Chapter 13. Affine Algebras, Theta Functions, and Modular Forms.

§ 13.1

(We develop a theory of theta functions in the following general framework (keeping in mind applications to affine algebras, we use notation which is identical to that used in previous chapters)

- Let $l \in \mathbb{Z}_+$ and let $\mathcal{H}_{\mathbb{R}}$ be an $(l+2)$-dim. v.s. over $\mathbb{R}$ with a nondegenerate symmetric bilinear form $(\cdot|\cdot)$ of index $(l+1, 1)$ (that is, under a certain basis, the matrix representation of $(\cdot|\cdot)$ is diag$(1, \dots, 1, 1, -1)$)
We will identify $\mathcal{H}_{\mathbb{R}}$ with $\mathcal{H}_{\mathbb{R}}^*$ via this form.
[since the bilinear form $(\cdot|\cdot)$ is nondegenerate, we have an isomorphism $\nu: \mathcal{H} \to \mathcal{H}^*$ defined by $\langle \nu(h), h_1 \rangle = (h|h_1)$, $h, h_1 \in \mathcal{H}$ and the
- induced bilinear form $(\lambda|\mu) = (\nu^{-1}(\lambda)|\nu^{-1}(\mu))$, $\lambda, \mu \in \mathcal{H}^*$]
- Fix a $\mathbb{Z}$-lattice M in $\mathcal{H}_{\mathbb{R}}$ of rank l, positive-definite and $\boxed{\text{integral.}}$ (i.e. $(a|b) \in \mathbb{Z}$ for all $a, b \in M$). Fix a vector $\delta \in \mathcal{H}_{\mathbb{R}}$ s.t. $(\delta|\delta) = 0$ and $(\delta|M) = 0$. [Denote $\mathring{\mathcal{H}}_{\mathbb{R}}^{\perp} = \{u \in \mathcal{H}_{\mathbb{R}} \mid (u|v) = 0 \text{ for } v \in \mathring{\mathcal{H}}_{\mathbb{R}}\}$. Then $\mathcal{H}_{\mathbb{R}} = \mathring{\mathcal{H}}_{\mathbb{R}} \oplus \mathring{\mathcal{H}}_{\mathbb{R}}^{\perp}$. Note that $(\cdot|\cdot)|_{\mathring{\mathcal{H}}_{\mathbb{R}}^{\perp}}$ is of index $(1,-1)$, that is there is a basis $\{u_1, u_2\}$ of $\mathring{\mathcal{H}}_{\mathbb{R}}^{\perp}$, s.t. $(u_1|u_1) = 1 = -(u_2|u_2)$, $(u_1|u_2) = 0$
- put $\mathring{\mathcal{H}}_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} M \subset \mathcal{H}_{\mathbb{R}}$, $\mathcal{H} = \mathbb{C} \otimes_{\mathbb{R}} \mathcal{H}_{\mathbb{R}}$, $\mathring{\mathcal{H}} = \mathbb{C} \otimes_{\mathbb{R}} \mathring{\mathcal{H}}_{\mathbb{R}} \subset \mathcal{H}$. (and extend put $\delta = u_1 + u_2$, $\Lambda_0 = \frac{1}{2}(u_1 - u_2)$)
$(\cdot|\cdot)$ to $\mathcal{H}$ by linearity. For $\lambda \in \mathcal{H}$, we denote by $\bar{\lambda}$ the orthogonal projection of λ on $\mathring{\mathcal{H}}$.)

Let $Y = \{v \in \mathcal{H} \mid \operatorname{Re}(\delta|v) > 0\}$. (The absolute convergence of ch$L(\lambda)$ if A is of affine type)

- For $\alpha \in \mathring{\mathcal{H}}_{\mathbb{R}}$, let t_α denote the automorphism of $\mathcal{H}$ defined by:

$$t_\alpha(v) = v + (v|\delta)\alpha - ((v|\alpha) + \tfrac{1}{2}(\alpha|\alpha)(v|\delta))\delta$$

(The element of group of translations T)

Note that the automorphism t_α is charaterized by the properties

a) $t_\alpha(\delta) = \delta$

b) $t_\alpha(v) = v + (v|\delta)\alpha \mod \mathbb{C}\delta$

c) $(\cdot|\cdot)$ is t_α-invariant.

