Recall from GTM 228.

- $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \to$ modular group

def 1: Let k be an integer, A meromorphic function $f: \mathcal{H} \to \mathbb{C}$ is weakly modular of weight k if: $f(A\cdot\tau) = (c\tau+d)^k f(\tau)$, $\tau \in \mathcal{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$

def 2: Let k be an integer. A function $f: \mathcal{H} \to \mathbb{C}$ is a modular form of weight k if

(1) f is holomorphic on $\mathcal{H}$

(2) f is weekly modular of weight k

(3) f is holomorphic at ∞

The set of modular forms of weight k is denote by $\mathcal{M}_k(SL_2(\mathbb{Z}))$

Example: ① $f=0$ is a modular forms of weight 0

② 对上半平面 $\mathscr{H}$ 赋予 Riemann 度量 $\frac{dx^2+dy^2}{y^2}$, 其曲率为常数 -1, 这是双曲几何常见的模型. 具有整数权 k 和级 $SL(2,\mathbb{Z})$ 的模形式是定义在 $\mathscr{H}$ 上的一类全纯函数, 按定义, 这样一个函数 f 必须

(a) 满足 $(c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right) = f(\tau)$, 其中 $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ 取遍 $SL(2,\mathbb{Z})$,

(b) 具有 Fourier 展开 $f(\tau) = \sum_{n\geq 0} a_n(f) q^n$, 其中 $q := e^{2\pi i\tau}$; Fourier 系数 $a_n(f)$ 往往蕴藏微妙的算术信息.

- Fix a subgroup Γ of finite index in $\Gamma(1) = SL_2(\mathbb{Z})$, a function $f: \mathcal{H} \to \mathbb{C}$ is said a modular form of weight k and multiplier system χ for Γ, if f is holomorphic on $\mathcal{H}$ and $f(A\cdot\tau) = \chi(A)(c\tau+d)^k f(\tau)$ for all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $\tau \xrightarrow{\text{\tau}} \in \mathcal{H}$

(\Gamma)

$|\chi(A)| = 1$ $\chi: \Gamma \to \mathbb{C}^\times$

- Let f be such a modular form, Since Γ has a finite index in $SL_2(\mathbb{Z})$. $\Rightarrow$ $T^s \in \Gamma$ for some positive integer s and hence $f(\tau+s) = e^{2\pi i c} f(\tau)$ for some $c \in \mathbb{R}$

$f(T^s(\tau)) = \chi(T^s) f(\tau)$

$\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ $\quad A\cdot\tau = \frac{a\tau+b}{c\tau+d}$ $\quad \chi(T^s) = e^{2\pi i c}$ for $c \in \mathbb{R}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

(给出 f 在相关 Γ 下的某种周期性)

Set $F(e^{2\pi \tau i/s}) = e^{-2\pi i c\tau/s} f(\tau)$

Then F is well-defined holomorphic function in $Z = e^{2\pi i\tau/s}$ on the punctured dis $0 < |z| < 1$

去心圆盘

$\tau+s$ $\quad Z_\tau = e^{2\pi i\tau/s}$

$F(Z_{\tau+s}) = e^{-2\pi i c(\tau+s)} f(\tau+s)$ $\quad \tau, s \in \mathcal{H}$

$= e^{-2\pi i c\tau/s} f(\tau) = F(Z_\tau)$

$0 \le \operatorname{Re}\tau < s \quad \tau \in \mathcal{H}$ $\qquad \operatorname{Im}\tau \to +\infty$ $\qquad f(\tau)$ 在 ∞ 处性

$\downarrow$ $\qquad\qquad \downarrow$ $\qquad\qquad \downarrow$

$0 < Z_\tau < 1$ $\qquad Z \to 0$ $\qquad F(z)$ 在 $Z=0$

Thm (Rudin Thm 10.16) For every open set Ω in the plane, every $f \in H(\Omega)$ is represented by power series in Ω $\Rightarrow$ $F(z)$ 在 $0<|z|<1$ 解析

$\Rightarrow$ 幂级数的展开 (收敛半径为 1).

