Chapter 14.

- The pricipal and homogeneous Vertex operator Constructions
- Boson - Formion Correspondence
- Solition equation

note: The highest - module $L(\Lambda_0)$ over an affine algebra $g(A)$ is called the basic representation of $g(A)$

§14.1.

$g(A) \longrightarrow$ an affine algebra of type $X_N^{(r)}$ and rank $l+1$

$g(A) = g'(A) + Cd$

$g'(A) =$ generated by $\{e_i, f_i \ (i=0 \ldots l)\}$

$\bar{g}(A) = g'(A)/CK$. $\quad \pi: g'(A) \longrightarrow \bar{g}(A)$

- let $\deg e_i = -\deg f_i = 1 \quad (i=0,\ldots l) \quad \mathbb{1} = (1 \ldots 1)$

define the principal gradation $g'(A) = \bigoplus_{j\in\mathbb{Z}} g_j(\mathbb{1})$

$$g_j = \bigoplus_{\substack{\alpha : \alpha = \sum_i k_i \alpha_i \\ \sum k_i = j}} g_\alpha$$

$$\bar{g}(A) = \bigoplus_{j\in\mathbb{Z}} \bar{g}_j(\mathbb{1})$$

- $\bar{e} = \sum_{i=0}^{l} \pi(e_i) \in \bar{g}_1(\mathbb{1})$ is called the cyclic ~~me~~ ~~elemen~~t of $\bar{g}(A)$
- $\bar{s} = \{x \in \bar{g} \mid [x, \bar{e}] = 0\}$

$$\bar{s} = \bigoplus_{j\in\mathbb{Z}} \bar{s}_j$$

- $\boxed{s = \pi^{-1}(\bar{s})}$ is called the principal subalgebra $g'(A)$

§14.4 $L(\Lambda_0) \to \mathfrak{g}(A)$ $\quad \Lambda_0 \in \mathfrak{h}^*$ by

$\uparrow$ $\mathfrak{s}$ $\quad \langle \Lambda_0, \alpha_i^\vee \rangle = \delta_{0,i}$

$\langle \Lambda_0, d \rangle = 0$

$\{\alpha_0 \cdots \alpha_l, \Lambda_0\} \to \mathfrak{h}^*$

Recall: the definition of the infinite-dimensional Heisenberg alge (§ 9.13)

$\mathfrak{s}$ basis $\{p_i, q_i, i=1,2\cdots\}$ and c with

$[p_i, q_i] = c \quad (i=1 \cdots$

$\underline{\mathbb{C}c}$

Lemma 14.4 The pricipal subalgebra $\mathfrak{s}$ of $\mathfrak{g}(A) \cong \mathfrak{s}$

Pf: $p_i \longrightarrow e_i$

$q_i \longrightarrow f_i$

$c \longrightarrow k$

$\mathfrak{s} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{s}_j$ $\quad \mathfrak{s}_j \subset \mathfrak{g}_j$

By prop 14.3(b) $\quad [\mathfrak{s}_j, e] = 0$

$[\mathfrak{s}_i, \mathfrak{s}_j] \subset \mathbb{C}K$ for all $i, j \in \mathbb{Z}$

prop 14.2 (c) + (14.3.6)

$[\mathfrak{s}_i, \mathfrak{s}_j] = 0$ for $i \neq -j$

$[\mathfrak{s}_i, \mathfrak{s}_{-i}] = \mathbb{C}K$

$\{a_n, n \in \mathbb{Z}, \hbar\}$

$[\hbar, a_n] = 0 \quad n \in \mathbb{Z}$

$[a_n, a_m] = \delta_{m-n,0}\, \hbar$

In particular $[a_0, a_n] = 0$

Now we make the first step in the "principal construction" of the basic representation

Prop 14.4

$\mathfrak{g}(A) \longrightarrow$ affine alge $A \in X_N^{(r)}$, $A = A, D,$ or E

$\mathfrak{s} \longrightarrow \mathfrak{g}'(A)$ be the principal subalgebra of $\mathfrak{g}'(A)$

$\Rightarrow$ The $\mathfrak{g}'(A)$-mod $L(\Lambda_0)$, casider an $\mathfrak{s}$-module, remain irreducible.