It follows that $t_\alpha t_\beta = t_{\alpha+\beta}$.

补充上面写的δ和Λ_0之定义.
$\{\alpha_0, \dots, \alpha_l, \Lambda_0\}$.

- For $\alpha \in \mathring{\mathcal{H}}_{\mathbb{R}}$, let $p_\alpha(v) = v + 2\pi i\alpha$, $v \in \mathcal{H}$. All the transformations p_α and t_β ($\alpha, \beta \in \mathring{\mathcal{H}}_{\mathbb{R}}$) of $\mathcal{H}$ generate a group N called the Heisenberg group. 来说生成非Abel群.
两个变量和第三个变量有联系.

More explicitly, $N = \mathring{\mathcal{H}}_{\mathbb{R}} \times \mathring{\mathcal{H}}_{\mathbb{R}} \times i\mathbb{R}$ with multiplication:

Abel群的中心扩张.

$$(\alpha, \beta, u)(\alpha', \beta', u') = (\alpha + \alpha', \beta + \beta', u + u' + \pi i((\alpha|\beta') - (\alpha'|\beta)))$$

① $(0,0,0)$ is an identity ② $(-\alpha, -\beta, -u)$ is $(\alpha, \beta, u)^{-1}$ ③ associativity law. (背面

The action of N on $\mathcal{H}$ is given by

(13.1.1) $$(\alpha, \beta, u)(v) = t_\beta(v) + 2\pi i\alpha + (u - \pi i(\alpha|\beta))\delta.$$

($\mathcal{H}$ forms an N-module with respect to the action (13.1.1)).

- $(0,0,0)(v) = v$.

so that $(\alpha, 0, 0)(v) = v + 2\pi i \alpha = p_\alpha(v)$.

$(0, \beta, 0)(v) = t_\beta(v)$

$(0, 0, u)(v) = v + u\delta$.

and Y is N - invariant.

$\because Y = \{v \in H \mid Re(\delta \mid v) > 0\}$, need to verify $\forall (\alpha, \beta, u) \in N$,

s.t. $Re((\alpha, \beta, u)(v) \mid \delta) > 0$. since $(2\pi i \alpha \mid \delta) = 0$. $(\delta \mid \delta) = 0$.

and $(t_\beta(v) \mid \delta) = (t_\beta(v) \mid t_\beta(\delta)) = (v \mid \delta)$ ┘

we get the conclusion

. Denote by $N_{\mathbb{Z}}$ the subgroup of N generated by all $(\alpha, 0, 0)$, $(0, \beta, 0)$ with $\alpha, \beta \in M$ and $(0, 0, u)$ with $u \in 2\pi i \mathbb{Z}$.

Then $N_{\mathbb{Z}} = \{(\alpha, \beta, u) \in N \mid \alpha, \beta \in M.\ u + \pi i (\alpha \mid \beta) \in 2\pi i \mathbb{Z}\}$

若 $(\alpha \mid \beta) = 1$?

要求 $(\alpha \mid \beta) \in 2\mathbb{Z}$. ?

' The Heisenberg group H, is the group of 3×3 upper triangular matrices of the form : $\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$ under the operation of matrix multiplication:

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a' & c' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+a' & c'+c+ab' \\ 0 & 1 & b+b' \\ 0 & 0 & 1 \end{pmatrix}$$

and inverse are given by

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a & ab-c \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix}$$

§13.2.

- Fix a nonnegative integer k.

A $\boxed{\text{theta function of degree } k}$ is a holomorphic function F on the domain Y s.t. the following two conditions hold for all $v \in Y$:

(T1) $F(n.v) = F(v)$ for all $n \in N_{\mathbb{Z}}$.

(T2) $F(v + a\delta) = e^{ka} F(v)$ for all $a \in \mathbb{C}$.