F has a Laurent expansion $F(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ converges absolutely

(在 $Z_\tau=0$ 的孤立奇点处洛伦展开)

F has a Laurent expansion $F(z)=\sum_{n\in\mathbb{Z}} a_n Z_\tau^n$ converging absolutely for $0<|Z_\tau|<1$. Hence we have the Fourier expansion:

$$f(\tau)=\frac{F(z)}{e^{-2\pi i c\tau/s}}=\sum_{n\in\mathbb{Z}} a_n e^{2\pi i c\tau/s} Z_\tau^n \quad (*)$$

$$=\sum_{n\in\mathbb{Z}} a_n e^{2\pi i(n+c)\tau/s} \quad \text{for } \tau\in\mathcal{H}$$

(此时在变量替换回来, 得到 $f(\tau)$ 的 Fourier 展开)

Def: (1) we call f meromorphic at $i\infty$ $\longrightarrow$ if $a_n=0$ for $n\ll 0$

(ie $\exists\, n_0$ s.t $a_n\neq 0$ $n<n_0\in\mathbb{Z}$ $a_n=0$)

☆ (2) . . . . holomorphic at $i\infty$ $\longrightarrow$ if $a_n\neq 0 \Rightarrow n+c\geq 0$

$n_0+c\geq 0$

(3) . . . . vanishing at $i\infty$ $\longrightarrow$ if $a_n\neq 0 \Rightarrow n+c>0$

when f is holomorphic at $i\infty$. We say that the value of f at $i\infty$ is a_{-c}

(intepreted as 0 if $c\notin\mathbb{Z}$)

then $\lim_{y\to\infty} f(iy)=\lim_{y\to\infty}\left(\sum_{n\in\mathbb{Z}} a_n e^{-2n(n+c)\pi y/s}\right)$

$$=\delta_{n+c,0}\, a_n=a_{-c}$$

If $n_0=\min\{n \mid a_n\neq 0\}$, we let $r=\left|n_0+c/s\right|$ and we say f it has zero (polo) of order r at $i\infty$ if $r\geq 0$ (resp $r<0$)

Cusp: (尖点)

Recall: for $\tau \in \mathcal{H}$, fixed point of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

if $\tau = A(\tau) = \frac{a\tau+b}{c\tau+d}$ $\quad \varphi: \tau \to A\tau$

Fact 2: $A\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \lambda\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

then (w_1, w_2) 为 A 特征向量 $\Leftrightarrow$ $\tau = \frac{w_1}{w_2}$ are fixed points of A

$\left(\text{ie } A\left(\frac{w_1}{w_2}\right) = \frac{a\frac{w_1}{w_2}+b}{c\frac{w_1}{w_2}+b} = \frac{aw_1+bw_2}{cw_1+dw_2} = \frac{\lambda w_1}{\lambda w_2} = \frac{w_1}{w_2} \right)$

when $c \neq 0$. $\quad \frac{a\tau+b}{c\tau+d} = A(\tau) = \tau$

ie $\quad c\tau^2+d\tau+a\tau+b=0 \quad c\tau^2+(d-a)\tau-b=0$

$\Delta(A) := (a-d)^2+4bc = (a+d)^2-4$

$|tr(A)| = |a+d| = \begin{cases} <2 \longrightarrow \varphi: \tau \to A\tau \text{ 椭圆} \\ =\pm 2 \longrightarrow \varphi \text{ parabolic (抛物线)} \\ >2 \longrightarrow \varphi \text{ 双曲} \end{cases}$

Corresponding τ $\begin{cases} |trA|<2 \longrightarrow \\ |tr(A)|=2 \longrightarrow \text{parabolic point (or cusp} \to \text{尖点)} \\ |tr(A)|>2 \end{cases}$

Rmk: ① Γ be a subgroup of $SL_2(\mathbb{Z})$ of finite index.
The quotient topological space $G\backslash\mathcal{H}$ can be showed to be a Hausdorff space. Typically it's not compact But can be compactified by adding a finite number of points called cusps. These are points at the boundary of $\mathcal{H}$ in $\mathbb{Q}\cup\{i\infty\}$. Such that there is porobolic element φ fixed the point, This yields a compact topolocial space $G\backslash\mathcal{H}^*$, $\longrightarrow$ it can be endowed with the structure

of a Rimann Surface $\Rightarrow$ which allows one to speak of holomorphic and meromorphic fun..

