6.30

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4.2 Diagonal type

Let V be a braided vector space of diagonal type.
$\dim V = \theta < \infty$

Let $(x_i)_{i \in \mathbb{I}_\theta}$ be a basis of V. $q = (q_{ij})_{i,j \in \mathbb{I}_\theta} \in \Bbbk^{\theta \times \theta}$ s.t
$q_{ii} \neq 0$ $\quad c(x_i \otimes x_j) = q_{ij}\, x_j \otimes x_i \quad \forall i,j \in \mathbb{I}_\theta$.

$B(V)$

Let $(\alpha_i)_{i \in \mathbb{I}_\theta}$ be a canonical basis of $\mathbb{Z}^\theta$.
χ the bilinear form on $\mathbb{Z}^\theta$ st $\chi(\alpha_i, \alpha_j) = q_{ij}$.
Set $\underline{q_{\alpha,\beta}} = \chi(\alpha, \beta)$ $\quad \alpha, \beta \in \mathbb{Z}^\theta$

$B(V)$ is a $\mathbb{Z}^\theta$-graded with $\deg x_i = \alpha_i$
$T(V)$ is ......

$$\deg x_1 x_2 = \deg x_1 + \deg x_2 = \alpha_1 + \alpha_2$$
$$\deg (a x_1) = \alpha_1$$
$$\deg \left(\prod x_i^{a_i} \right) = \sum a_i \alpha_i$$

Since c is $\mathbb{Z}^\theta$-graded $B(V)$ inherits the $\mathbb{Z}^\theta$-grading of $T(V)$.

There is a totally ordered subset $L \subset B(V)$ consisting of $\mathbb{Z}^\theta$-homogeneous elts. st.

$$\{ l_1^{m_1} \cdots l_k^{m_k} \mid k \in \mathbb{N}_0 ;\ l_1 > \cdots > l_k \in L,\ 0 < m_i < N_{l_i} \ \forall i \in \mathbb{I}_k \}$$

is a basis of $B(V)$ (called <u>restricted PBW basis</u>)

$$N_l = \min \{ n \in \mathbb{N} \mid (n)_{q_{\deg l, \deg l}} = 0 \} \in \mathbb{N} \cup \infty.$$

Called ~~the~~ <u>height of l</u>.

Thm 1. If a Hopf alg is generated by a set of <u>~~skew~~ elts.</u> <u>and</u> by an abelian group G of all group-like elts.
$\deg l \in \mathbb{Z}^\theta$
$\Rightarrow$ the set of all monotonic restricted G-super-words

$\Rightarrow$ the set of all monotonic restricted G-super-words in hard letters constitute a basis for H.

$\Delta^+(B(V))$ denote the set of degrees of (restricted) PBW generators of $B(V)$ $\quad \Delta^+(B(V)) \subset \mathbb{N}^\theta$

$\Delta(B(V)) := \Delta^+(B(V)) \cup -\Delta^+(B(V))$

$m_{ij} := \min\{m \in \mathbb{N}_0 \mid (m+1)_{q_{ii}} (q_{ii}^m q_{ij} q_{ji} - 1) = 0\} < \infty$

$M_{ij} := \{(\mathrm{ad}_c\, x_i)^m (x_j) \mid m \in \mathbb{N}_0\}$

$x \in P(H) \qquad \mathrm{ad}_c\, x(y) = xy - \mu \circ c(x \otimes y)$

m_{ij} is well-def $\Leftarrow$ M_{ij} is finite.

$s_i : \mathbb{Z}^\theta \longrightarrow \mathbb{Z}^\theta$

$s_i(\alpha_j) := \begin{cases} -\alpha_i, & j = i \\ \alpha_j + m_{ij}\alpha_i & \text{if } j \neq i \end{cases}$

called pseudo-reflection.

$s_i^2 = id$.

Def. (groupoid) $\mathcal{G} \neq \phi$, $\phi \neq D \subset \mathcal{G} \times \mathcal{G}$. $\circ : D \to \mathcal{G}$

$(\mathcal{G}, \circ)$ is called a groupoid if

(1) if $(x,y) \in D$ then each of $x, y, x \circ y$ is uniquely determined by the other two

(2) if $(x,y), (y,z) \in D, \Longrightarrow (x\circ y, z), (x, y\circ z) \in D$ and $(x\circ y)\circ z = x \circ (y \circ z)$

(3) if $(x,y), (x\circ y, z) \in D, \Rightarrow (y,z), (x, y\circ z) \in D$ and $\sim \ldots$

(4) if $(y,z), (x, y\circ z) \in D \Longrightarrow (x,y), (x\circ y, z) \in D$

(4) if $(y,z), (x, y\circ z) \in D \Longrightarrow (x,y), (x\circ y, z) \in D$
and ...