- Let $\boxed{Th_k}$ denote the space (over $\mathbb{C}$) of all theta functions of degree k. Then

$$\widetilde{Th} = \bigoplus_{k \geq 0} \widetilde{Th}_k \qquad (\widetilde{Th}_s \cdot \widetilde{Th}_t \subset \widetilde{Th}_{s+t} \text{ by (T1)(T2)})$$

is a graded associative $\mathbb{C}$-algebra, called $\boxed{\text{the algebra of theta functions}}$

- (In order to produce examples of theta functions,) let $M^* = \{\lambda \in \mathfrak{h}_{\mathbb{R}} \mid (\lambda|\alpha) \in \mathbb{Z} \text{ for all } \alpha \in M\}$ be the lattice dual to M; for a positive integer k, put:

$$P_k = \{\lambda \in \mathfrak{h} \mid (\lambda|\delta) = k \text{ and } \bar{\lambda} \in M^*\}.$$

- Given $\lambda \in P_k$, we define the $\boxed{\text{classical theta function of degree } k}$ with characteristic $\bar{\lambda}$ by the series

$$(13.2.1) \quad \Theta_\lambda = e^{-\frac{(\lambda|\lambda)}{2k}\delta} \sum_{\alpha \in M} e^{t_\alpha(\lambda)} \qquad \text{Prop } (12.7.2)$$

Equivalently, we take $\lambda' \equiv \lambda \mod \mathbb{C}\delta$ s.t. $(\lambda'|\lambda') = 0$. [let $\lambda' := \lambda - \frac{|\lambda|^2}{2k}\delta \Rightarrow (\lambda'|\lambda') = |\lambda|^2 - \frac{2|\lambda|^2(\lambda|\delta)}{2m} = 0$] and let $\Theta_\lambda = \sum_{\alpha \in M} e^{t_\alpha(\lambda')}$. (Note that this function (13.2.1) is exactly the one defined by (12.7.2) (which naturally arises in the theory of affine algebras).)

[(12.7.2) $\Theta_\lambda = e^{-\frac{|\lambda|^2}{2k}\delta} \sum_{t \in T} e^{t(\lambda)}$, $\left(A_\lambda = \sum_{w \in \mathring{W}} \varepsilon(w) \Theta_{w(\lambda)}\right)$]

As in §12.7, we can rewrite Θ_λ in another form:

$$(13.2.2) \quad \Theta_\lambda = e^{k\Lambda_0} \sum_{\gamma \in M + k^{-1}\bar{\lambda}} q^{\frac{1}{2}k(\gamma|\gamma)} e^{k\gamma}.$$

As before, q stands for $e^{-\delta}$, and $\Lambda_0 \in \mathfrak{h}_{\mathbb{R}}$ is the unique isotropic vector s.t. $(\Lambda_0|\delta) = 1$ and $(\Lambda_0|M) = 0$.

[Prob. (12.7.3) $\Theta_\lambda = e^{k\Lambda_0} \sum_{\gamma \in M + k^{-1}\bar{\lambda}} e^{-\frac{1}{2}k|\lambda|^2\delta + k\gamma}$]

(It is clear that the series (13.2.2) converges absolutely on Y to a holomorphic function; (by P178 prop 10.6 b) (θ_λ 是 χ_λ 的一部分) and P200 prop 11.10))

Remark: (1) θ_λ is a theta function of degree k.

Γ · (T1): $\theta_\lambda((\alpha,\beta,u)(v)) = e^{-\frac{|\lambda|^2}{2k}\{\delta|(\alpha,\beta,u)(v)\}} \sum_{\alpha\in M} e^{\{t_\alpha(\lambda)|(\alpha,\beta,u)v\}}$

where $(\alpha,\beta,u)\in N_Z$ then by $(\alpha,\beta,u)(v) = t_\beta(v) + 2\pi i\alpha + (u - \pi i|\alpha|\beta)\delta$.

we have $\{\delta|(\alpha,\beta,u)(v)\} = \{\delta|v\}$ and

$\{t_\alpha(\lambda)|(\alpha,\beta,u)v\} = \{\lambda + (\lambda|\delta)\alpha - (\ \ )\delta \mid v + (v|\delta)\beta + 2\pi i\alpha + (\ )\delta\}$

$\cdots = \{t_\alpha(\lambda)|v\}$

hence $\theta_\lambda((\alpha,\beta,u)(v)) = \theta_\lambda(v)$ for all $(\alpha,\beta,u)\in N_Z$.

