.8) $[T_m, T_n] = 2(K+h^\vee)\left((m-n)T_{m+n} + \delta_{m,-n}\frac{m^3-m}{6}(\dim \mathring{g})K\right)$.

- we immediately obtain from (12.8.8) and Lem 12.8 the following.

Cor. 12.8 Let V be a restricted g'-module such that K is a scalar operator kI, $k \neq -h^\vee$, let:

(12.8.9) $L_n = \frac{1}{2(k+h^\vee)} T_n, \quad n \in \mathbb{Z}$

(12.8.10) $c(k) = \frac{k(\dim \mathring{g})}{k+h^\vee}$,

大到小的限制.

(12.8.11) $h_\Lambda = \frac{(\Lambda+2\rho|\Lambda)}{2(k+h^\vee)}$ if $V = L(\Lambda)$.

小的代数模扩张成大的代数模是不平凡的.

a) letting $d_n \mapsto L_n$, $c \mapsto c(k)$ extends V to a module over $g' + Vir$ (the semidirect sum defined in §7.3). In particular, V extends to a module over g $(= g' + \mathbb{C}d)$ by letting $d \mapsto -L_0$.

$\ulcorner$Recall: P99. The semidirect product of Lie algebras $Vir + \hat{\mathcal{L}}(\mathring{g})$ defined by (7.3.3), (7.2.1), (7.3.1) and $[c, \hat{\mathcal{L}}(\mathring{g})] = 0$.

(7.3.3) $[d_i, d_j] = (i-j)d_{i+j} + \frac{1}{12}(i^3-i)\delta_{i,-j}c \quad (i,j \in \mathbb{Z})$.

(7.2.1) $[a+\lambda K, b+\mu K] = [a,b]_0 + \psi(a,b)K$, $(a, b \in \mathcal{L}(\mathring{g}),\ \lambda, \mu \in \mathbb{C})$

(7.3.1). $d_s|_{\mathcal{L}(\mathring{g})} = -t^{s+1}\frac{d}{dt}$, $d_s(K) = 0$. $\lrcorner$

comparing the result of $[L_n, L_m]$ with (7.3.3)

(在 g 中构造出 L_n 的地位等同于 d_i)

moreover comparing $[L_n, x^{(m)}]$ and $[T_n, x^{(m)}]$ with (7.3.1)

$\ulcorner$ $[x^{(m)}, T_j] = 2(K+h^\vee)m x^{(m+j)}$

$[L_n, x^{(m)}] = -m x^{(m+j)}$ $\lrcorner$

g'-mod $\to U_c(g')$-mod

$g' \curvearrowright V \Leftrightarrow U_c(g') \curvearrowright V$.

$\varphi: U_c(g') \to \mathrm{End}(V)$. #

b) Let V is the g-module $L(\Lambda)$, then $L_0 = h_\Lambda I - d$. ? 原因 $x^{(m)}$ or T_n 都可以看成 $U_c(g')$

proof: $L_0(v) = \frac{1}{2(k+h^\vee)} T_0(v) \overset{\text{by Lem (12.8) b)}}{=\!=} \frac{(\bar\Lambda|\bar\Lambda+2\bar\rho)}{2(k+h^\vee)} v$

$(h_\Lambda I - d)(v) = \frac{(\Lambda+2\rho|\Lambda)}{2(h^\vee+k)} v - \Lambda(d)v =$

by $\Lambda = \bar\Lambda + \langle\Lambda, K\rangle\Lambda_0 + \frac{(\Lambda|\Lambda)}{\Lambda(d)}\delta$