B 等价的尖点 i.e. $\exists A \in SL_2(\mathbb{Z})$ s.t $\alpha = A\beta$
本质上相同，因此只要取一个代表点来讨论.

A cusp of a subgroup Γ of finite index in $SL_2(\mathbb{Z})$ is an orbit of Γ in $\mathbb{Q} \cup \{i\infty\}$, where $\frac{a}{0}$ is interpreted as $i\infty$ for $a \in \mathbb{Q}$, $a \neq 0$.

Since for $\underset{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}{A} \in SL_2(\mathbb{Z}) = \Gamma(1)$ $\quad A(iy) = \frac{aiy + b}{ciy + d}$

$$A(i\infty) = \lim_{y \to \infty} \frac{aiy + b}{ciy + d} = \frac{a}{c} \in \mathbb{Q}$$

$i\infty$ $\quad T^m(iy) = \frac{m}{0} \to i\infty$

$y \to \infty$ $\quad T^m(i\infty) = i\infty$

Since $\Gamma(1) = SL_2(\mathbb{Z})$ acts transitively on $\mathbb{Q} \cup \{i\infty\}$

$G = \Gamma(1)(i\infty) \to$ 一条轨道 $\quad G$

$\Rightarrow$ the set of cusps of Γ is finite

the numbers of cusp of $\Gamma \leq$ index of Γ in $SL_2(\mathbb{Z})$

Sometimes we speak of the cusp $\alpha \in \mathbb{Q} \cup \{i\infty\}$ of Γ;
this means the orbit of α under Γ.

for example $\Gamma(1) \longrightarrow$ has one cusp $i\infty$, $\Gamma_0 \to i\infty, -1$

$\Gamma_0(k)$ for prime $k \longrightarrow i\infty, 0$

i∞处 $\alpha \in \Gamma \cdot (i\infty)$

Let f be above and consider a cusp α of Γ

Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) = SL_2(\mathbb{Z})$ be such that $= (c\tau+d)^{k} \chi(B) f(\tau)$

$B(i\infty) = \alpha$. Then $f_0(\tau) := (c\tau + d)^{-k} f(B\tau)$

(本质上与 $f(\tau)$ 相关 封) 进行变量替换 $i\infty \longrightarrow \alpha$.

从而讨论 α 处的全纯.亚纯定义 (由上而 $i\infty$)

is a modular form of weight k and some multiplier system χ_0 for $B^{-1}\Gamma B$

τ 尖点 $B \in SL_2(\mathbb{Z}) \rightarrow B\tau$ 尖点

$\varphi: \tau \rightarrow A\tau$ 抛物元.

$\Rightarrow BAB^{-1}$ 抛物 . $B(\tau)$

$BAB^{-1}(B\tau) = B(A \cdot \tau) = B\tau$

A holomorphic modular form of weight 0 is constant:

Def: ① We say f is meromorphic, holomorphic or has zero or pole of order r at α if f_0 is --- at $i\infty$

② we say f is R-singular if orders of poles of at all cusps $\leq R$ (在 holomorphic 条件下 才有这个定义)

综上:

$\Rightarrow$ Def. A modular form of weight k and multiplier system χ for Γ is called a meromophic modular form, a holomorphic modular form or a cusp form if it is meromorphic, holomorphic or vanishes at all cusps of Γ. respectively
(包括 $i\infty$ 及 $\mathbb{Q}$ 中 cusp 所以点)

(Question:) A holomorphic modular form of weight 0 is constant?

$f(A\tau) = \chi(A) f(\tau)$? f $a_{n_0} \neq 0$

This allows one to identify modular forms. $n>0$ $a_n = 0$

- $q = e^{2\pi i \tau}$

$Y =$

- using various specialization of classical theta functions, we can construct modular forms.