· (T2) $\theta_\lambda(v+a\delta) = e^{-\frac{|\lambda|^2}{2k}\{\delta|v+a\delta\}} \sum_{\alpha\in M} e^{\{t_\alpha(\lambda)|v+a\delta\}}$

$= e^{-\frac{|\lambda|^2}{2k}\{\delta|v\}} \sum_{\alpha\in M} e^{\{t_\alpha(\lambda)|v\}} e^{a(\lambda|\delta)}$

$= e^{ka}\theta_\lambda(v)$, where $a\in\mathbb{C}$, $(\lambda|\delta) = k$ since $\lambda\in P_k$ $\lrcorner$

(2) Note that

(13.2.3) $\theta_{\lambda+k\alpha+a\delta} = \theta_\lambda$ for $\alpha\in M$, $a\in\mathbb{C}$.

Γ $\theta_{\lambda+\delta} = e^{-\frac{|\lambda+\delta|^2}{2k}\delta} \sum_{\alpha\in M} e^{t_\alpha(\lambda+\delta)} = e^{-\frac{|\lambda|^2}{2k}\delta - \frac{2(\lambda|\delta)}{2k}\delta} \sum_{\alpha\in M} e^{t_\alpha(\lambda)+\delta}$

$= e^{-\frac{|\lambda|^2}{2k}\delta - \delta} \cdot e^{\delta} \sum_{\alpha\in M} e^{t_\alpha(\lambda)} = \theta_\lambda$.

· Let $\theta_\lambda = e^{k\Lambda_0} \sum_{\gamma\in kM+\bar\lambda} q^{\frac{1}{2k}|\gamma|^2} e^{\gamma}$, then we can see $\theta_{\lambda+k\alpha} = \theta_\lambda$ $\lrcorner$

· Choose an orthonormal basis $v_1,\dots,v_\ell$ of $\mathfrak{h}_\mathbb{R}$ and coordinate $\mathfrak{h}$ by (13.2.4) $v = 2\pi i\left(\sum_{s=1}^{\ell} z_s v_s - \tau\Lambda_0 + u\delta\right)$ $z_s\in\mathbb{C}$ or $\mathbb{R}$

so that we shall often write (τ, z, u), where $z = \sum_{s=1}^{\ell} z_s v_s \in \mathring{\mathfrak{h}}$.

$\tau, u\in\mathbb{C}$, in place of $v\in\mathfrak{h}$.

Rem: (1) $Y = \{(\tau, z, u) \mid z\in\mathring{\mathfrak{h}};\ \tau,u\in\mathbb{C},\ \mathrm{Im}\,\tau>0.\}$

Γ $\mathrm{Re}(\delta|v) = \mathrm{Re}(2\pi i(-\tau\Lambda_0 + u\delta)|\delta) = \mathrm{Re}(-2\pi i\tau) > 0$

Recall: $Y = \{v\in\mathfrak{h} \mid \mathrm{Re}(\delta|v)>0\}$ $\lrcorner$

(2) we can rewrite θ_λ in its classical form

(13.25) $\theta_\lambda(\tau, z, u) = e^{2\pi i k u} \sum_{\gamma \in M + k^{-1}\bar{\lambda}} e^{\pi i k \tau (\gamma|\gamma) + 2\pi i k (\gamma|z)}$

where $(\gamma|z) = \sum_{i=1}^{l} \gamma_i z_i$, $\gamma = \sum_{i=1}^{l} \gamma_i v_i$