① $\mathfrak{s} \cong \mathfrak{S}$

② recall prop 10.10: $\mathfrak{g}(A)$ $\boxed{s =} (\langle \Lambda, \alpha_1^\vee \rangle, \dots \Lambda(\alpha_l^\vee))$ $\Lambda \in P_+$

Consider the principal gradation $L(\Lambda) = \bigoplus_{j \in \mathbb{Z}_+} L_j(\mathbb{1})$

$$\dim_q L(\Lambda) = \prod_{j \geq 1} (1-q^j)^{\dim \mathfrak{g}_j(s+1) - \dim \mathfrak{g}_j(\mathbb{1})} \qquad \Lambda \in P_+$$

③ Lema 9.13(a)

V be an $\mathfrak{S}$-mod, such that $c = a I_V$, where $a \neq 0$.

which has a vacuum vector $v_0 \neq 0$ s.t. $V = U(\mathfrak{S}_-)(v_0)$

$\Rightarrow V \cong R_a = (R, a)$

$R = \mathbb{C}[x_1 \cdots x_n \cdots]$

$$\left\{ \begin{aligned} \underline{\sigma_a(p_i)} &= a \frac{\partial}{\partial x_i} \\ \underline{\sigma_a(q_i)} &= \underline{x_i} \\ \underline{\sigma_a(c)} &= \underline{a I_R} \end{aligned} \right.$$

Called the Canonical commutation relation repre.

Let $L(\Lambda_0) = \bigoplus_{j \geq 0} (L(\Lambda_0))_j$ the principal gradation of $L(\Lambda_0)$

$$L(\Lambda_0)_j = \bigoplus_{\lambda : \deg \lambda = j} L(\Lambda_0)_\lambda \qquad s = (1, 0, \dots, 0)$$

$$\deg (\Lambda - \sum k_i \alpha_i) = \sum_i k_i s_i$$

and Let $\dim_q(L(\Lambda_0)) = \sum_{j\geq 0} \dim (L(\Lambda_0))_j\, q^j$

be the q-dimension of $L(\Lambda_0)$

By prop 10.10

$$\dim_q(L(\Lambda_0)) = \prod_{j\geq 1} (1-q^j)^{\underbrace{\dim \mathfrak{g}_j^{+}(2,1,\ldots,1)}_{L} - \underbrace{\dim \mathfrak{g}_j(1)}_{L+\dim s_j}}$$

$$= \prod_{j\geq 1} (1-q^j)^{-\dim s_j}$$

On the other hand, the $\mathbb{Z}$-gradation $s = \bigoplus_{j\in\mathbb{Z}} s_j$

induce a $\mathbb{Z}$-gradation $U(s) = \bigoplus_{j\in\mathbb{Z}} \underbrace{U(s)_j}_{U(s_j)}$

Set $U_j = U(s_{-j})(v_0)$, $v_0 \to v_0 \in L(\Lambda_0)_{\Lambda_0}$

$V = U = \bigoplus_{j\in\mathbb{Z}} U_j \Rightarrow$ Then $U_j \subset L(\Lambda_0)_j$

But By Lemma 9.13 (a) $\Rightarrow$ $U \simeq R_a$ as s-module

$$\dim_q U = \sum_{j\geq 1} (\dim U_j)\, q^j = \prod_{j\geq 1} (1-q^j)^{\dim s_j} = \dim_q (L(\Lambda_0))$$