(5) $\forall x \in \mathcal{G}$, $\exists!\ e, f, y \in \mathcal{G}$ s.t. $(e,x), (x,f)$
$(y,x) \in D$ $\quad e\circ x = x\circ f = x$, $\quad y\circ x = f$.

(6) If $e\circ e = e$ $\quad f\circ f = f$ for certain $e, f \in \mathcal{G}$
$\Rightarrow \exists\ x \in \mathcal{G}$ s.t. $e\circ x = x\circ f = x$.

(Small cat. homomorphism are bijective)

$W(V) := \{ (s, E) \mid s \in \mathrm{Aut}(\mathbb{Z}^\theta)$, E is an ordered basis
of $\mathbb{Z}^\theta$, there exist $m_1, m_2 \in \mathbb{N}_0$, $m_1 \le m_2$,
$i_1, \cdots, i_{m_2} \in \{1, \cdots, \theta\}$. s.t.
$s_{i_m} \cdots s_{i_1}(E_0)$ is well-def. for all $m \le m_2$ and
$s_{i_{m_1}} \cdots s_{i_1}(E_0) = E$, $s = s_{i_{m_2}} \cdots s_{i_{m_1+1}} \}$

$E_0 := \{ \alpha_i \mid \in \mathbb{Z}^\theta$. $\quad (1, 0 \cdots 0) \ \cdots \ (0, \cdots, 1)$

V is of Cartan metrix.

Thm 5 $\quad \dim B(V) < \infty \Leftarrow A$ is a finite Cartan metrix.

$$c(x_i \otimes x_j) = q_{ij}\, x_j \otimes x_i$$

if

$$q_{ij} q_{ji} = q_{ii}^{a_{ij}} \qquad a_{ii} = 2.$$

(a_{ij}) $\quad$ GCM. $\quad$ (V, c) is Cartan type.

Conj.: $\mathrm{GKdim}\, B(V) < \infty \Rightarrow |W(V)| < \infty$

Thm. If either its weyl groupoid is infinite and $\dim V = 2$
or else V is affine Cartan type then
$\mathrm{GKdim}\, B(V) = \infty$

Prop 3.1 [5] If A is of affine type then $\mathrm{GKdim}\, B(v) = \infty$.

Pf. Δ^{re} : the set of real roots corresp. A

$\exists$ positive imaginary root δ s.t.

$$\Delta^{re} + \delta = \Delta^{re} \qquad [\text{Kac Prop } 6.3]$$

Let m = height of δ, α is a simple root.

Choose a homogeneous restricted PBW basis of $B(v)$.

Then for all $k \geq 0$ there exists a PBW generator y_k of degree $\alpha + k\delta$, hence $\underline{\deg y_k = mk + 1}$.. (height.)

Therefore $\mathrm{GKdim}\, B(v) = \infty$. by Lemma.

Lemma. Let $B = \bigoplus_{n \geq 0} B^n$ be a f.g. graded alg. $B^0 = \mathbb{K}$.

Let $(y_k)_{k \geq 0}$ be a family homogeneous elts of B. s.t.

$$\{ y_{i_1} \cdots y_{i_l} : i_j \in \mathbb{N},\ i_1 < \cdots < i_l \}$$

is a family of linearly indep. elts. If $\exists\ m, p \in \mathbb{N}$ s.t. $\underline{\deg y_i \leq mi + p}$ $\forall\, i \in \mathbb{N}$. then $\mathrm{GKdim}\, B = \infty$.

Pf. $1 \leq i_1 < \cdots < i_l$ we have

$$\begin{aligned} \deg y_{i_1} \cdots y_{i_l} &\leq m(i_1 + \cdots + i_l) + lp \\ &\leq m \frac{i_l(i_l+1)}{2} + lp \\ &\leq (m+p)\, i_l^2 \end{aligned}$$

$$\begin{aligned} \deg y_{i_1} \cdots y_{i_l} &= \deg y_{i_1} + \cdots + \deg y_{i_l} \\ &\leq (m i_1 + p) + \cdots + (m i_l + p) \end{aligned}$$

$$\deg y_{i_1} \cdots y_{i_l} \leq M$$

Let $i_l \leq \lfloor \sqrt{M/(m+p)} \rfloor$

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Let G be a finite set of homogeneous generator of B. B_m is the subspace generated by products of j elts in G. $0 \le j \le M$.