In fact. given a holomorphic function F on Y and $\alpha, \beta \in \mathbb{R}$ we define a holomorphic function $F^{\alpha,\beta}(\tau)$ on $\mathcal{H}$ by

(13.6.1) $F^{\alpha,\beta}(\tau) := \left(F|_{(\alpha,\beta,0)}\right)(\tau, 0, 0) = F(\tau, -\alpha+\tau\beta, -\tfrac{1}{2}\beta(-\alpha+\tau\beta))$

$(\tau, 0, 0) \in Y$
$\downarrow \in H$

$n = (\alpha, \beta, u_0) \in N \qquad v = (\tau, z, u) \in Y$

$n(v) = (\alpha, \beta, u_0)(\tau, z, u) = (\overset{0}{\tau}, \overset{0}{z} - \tau\beta + \alpha, u + u_0 - (z|\beta) + \frac{(\beta|\beta) - (\alpha|\beta)}{2})$

$(\alpha, \beta, 0)(\tau, 0, 0) = (\tau, -\tau\beta + \alpha, \frac{1}{2}(\beta | \tau\beta - \alpha))$

$F^{\alpha, \beta}(\tau) = F(\tau, \alpha - \tau\beta, \frac{1}{2}(\beta | \beta\tau - \alpha))$

For example $\theta_\lambda^{\alpha, \beta}(\tau) = e^{\pi i k(\alpha|\beta)} \sum_{\gamma \in M + k^{-1}\lambda - \beta} e^{2\pi i k(\alpha|\gamma)} q^{\frac{k|\gamma|^2}{2}}$

(13.6.2)

$\theta_\lambda^{\alpha, \beta}(\tau) = \theta_\lambda(\tau, \alpha - \tau\beta, \frac{1}{2}(\beta | \tau\beta - \alpha)) \; (\cancel{***})$

$\theta_\lambda(\tau, z, u) = e^{ku_0} \sum_{\gamma \in M + k^{-1}\lambda} e^{-\frac{1}{2}k(\gamma|\gamma)\delta} e^{t_\gamma}(\tau, z, u)$

$= e^{2\pi i k u} \sum_{\gamma \in M + k^{-1}\lambda} e^{\pi i k \tau(\gamma|\gamma) + 2\pi i(\gamma|z)}$

$(***) = e^{\pi i k(\beta | \tau\beta - \alpha)} \sum_{\gamma \in M + k^{-1}\lambda} e^{\pi i k \tau(\gamma|\gamma) + 2\pi i k(\gamma | \alpha - \tau\beta)}$

$\_\_\_\_ \sum_{\gamma \in M + k^{-1}\lambda - \beta} \_\_\_\_\_\_$

$= \_\_\_\_$

Furthermore, it is clear by 13.4.3 ($A(\alpha.\beta u)A^{-1} = (a\alpha+b\beta, c\alpha+d\beta, u)$

that

(13.6.3) $\left\{ \begin{array}{l} (F|_{(A,j)})^{\alpha.\beta}(\tau) = F^{a\alpha+b\beta,\, c\alpha+d\beta}(\tau)\,|_{(A,j)} \\ \text{where } \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, j\right) \in M_{\beta}(2.R) \end{array} \right.$

pf: $(F|_{(Aj)})^{\alpha.\beta}(\tau) = (F|_{(Aj)})_{(\alpha.\beta.0)}(\tau)$

$= (F|_{(A,j)})((\alpha.\beta.0)(\tau))$

$= (j(\tau))^{-1} F(A(\alpha.\beta.0)A^{-1}(A\tau)$

$= (j(\tau))^{-1} F|_{(a\alpha+b\beta,\, c\alpha+d\beta,\, 0)}(A\tau))$

$= \left(F^{a\alpha+b\beta,\, c\alpha+d\beta}(\tau)\right)|_{(A,j)}$

$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\quad S(\alpha.\beta.0)S^{-1} = (-b.\beta,\, \alpha.0)$

A special case of this is

$$(F|_{(S,j)})^{\alpha.\beta}(\tau) = F^{-\beta.\alpha}(\tau)|_{(S,j)} \quad (13.6.4)$$

$nM^* \subset M \quad n(\delta|\delta) \in 2\mathbb{Z} \qquad n: \qquad k=m \quad \Gamma(kn)$

Prop 13.6 give positive integer s and m put

$\alpha,\beta \in \mathfrak{g}_k$

$$\mathcal{F}_{m,s} = \{ \theta^{\alpha,\beta}(\tau) \mid \theta \in Th_m, \ s\alpha \in M, \ s\beta \in M \}$$

Then every function from $\mathcal{F}_{m,s}$ is holomorphic modular form of weight $\frac{1}{2}L$ for $\Gamma(mn) \cap \Gamma(s)$ (resp $\Gamma(mn) \cap \Gamma(s) \cap \Gamma_0 \cap \Gamma_s$) if the lattice M is even (resp. odd)