Γ $\theta_\lambda(\tau, z, u) = e^{k\Lambda_0}|_{2\pi i(z - \tau\Lambda_0 + u\delta)} \sum_{\gamma \in M + k^{-1}\bar{\lambda}} e^{(-\frac{1}{2}k(\gamma|\gamma)\delta + k\gamma)|_{2\pi i(z - \tau\Lambda_0 + u\delta)}}$

$= e^{2\pi i k u} \sum_{\gamma \in M + k^{-1}\bar{\lambda}} e^{\frac{1}{2}k(\gamma|\gamma) 2\pi i \tau + 2\pi i k (\gamma|z)}$

$= e^{2\pi i k u} \sum_{\gamma \in M + k^{-1}\bar{\lambda}} e^{\pi i k \tau (\gamma|\gamma) + 2k\pi i (\gamma|z)}$ $\lrcorner$

Note also that in these coordinates, we have: $q = e^{-\delta} = e^{2\pi i \tau}$.

Γ $e^{-\delta}(v) = e^{2\pi i \tau}$ $\lrcorner$

. let $\mathcal{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$ be the $\boxed{\text{Poincaré upper half-plane.}}$

lemma 13.2. Th_0 is the algebra of holomorphic functions in $\tau \in \mathcal{H}$

proof: Recall: $\widetilde{Th_0}$ is the space (over $\mathbb{C}$) of all theta functions of degree 0. i.e. $F(v + a\delta) = F(v)$ if $F \in \widetilde{Th_0}$, for all $a \in \mathbb{C}$.

. suppose that $F = f(\tau)$ is independent of z and u, and is a holomorphic function in τ. Since

$(\alpha, \beta, u)(v) \equiv v \pmod{\mathfrak{h} + \mathbb{C}\delta}$ for $\alpha, \beta \in M$, $v \in Y$, $u \in \mathbb{C}$.

Thus $F[(\alpha, \beta, u)(v)] = f(\tau)$ 此 $F(v)$ for $v \in Y$.

Thus $F \in \widetilde{Th_0}$. Γ $v = 2\pi i(\sum z_s v_s - \tau\Lambda_0 + u\delta)$ $\lrcorner$

. let $F \in \widetilde{Th_0}$. since $F(v + a\delta) = F(v)$ for any $a \in \mathbb{C}$. Then F is independent of u. Now

$(\alpha, \beta, 0)(2\pi i(z - \tau\Lambda_0)) = 2\pi i(t_\beta(z - \tau\Lambda_0) + \alpha) - \pi i(\alpha|\beta)\delta$

$= 2\pi i(z - \tau\beta + \alpha - \tau\Lambda_0) \pmod{\mathbb{C}\delta}$.

According to (T1): $F(n \cdot v) = F(v)$ for all $n \in N_\mathbb{Z}$, Then we have

$F((\alpha, \beta, 0)(2\pi i(z - \tau\Lambda_0))) = F(2\pi i(z - \underline{\tau\beta + \alpha} - \tau\Lambda_0)) = F(2\pi i(z - \tau\Lambda$

i.e. F is periodic in z for each fixed $\tau \in \mathcal{H}$. with periods in $M + \tau M$. Hence F is a holomorphic function on the compact complex manifold $\underline{\mathfrak{h}/(M + \tau M)}$ for fixed $\tau \in \mathcal{H}$. $\underline{\text{Hence } F \text{ is}}$ (类似于环面 $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$, which is compact

$\underline{\text{independent of } z}$.

That is, F is purely a holomorphic function in τ.

Any holomorphic function on a compact Riemann surface is constant!! (Liouville's theorem)

Prop 13.2. Let Θ_λ and Θ_μ be classical theta functions of degree m and n, respectively. Then

$$\Theta_\lambda\Theta_\mu = \sum_{\alpha\in M \bmod (m+n)M} \Theta_{\lambda+\mu+n\alpha}\,\psi_\alpha.$$

where $\psi_\alpha = \sum_{\gamma\in(m+n)M} q^{|n\bar\lambda - m\bar\mu - mn(\alpha+\gamma)|^2/2mn(m+n)}$.