$\Rightarrow U = L(\Lambda_0)$

$\Rightarrow L(\Lambda_0)$ is irreducible as s-module. #

$\boxed{\S 14.5}$

$R = \mathbb{C}[x_1, x_2 \cdots \quad]$

$\hat{R} = \mathbb{C}[[x_1 \cdots \quad x_n \cdots]]$

A differential operator on R is sum of the form

$$\sum_{r \geq 0} \sum_{1 \leq i_1 \cdots \leq i_r} P_{i_1} \cdots P_{i_r} \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_r}}, \text{ when } P_{i_1}, \cdots P_{i_r} \in \hat{R}$$

$(R) \subset \hat{R}$

- differential operator on R with values $\hat{R}$

example. $\frac{\partial}{\partial x_i}$ $\quad \underline{\prod_i \frac{1}{m_i!} \left(\frac{\partial}{\partial x_i}\right)^{m_i}}$

$m = (m_1 \cdots \quad m_2 \cdots \quad)$

- $D_m : R \longrightarrow \hat{R}$ by $D_m = \frac{1}{m_i!} \left(\frac{\partial}{\partial x_i}\right)^{m_i}$

- also allow such operators combined with multiplication by ele. of $\hat{R}$

$\Rightarrow \sum_m P_m D_m : R \longrightarrow \hat{R}$

Prop. each linear map from $R \longrightarrow \hat{R}$ has this form

$\underbrace{\sum_m P_m D_m}_{D}$ for a unique set of elements $P_m \in \hat{R}$

. Operator $T_\lambda : R \longrightarrow \hat{R}$ defined by

$$(T_\lambda f)(x_1 \cdots x_n \cdots) = f(x_1+\lambda_1, x_2+\lambda_2 \cdots)$$

note that

$$(14.5.1) \quad T_\lambda = \exp\left(\sum_{i \geq 1} \lambda_i \frac{\partial}{\partial x_i}\right)$$

By Taylor's formular.

$m = (m_1 \cdots m_n \cdots)$

$$f(x_1+\lambda_1, x_2+\lambda_2 + \cdots)$$

$$= \sum_m \frac{\lambda_1^{m_1}}{m_1!} \cdot \frac{\lambda_2^{m_2}}{m_2!} \cdots \left(\frac{\partial}{\partial x_1}\right)^{m_1} \cdots \left(\frac{\partial}{\partial x_2}\right)^{m_2} \cdots f(x_1 \cdots x_n \cdots)$$

$$= \sum_m \left(\prod_i \lambda_i^{m_i}\right) D_m$$

$$\left(\sum_{m_1} \frac{\lambda_1^{m_1}}{m_1!}\left(\frac{\partial}{\partial x_1}\right)^{m_1}\right)\left(\sum_{m_2} \frac{\lambda_2^{m_2}}{m_2!}\left(\frac{\partial}{\partial x_2}\right)^{m_2}\right) \cdots f(x_1 \cdots x_n)$$

$$= \exp\left(\lambda_1 \frac{\partial}{\partial x_1}\right) \exp\left(\lambda_2 \frac{\partial}{\partial x_2}\right) \cdots f(x_1 \cdots x_n)$$

$$= \exp\left(\lambda_1 \frac{\partial}{\partial x_1} + \lambda_2 \frac{\partial}{\partial x_2} \cdots + \right) f(x_1 \cdots x_n)$$

$$\Rightarrow \boxed{T_\lambda = \exp\left(\sum_{i \geq 1} \lambda_i \frac{\partial}{\partial x_i}\right)}$$

Lemma 14.5

Let $D: R \to \hat{R}$ is a linear map

(1) $[x_i, D] = \lambda_i D$, $i = 1 \cdots$ $\quad (x_i D(f)) - D(x_i f) = \lambda_i D(f)$

$\Rightarrow D = D(1) \exp(-\sum_i \lambda_i \frac{\partial}{\partial x_i}) = D(1) T_{-\lambda}$

(2) If $[\frac{\partial}{\partial x_i}, D] = u_i D$ $\quad i = 1 \cdots$ $\longleftrightarrow$ $\frac{\partial}{\partial x_i} D(f) - D(\frac{\partial f}{\partial x_i}) = u_i D(f)$