$A\in$

$\Gamma(s) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod s$

pf: (第一步) let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(m,n) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod mn \}$

resp $\Gamma_0 = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid ac \text{ and } bd \text{ are even} \}$

Then By Thm 13.15 (Jacobi) + 13.6.3

$\longrightarrow A \in \Gamma(mn)$

(Jacobi) $\quad k=m$ $\quad A \in \Gamma(mn)$, $\theta_\lambda|_{(A,j)} = v(A,j,m)\,\theta_\lambda$

where $|v(A,j,m)| = 1$

(13.6.3) $\quad (F|_{A,j})^{\alpha,\beta}(\tau) = F^{a\alpha+b\beta,\ c\alpha+d\beta}(\tau)\big|_{(A,j)}$

$\Rightarrow$ We have:

$(\square)\quad v(A)\,\theta_\lambda^{\alpha,\beta}(\tau) = \theta_\lambda^{a\alpha+b\beta,\ c\alpha+d\beta}(A\tau)\,[c\tau+d]^{-\frac{L}{2}}$

where $v(A) \in \mathbb{C}$, $|v(A)|=1$. On the other hand by (13.6.2)

$N = \overset{0}{\gamma} \times \overset{0}{\gamma} \times \underline{\Omega}$

we have

$v(\square\square) \; \theta_\lambda^{\alpha,\beta}(\tau) = \pm \theta_\lambda^{a\alpha+b\beta,\, c\alpha+d\beta}(\tau)$ if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(S)$

$\downarrow A\tau$ $\quad \downarrow A\tau$ i.e $b \equiv c \equiv 0 \bmod S$

By $(\square) + (\square\square) \Rightarrow$

$$\begin{cases} \theta_\lambda^{\alpha,\beta}(A\cdot\tau) = \pm\theta_\lambda^{a\alpha+b\beta,\, c\alpha+d\beta}(A\cdot\tau) \\ \quad \downarrow (\square) \\ \quad = \pm v(A)(c\tau+d)^{\frac{L}{2}}\theta_\lambda^{\alpha,\beta}(\tau) \\ \text{where } |\pm v(A)| = 1 \end{cases}$$

$j(\tau) = (c\tau+d)$

Satify modular form of degree $\frac{L}{2}$

The proof of $(\square)$ and $(\square\square)$

for $(\square)$ need $A \in \Gamma(mn)$ $\quad$ for $(\square\square)$ need $A \in \Gamma(S)$

(if M is not even need $A \in \Gamma_\theta$)

$v(A)\theta_\lambda = \theta_\lambda|_{(A,j)}$

$$\Rightarrow \; v(A)\,\theta_\lambda^{\alpha,\beta}(\tau) = \left(\theta_\lambda|_{(A,j)}\right)^{\alpha,\beta} = \theta_\lambda^{a\alpha+b\beta,\, c\alpha+d\beta}|_{(A,j),(\tau)} = (j(\tau))^{-\frac{L}{2}}\,\theta_\lambda^{a\alpha+b\beta,\, c\alpha+d\beta}(A\cdot\tau)$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (c\tau+d)^{-\frac{L}{2}}$

$$= \theta_\lambda^{\alpha,\beta}(\tau) = e^{\pi i m(\;)}\sum_{\gamma\in M+k^{-1}\lambda-\beta}$$

For $(\square\square)$

$\downarrow$ 用 λ (13.6.2)

$$\pm\,\theta_\lambda^{a\alpha+b\beta,\, c\alpha+d\beta}(\tau) = e^{\pi i m(a\alpha+b\beta \mid c\alpha+d\beta)}\sum_{\gamma\in M+k^{-1}\lambda-(c\alpha+d\beta)} e^{2\pi i m(a\alpha+b\beta\mid\gamma)+\pi i\tau m(\gamma\mid\gamma)}$$

$$= e^{m\pi i(ac(\alpha|\alpha)+bd(\beta|\beta)+(ac+bd)(\alpha|\beta))}\sum_{\gamma\in M+k^{-1}\lambda-\beta} e^{2\pi i m(a\alpha+b\beta\mid\gamma+c\alpha+d\beta-\beta)+\pi i\tau m(\gamma+c\alpha+d\beta-\beta\mid\gamma+c\alpha+d\beta-\beta)}$$