Proof: We may assume that $|\lambda|^2 = |\mu|^2 = 0$, Then

$$\Theta_\lambda = \sum_{\alpha\in M} e^{t_\alpha(\lambda)}, \quad \Theta_\mu = \sum_{\beta\in M} e^{t_\beta(\mu)} \quad \text{and}$$

$$\Theta_\lambda\Theta_\mu = \sum_{\alpha,\gamma\in M} e^{t_\gamma(\lambda)+t_\gamma t_\alpha(\mu)} = \sum_{\alpha,\gamma\in M} e^{t_\gamma(\lambda+t_\alpha(\mu))} \quad (\text{where } \gamma+\alpha=\beta).$$

$$= \sum_{\alpha\in M} q^{-\frac{|\lambda+t_\alpha(\mu)|^2}{2(m+n)}}\,\Theta_{\lambda+t_\alpha(\mu)} \quad \left(\begin{array}{l}\text{Note that: } \Theta_{\lambda+t_\alpha(\mu)} = e^{\frac{|\lambda+t_\alpha(\mu)|^2}{2(m+n)}} \sum_{\gamma\in M} e^{t_\gamma(\ )} \\ \& \ (\lambda+t_\alpha(\mu)|\delta) = m+(\mu|\delta) = m+n, \ q = e^{-\delta}\end{array}\right)$$

$$(*) = \sum_{\alpha\in M \bmod (m+n)M} \Theta_{\lambda+t_\alpha(\mu)} \underbrace{\sum_{\gamma\in(m+n)M} q^{-|\lambda+t_{\alpha+\gamma}(\mu)|^2/2(m+n)}}_{} \quad (\text{by } \Theta_{\lambda+ka+a\delta} = \Theta_\lambda)$$

(加回來 mod 之前)

· since $\lambda = \bar\lambda + m\Lambda_0 - \frac{|\bar\lambda|^2}{2m}\delta$, $\mu = \bar\mu + n\Lambda_0 - \frac{|\bar\mu|^2}{2n}\delta$ by (6.2.6)

[Recall: (6.2.6) $\lambda - \bar\lambda = \langle\lambda, K\rangle\Lambda_0 + \langle 2\lambda, K\rangle^{-1}(|\lambda|^2 - |\bar\lambda|^2)\delta$.]

so $|\lambda+\mu|^2 = 2(\lambda|\mu) = 2(\bar\lambda|\bar\mu) - \frac{m|\bar\mu|^2}{n} - \frac{n|\bar\lambda|^2}{m} = -\frac{|n\bar\lambda - m\bar\mu|^2}{mn}$. (☆)

and $|t_\alpha(\mu)| = |\mu| = 0$, $t_\alpha(\mu) \equiv \mu + n\alpha \pmod{\mathbb{C}\delta}$ for $\alpha\in M$.

Thus $\overline{t_\alpha(\mu)} = \bar\mu + n\alpha \Rightarrow |\lambda + t_{\alpha+\gamma}(\mu)|^2 = -\frac{|n\bar\lambda - m\bar\mu - mn(\alpha+\gamma)|^2}{mn}$ (☆☆)

hence

$$(*) = \sum_{\alpha\in M \bmod (m+n)M} \Theta_{\lambda+\mu+n\alpha} \sum_{\gamma\in(m+n)M} q^{|n\bar\lambda - m\overline{t_{\alpha+\gamma}(\mu)}|^2/2mn(m+n)} \quad \text{by (☆)}$$

$$= \sum_{\alpha\in M \bmod (m+n)M} \Theta_{\lambda+\mu+n\alpha} \sum_{\gamma\in(m+n)M} q^{|n\bar\lambda - m\bar\mu - mn(\alpha+\gamma)|^2/2mn(m+n)} \quad \text{by (☆☆)}$$

$$= \sum_{\alpha\in M \bmod (m+n)M} \Theta_{\lambda+\mu+n\alpha}\,\psi_\alpha.$$

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