$\Rightarrow D(1) = c \exp(\sum_i u_i x_i)$ $\quad c \in \mathbb{C}$ $\quad \forall f \in R$

(3) if $[x_i, D] = \lambda_i D$ and $[\frac{\partial}{\partial x_i}, D] = u_i D$, $i = 1 \cdots n$

then $D = c \exp(\sum_i u_i x_i) \exp(-\sum_i \lambda_i \frac{\partial}{\partial x_i})$ for some $c \in \mathbb{C}$

(1) Pf: show $D(f) = \cdots (f)$ for all monomials $f \in R$

degree of f

$\deg f = 0$, $f = c \in \mathbb{C}$

$D(1) T_{-\lambda}(c) = D(1)\, c = D(c) = D(f)$ ✓

assum that for f ✓ $\quad (x_i f)$?

$D(x_i f) = (x_i - \lambda_i) D(f) = (x_i - \lambda_i) D(1) T_{-\lambda} f$

$\|$

$$(T_{-\lambda}(x_i)\; T_\lambda(f))$$

$$D(1)\, T_{-\lambda}(x_i f) = D(1)(x_i - \lambda_i)\, T_\lambda f$$

$$= D(1)\,(x_i - \lambda_i)\, T_\lambda f$$

(2). $\boxed{\frac{\partial}{\partial x_i}} D(f) - D\left(\frac{\partial f}{\partial x_i}\right) = u_i Df \quad f \in R$ ☆

Consider $\exp\left(-\sum u_i x_i\right) D(f) \in \hat{R}$

$$\frac{\partial}{\partial x_i}\left(\exp\left(\sum_i u_i x_i\right) D(f)\right)$$

$$= -u_i \exp\left(\sum_i - u_i x_i\right) D(f) + \exp\left(\sum_i - u_i x_i\right) \frac{\partial}{\partial x_i} D(f)$$

$$= \exp\left(\sum_i - u_i x\right)\left(\frac{\partial}{\partial x_i} - u_i\right) D(f)$$

$$\exp\left(\sum_i - u_i x\right) D\left(\frac{\partial f}{\partial x_i}\right)$$

$$= \exp\left(\sum_i - u_i x\right)\left(\frac{\partial}{\partial x_i} - u_i\right) D(f).$$

$$\Delta = \exp\left(\sum_i - u_i x_i\right) D$$

$\Delta \frac{\partial f}{\partial x_i} = \frac{\partial}{\partial x_i} \Delta (f)$ for each i and f

$f = 1 \qquad \frac{\partial}{\partial x_i}\left(\Delta(1)\right) = 0$

$\Rightarrow \Delta(1) = c$ for some $c \in \mathbb{C}$

$\underline{D(1)} = \underline{\exp\left(\sum_i u_i x_i\right)} \; \underbrace{\boxed{\Delta(1)}}_{c}$

$= c \exp\left(\sum_i u_i x_i\right)$

def: Differential operators $D: R \to \hat{R}$ of the form $\exp\left(\sum_j \lambda_j x_j\right) \exp\left(-\sum_j u_j \frac{\partial}{\partial x_j}\right)$ are called vetex operator

§14.6

$\tau^{\beta} \longrightarrow$

$\tau^{\beta}(z) = \sum_{j \in \mathbb{Z}} \tau_{\beta, j} z^{j}$

$\tau^{\beta}(z) = \underbrace{\langle \lambda_0, A_{\beta,0} \rangle}_{\in \mathbb{C}} \overbrace{\left(\exp \sum_{j=1}^{\infty} \lambda_{\beta j} z^{b_j} x_j\right)}^{R \to \hat{R}}$

根向量

$\underbrace{\left(\exp -\sum_{j=1}^{\infty} \lambda_{\beta, N+1-j} b_j^{-1} z^{-b_j} \frac{\partial}{\partial x_j}\right)}_{R \to R}$

Thm 14.6: Let $\mathfrak{g}(A)$ be an affine algebra such that either A is symmetic (from table Aff 1), or ~~from~~ from Aff 2 or 3.