为c的倍数

$\gamma + c\alpha + (d-1)\beta$ $\qquad$ $b \equiv c \equiv \bmod S$

$$= e^{m\pi i(ac(\alpha|\alpha)+bd(\beta|\beta))}\sum_{\gamma\in M+k^{-1}\lambda-\beta} e^{\pi i m\tau(\gamma|\gamma)+2m\pi i(\gamma|\alpha)}$$

$$= e^{m\pi i(ac(\alpha|\alpha)+bd(\beta|\beta))}\,\theta_\lambda^{\alpha,\beta}(\tau)$$

where $|\nu(A)\, e^{m\pi i(ac(\alpha|\alpha)+bd(\beta|\beta))}|=1$

where the lattice M is even and for $\Gamma(m,n)\cap\Gamma(S)\cap\Gamma(\theta)$

when the lattice M is odd.

(第二步) 证明 $f\to$ holomorphic. $i\infty$ (2)

Furthermore, it is clear from (13.6.2) $\theta_\lambda^{\alpha,\beta}(\tau)=e^{\pi i(\alpha|\beta)}\sum_{\gamma\in M+k^{-1}\lambda-\beta} e^{\pi i k(\gamma|\alpha)}q^{k|\gamma|^2/2}$ ①

that $\theta_\lambda^{\alpha,\beta}(\tau)\Big|_{(T,j)} = j(\tau)^{-l}\theta_\lambda^{\alpha,\beta}(T.\tau)$

$= \theta_\lambda^{\alpha,\beta}(T.\tau) = \theta_\lambda^{\alpha,\beta}(\tau+1) = \theta_\lambda^{\alpha,\beta}$

$\theta_\lambda^{\alpha,\beta}(\tau)\Big|_{(T^2,j)} = \theta_\lambda^{\alpha,\beta}(T^2.\tau) = \theta_\lambda^{\alpha,\beta}(\tau+2) = \theta_\lambda^{\alpha,\beta}$

$S=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$

$j(\tau)=(\tau)$

according M is even or odd ✓

By (13.6.4) $(F|_{(S,j)})^{\alpha,\beta}(\tau) = F^{-\beta,\alpha}(\tau)|_{(S,j)} \longrightarrow \theta_\lambda^{\alpha,\beta}(\tau)\Big|_{(S,j)}$

$= (\tau)^{-l}$

(13.4.1) T and $S\longrightarrow SL_2(\mathbb{R})$

T^2 and $S\longrightarrow \Gamma_\theta$

$\Rightarrow$ the linear span of $\mathcal{F}_{m,s}$ is invariant under $Mp_2(\mathbb{Z})$ if M is even

then By prop 13.4 (b) $Th\Big|_{(A,j)} = Th$ if the lattice M is even.

$\Rightarrow \theta_\lambda^{\alpha,\beta}\in Th$ and all function from $\mathcal{F}_{m,s}$ is are holomorphic at the cusp $i\infty$, By Lemma 13.12 in §13.12 $\Rightarrow$

all of them are holomorphic modular forms. provided M is even

The general case is reduced to this one by $\tau\to 2\tau$.

i.e. When M is odd. without loss of generality, we can assume that $(\alpha_1|\alpha_1)$ is an odd. , Replacing α_i by $\alpha_i - \alpha_1$ for $i \geq 2$, if $(\alpha_i|\alpha_i)$ is odd.
then $(\alpha_i|\alpha_i) \in 2\mathbb{Z}$. for $i = 2 \cdots l$

Now, $M' = 2\mathbb{Z}\alpha_1 + \sum_{i=2}^{l} \mathbb{Z}\alpha_i$ is even sublattice of M

and $M = M' \cup (\alpha_1 + M')$

$$\theta_\lambda^{\alpha,\beta} = \theta_\lambda'^{\alpha\beta} + \theta_\lambda'^{\alpha,\beta}$$