$R = \mathbb{C}[x_1, \ldots x_n \ldots]$

$\Rightarrow$ Then identity operator, x_i, $\frac{\partial}{\partial x_j}$ $(j=1 .. l)$

☆ $\boxed{\tau_{\beta,j}}$ $(i=1 - l,\ j\in\mathbb{Z})$ form a basis of a Lie algebra of the algebra of differential operators ~~perp~~ preserving R.

$\langle \text{identity}, x_i, \frac{\partial}{\partial x_j}, \tau_{\beta,j} \rangle \cong g'(A)$ ☆

$\Downarrow$

The realization of the basic representation is called the principal vertex operator construction

☆ 14.8

$L(\Lambda_0)$

$g \swarrow$

$t \swarrow$

§ 14.8. $\hat{L}(g) \longrightarrow \mathrm{End}(V)$

$\left(t^{-m}\otimes v \otimes e^{\alpha}\right)$

$$V = S\Big(\bigoplus_{j<0} t^j \otimes h\Big) \otimes_{\mathbb{C}} \mathbb{C}[Q]$$

- $g = h \oplus \Big(\bigoplus_{\alpha\in\Delta} \mathbb{C}E_\alpha\Big) \rightarrow$ Simple f.d. Lie alge of A_l, D_l or E_l

with commutation relations defined by (7.8.5)

$$\begin{cases} [h, h'] = 0 & \text{if } h, h' \in h. \\ [h, E_\alpha] = (h|\alpha) E_\alpha & \cdots \\ [E_\alpha, E_{-\alpha}] = -\alpha & \text{if } \alpha\in\Delta \end{cases}$$

$[E_\alpha, E_\beta] = 0$ if $\alpha, \beta\in\Delta$, $\alpha+\beta \notin \Delta\cup\{0\}$

$[E_\alpha, E_\beta] = \varepsilon(\alpha,\beta) E_{\alpha+\beta}$, $\alpha+\beta\in\Delta$

$\varepsilon : Q \times Q \longrightarrow \pm\{1\}$

$\varepsilon(h'+h'', h) = \varepsilon(h', h)\,\varepsilon(h'', h)$

$(h|E_\alpha) = 0 \qquad (E_\alpha|E_\beta) = -\delta_{\alpha+\beta,0}$

Let $\hat{L}(g) = \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} + \mathbb{C}K + \mathbb{C}d.$

Conside the complex commutative associative algebra

(14.8.1) $V = S\Big(\underbrace{\bigoplus_{j<0} g_{j\delta}}\Big) \otimes_{\mathbb{C}} \mathbb{C}[Q].$

basis $t^j \otimes \alpha_i \quad i = 1 \cdots l,\ j<0$

. is isomorphic to the polynomial ring over $\mathbb{C}$.

in variable $t^j \otimes \alpha_i$

$j<0,\ \alpha_i \quad i = 1 \cdots l$

the group algebra

of the root lattice $Q \subset \mathfrak{h}$ of $g(X_l)$

. $\mathbb{C}[Q]$ is isomorphic to the algebra of Laurent polynomials over $\mathbb{C}$ in the variables $\alpha_1 \cdots \alpha_l$

$$\sum_{m_1 \ldots m_l \in \mathbb{Z}} \lambda_{m_1 \ldots m_l}\, \alpha_1^{m_1} \cdots \alpha_l^{m_l}$$

$\Rightarrow V = \mathbb{C}\,[\alpha_1 \cdots \alpha_l,\ \alpha_1^{-1}, \alpha_2^{-1} \cdots \alpha_l^{-1},\ t^j \otimes h_i]$