全纯. To

#

Cor 13.6: Let M be $\mathbb{Z}$-lattice of rank l. Let $(\cdot|\cdot) \to \mathbb{Q}$-valued. $\varepsilon: M \to \mathbb{C}$ be constant on cosets of some sublattice of finite index and let $\alpha \in \mathbb{Q} \otimes_{\mathbb{Z}} M$ Then

$$f(\tau) = \sum_{\gamma \in M} \varepsilon(\gamma)\, q^{|\gamma+\alpha|^2} = \theta_{\lambda}^{0,-\alpha}$$

is a holomorphic modular form of wight $\frac{l}{2}$ for some $\Gamma(N)$ and some multiplier system

pf: replacing M by a sublattice M' of finite index.
Supose that ε is on the coset of sublattice M' is constant.
Since $(\cdot|\cdot)$ on M' is $\mathbb{Q}$-valued and M' is f.g. $\mathbb{Z}$ mod. $\exists$ a positive integer m_0 such

that $M_0 = m_0 M'$ is even integual sublattice of M.

$k_0 = |M/M_0| = \underline{|M/M'|}\,\underline{|M'/M_0|} < \infty$

Take $\{\beta_1 \cdots \beta_{k_0}\} \subset M$ s.t

$$M = \bigcup_{i=0}^{k_0} (\beta_i + M_0) \qquad q = e^{2\pi i \tau}$$

the $f(\tau) = \sum_{\gamma \in M} \varepsilon(\gamma)\, q^{|\gamma+\alpha|^2}$

$$= \sum_{i=1}^{k_0} \left(\sum_{\gamma \in M_0} \underline{(\varepsilon(\beta_i))}\, e^{k\pi i (\gamma+\alpha+\beta_i | \gamma+\alpha+\beta_i)} \right)$$

$$= \sum_{i=1}^{k_0} \underset{\shortparallel\ 1}{\varepsilon(\beta_i)} \left(\sum_{\gamma \in M_0} e^{k\pi i |\gamma+\alpha+\beta_i|^2} \right)$$

$$= k_0\, q^{|\gamma+\alpha+\beta_i|^2}$$

Since $\alpha \in \mathbb{Q} \otimes_{\mathbb{Z}} M$. there exist a positive integer s.

$$\begin{cases} s(\beta_i+\alpha) \in M_0 \\ s(\beta_i+\alpha) \in 2\mathbb{Z} \end{cases} \quad \text{for } i = \overline{1 \cdots k_0}$$

By prop 13.6. $\underline{M = M_0}$ taking $m = k$

$$f(\tau) = \sum_{i=0}^{k_0} \varepsilon(\beta_i)\, \theta_{\lambda_0}^{0, -\alpha-\beta_i} \qquad \begin{cases} \alpha = 0 \\ \beta = -\alpha - \beta_i \end{cases}$$

$$f(\tau) = \underset{1}{\sum_{i=0}^{k_0} \underline{\varepsilon(\beta_i)}}\ \underset{\frac{1}{k}}{\sum_{\gamma \in M-\alpha-\beta_i} q^{|\gamma|^2}} = \theta_{\lambda_0}^{0, -\alpha-\beta_i}$$

Recall r Let $N =$ ~~be~~ the least common multiple of s and r

the all $\theta_{\lambda}^{0,(\alpha-\beta_i)}$ are holomorphic forms of weight $\frac{l}{2}$ for $\Gamma(N)$

The most popular example of holomorphic modular forms are these

$$f_{n,m}(\tau) = \sum_{j\in\mathbb{Z}} q^{m(j+\frac{n}{2m})^2} = \theta_{n,m}(\tau,0,0)$$

$$g_{n,m}(\tau) = \sum_{j\in\mathbb{Z}} (-1)^j q^{m(j+\frac{n}{2m})^2}$$, where $m,n \in \frac{1}{2}\mathbb{Z}$, $m>0$

We obtain from (13.5.6), provided that $n\in\mathbb{Z}$

$$f_{n,m}\left(-\frac{1}{\tau}\right) = \left(\frac{-i\tau}{2m}\right)^{1/2} \sum_{k\in\mathbb{Z} \bmod 2m\mathbb{Z}} e^{-i\pi kn/m} f_{k,m}(\tau)$$

$$\theta_{n,m}(\tau,z,u) = e^{2\pi i m u} \sum_{k\in\mathbb{Z}+\frac{n}{2m}} e^{2\pi i m(k^2\tau + kz)} \qquad (13.5.6)$$

$$\theta_{n,m}(\tau,0,0) = e^0 \sum_{k\in\mathbb{Z}+\frac{n}{2m}} e^{2\pi i m(k^2\tau)} = f_{n,m}(\tau)$$

$$\theta_{n,m}\left(-\frac{1}{\tau}, \underset{0}{\frac{z}{\tau}}, u + \underset{0}{\frac{z^2}{2\tau}}\right) = (-i\tau)^{\frac{1}{2}} (2m)^{-\frac{1}{2}} \sum_{n'\in\mathbb{Z} \bmod 2m\mathbb{Z}} e^{\cdots} \theta_{n',m}(\tau,z,u)$$

$$\theta_{n,m}\left(-\frac{1}{\tau},0,0\right) = \left(\frac{-i\tau}{2m}\right)^{\frac{1}{2}} \sum_{k=0}^{2m-1} e^{-\pi kn/m} \theta_{k,m}(\tau)$$

$$\downarrow$$

$$f_{n,m}\left(-\frac{1}{\tau}\right) = \left(\frac{-i\tau}{2m}\right)^{\frac{1}{2}} \sum_{k=0}^{2m-1} e^{-\pi i kn/m} f_{k,m}(\tau)$$

In fact $13.56 \Rightarrow 13.65$ only for integral m

$\frac{m}{2}$ $(\alpha|\alpha)=1$

Since $g_{n,m} = f_{n,4m} - f_{(n+2m),4m}$ $f_{n,m}(-\frac{1}{\tau})$

$= f_{2n,4m} - f_{n+2m,4m}$ $g_{n,m}(-\frac{1}{\tau})$

$$f_{2n,4m} = \sum_{j\in\mathbb{Z}} q^{4m(j+2n/8m)^2}$$

$$f_{n+2m,4m} = \sum_{j\in\mathbb{Z}} q^{4m(j+2n+4m/8m)^2}$$

$$4m(j+2n/8m)^2 = m(2j+\frac{n}{2m})^2$$

$$4m(j+2(2m+n)/8m)^2 = m(2j+1+(\frac{n}{2m})^2)^2$$

$\Rightarrow$ $g_{n,m} = f_{2n,4m} - f_{(n+2m),4m}$

$$\Rightarrow \boxed{g_{n,m}(-\frac{1}{\tau}) = \left(\frac{-i\tau}{2m}\right)^{\frac{1}{2}} \sum_{j=1}^{2m} e^{-\frac{\pi(2j-1)ni}{2m}} g_{j-\frac{1}{2},m}(\tau)}$$

①

$\sum_{j=1}^{m}$

② $g_{n,m}(\tau) = - g_{2m-n,m}(\tau)$

$g_{m,m}(\tau) = -g_{m,m}(\tau) \Rightarrow g_{m,m}(\tau) = 0$

$$= \left(\frac{-i\tau}{2m}\right)^{\frac{1}{2}} \sum_{j=1}^{m} 2\left(e^{\frac{-\pi(2j-1)ni}{2m}}\right) g_{j-\frac{1}{2},m}(\tau)$$

$$= \left(\frac{-i\tau}{2m}\right)^{\frac{1}{2}} \sum_{\substack{j=1 \\ j\in\mathbb{Z}}}^{m} 2\cos\frac{(2j-1)n\pi}{2m} g_{j-\frac{1}{2},m}(\tau)$$

$$g_{\frac{1}{2},\frac{3}{2}}\left(-\frac{1}{\tau}\right) = (i\tau)^{\frac{1}{2}} g_{\frac{1}{2},\frac{3}{2}}(\tau)$$

$n=\frac{1}{2}$

$m=\frac{3}{2}$

$$(-i\tau)^{\frac{1}{2}} \sqrt{\tfrac{1}{3}} \left(2\cos\left(\frac{\frac{3}{2}\pi}{3}\right)\right) g_{\frac{3}{2},\frac{3}{2}}(\tau) = 0$$

$$+ (-i\tau)^{\frac{1}{2}} \sqrt{\tfrac{1}{3}} \left(2\cos\left(\frac{\pi}{6}\right)\right) g_{\frac{1}{2},\frac{3}{2}}(\tau) = 0$$

$$= (-i\tau)^{\frac{1}{2}} g_{\frac{1}{2},\frac{3}{2}}(\tau)$$

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