$i = 1 \cdots l \quad j<0$

- Let $\alpha \longmapsto e^{\alpha} \qquad Q \subset \mathbb{C}[Q]$
- As before $u^{(n)} = t^n \otimes u \quad (n \in \mathbb{Z},\ u \in g)$

for $n>0$, $u \in g$, $u(-n)$ the operator on V

multplication $u^{(-n)} = t^{-n} \otimes u$

. $n \geq 0$ $u \in \mathfrak{h}$ $u(n)$

denote by $u(n)$ the derivation of the algebra V defined by the formula:

$$(14.8.2) \quad u(n)\left((t^{-m} \otimes v) \otimes e^{\alpha} \right) = n\,\delta_{m+n,0}(u|v) \otimes e^{\alpha} + \delta_{n,0}(\alpha|u)(t^{-m} \otimes v) \otimes e^{\alpha}$$

$n \neq 0$ i.e. $U(n)$: $t^{-n} \otimes v \longmapsto n(u|v)$

$t^{j} \otimes v \longmapsto 0$ if $j \neq -n$

$1 \otimes v \longmapsto 0$

$n = 0$ $t^{j} \otimes v \longrightarrow 0$ $\forall j \neq 0$

$1 \otimes v \longrightarrow (u|u)\,v$

Choosing dual basis of u_i and u^i of $\mathfrak{h}$

$$D_0 = \sum_{i=0}^{l} \left(\frac{1}{2} u_i u^i + \sum_{n \geq 1} u_i(-n)\, u^i(n) \right)$$

Furthermore, for $\alpha \in Q$, define the sign operator

$$C_\alpha (f \otimes e^{\beta}) = \varepsilon(\alpha, \beta) f \otimes e^{\beta}$$

. for $u \in \Delta \subset Q$. introduce the vertex operator

$$(14.8.5) \quad \underset{\downarrow}{\Gamma_\alpha(z)} = \left(\exp \sum_{j \geq 1} \frac{-u(j)}{j} z^{j} \right) \left(\exp \sum_{j \geq 1} \frac{u(-j)}{j} z^{j} \right) e^{\alpha} z^{\alpha(0)} C_\alpha$$

$\mathrm{End}(V)$ $\qquad \exp \sum_{j \neq 0} \left(-\frac{u(j)}{j} z^{-j} \right)$

fact1: $\exp\left(\sum_{j\ge 1} \frac{-u(j)}{j} z^{-j}\right)$: $V \longrightarrow \mathbb{C}[z, \frac{1}{z}] \otimes V$

fact2: $z^{\alpha} \in \mathrm{End}\left(\mathbb{C}[z, z^{-1}] \otimes V\right)$ by

Here z is viewed as an ~~into~~ indeterminate.

expanding in δ power of z $\longrightarrow \in \mathrm{End}(V)$

$$\Gamma_\alpha(z) = \sum_{j \in \mathbb{Z}} \Gamma_\alpha^{(j)} z^{-j-1}$$

pf: $v \in V$,

$$v(M_m) = \prod_{\substack{i,j \\ j<0}} (t^j \otimes \alpha_i)^{m_{ij}} \prod_i \alpha_i^{m_i}$$

$\Gamma_\alpha(z)\,(M_m)$

$\underline{e^\alpha z^{\alpha_0} c_\alpha}(M_n) \in z^{n_\alpha} \otimes v$

$\exp\left(\sum_{j\ge 1} \frac{z^j}{j} u(-j)\right)(M_n) \in \sum_{k=0}^{K} z^{n_\alpha - k} \otimes v$

$z^{-j-1} = \underline{\Gamma_\alpha^{(j)}}$ $V \to V$

☆ Thm (14.8): The map $\sigma : \hat{L}(\mathfrak{g}) \longrightarrow \mathrm{End}\, V$ given by

$k \longmapsto 1$ $\quad E_\alpha^{(n)} \longmapsto \Gamma_\alpha^{(n)}$ for $\alpha \in \Delta$, $n \in \mathbb{Z}$

$u^{(n)} \longmapsto \underline{u(n)}$ $\quad d \longmapsto -D_0$

define the basic repn of $\hat{L}(\mathfrak{g})$ on $\hat{V}$

$L(\Lambda_0)$ #