$$B_M \supset \bigoplus_{k=0}^{M} B^k$$

$$\dim B_m \geq \sum_{k=0}^{M} \dim B^k \geq 2^{\lfloor \sqrt{M/(m+p)} \rfloor}$$

$\deg y_{i_1} \cdots y_{i_l} \le M$ $\qquad i_l \le \lfloor \sqrt{M/(m+p)} \rfloor = t$

$\{y_1, \ldots, y_t\}$ $\qquad 2^t$

$$\mathrm{GKdim}\, B = \overline{\lim_M} \log_M \dim B_M \geq \overline{\lim} \log_M 2^{\lfloor \sqrt{M/(m+p)} \rfloor}$$

$$= \overline{\lim} \left\lfloor \sqrt{\frac{M}{m+p}} \right\rfloor \frac{\ln 2}{\ln M}$$

$\frac{\sqrt{m}}{\ln m} \to \infty$

$$= \infty$$

Example 3.1. V : braid v.s. of dim 2. of diagonal type

$q = \begin{pmatrix} q & q \\ q & q \end{pmatrix}$ $\qquad c(x_i \otimes x_j) = q\, x_j \otimes x_i$

(1) $q = 1$ $\qquad B(V) \simeq S(V)$.

$c(x_i \otimes x_j) = x_j \otimes x_i$ $\qquad c^2 = id$. $\Rightarrow$ symmetric type.

$B(V) \simeq T(V) / \langle \ker(c+id) \rangle \overset{?}{=} T(V) / \langle xy - yx,\ x, y \in V \rangle$

$\{x_1, x_2\}$. basis.

$(c+id)(x_1 \otimes x_2 - x_2 \otimes x_1) = 0 \Rightarrow \langle x_1 \otimes x_2 - x_2 \otimes x_1 \rangle \subset \langle \ker(c+id) \rangle$

Conversely, if $y \in \langle \ker(c+id) \rangle$

$\cdot\ y \in T^2(V) \cap \ker(C+id)$ $\quad y = k_{11}\, x_1 \otimes x_1 + k_{12}\, x_1 \otimes x_2 + k_{21}\, x_2 \otimes x_1 + k_{22}\, x_2 \otimes x_2$

$$0 = (C+id)(y) = y + k_{11} x_1 \otimes x_1 + k_{12}\, x_1 \otimes x_1 + k_{21}\, x_1 \otimes x_1 + k_{22}\, x_2 \otimes x_2 .$$

$$= 2k_{11}\, x_1 \otimes x_1 + (k_{12}+k_{21})\, x_1 \otimes x_2 + (k_{21}+k_{12})\, x_2 \otimes x_1 + 2k_{22}\, x_2 \otimes x_2 = 0$$

$k_{22} = k_{11} = 0, \quad k_{12} = -k_{21}$

$$y = k_{12}\, x_1 \otimes x_2 - k_{12}\, x_2 \otimes x_1 \in \underline{\langle x_1 \otimes x_2 - x_2 \otimes x_1 \rangle}.$$

$\forall\ z \in \langle \ker(C+id) \rangle \cap T^n(V)$

$$z = \sum k\, x_1 \otimes \cdots \otimes \underset{=}{y} \otimes \cdots x_n \in \langle x_1 \otimes x_2 - x_2 \otimes x_1 \rangle$$

$\Rightarrow \langle \ker(C+id) \rangle = \langle x_1 \otimes x_2 - x_2 \otimes x_1 \rangle$

$\Rightarrow B(V) \simeq S(V).$

(2) $q = -1$ $\quad C(x_i \otimes x_j) = -x_j \otimes x_i$ $\quad C^2 = id$ $\quad$ Symmetric.

$B(V) \simeq \Lambda(V) = T(V) / \langle xy + yx,\ x, y \in V \rangle$

$xy + yx$ $\quad x_1 \otimes x_2 + x_2 \otimes x_1$ $\quad x_1 \otimes x_1, \quad x_2 \otimes x_2$

$\in \ker(C+id)$

$y \in T^2(V) \cap \ker(C+id)$

$$0 = (C+id)(y) = \underline{k_{11}}\, x_1 \otimes x_1 + k_{12}\, x_1 \otimes x_2 + k_{21}\, x_2 \otimes x_1 + \underline{k_{22}\, x_2 \otimes x_2}$$
$$- \underline{k_{11}\, x_1 \otimes x_1} - k_{12}\, x_2 \otimes x_1 - k_{21}\, x_1 \otimes x_2 - \underline{k_{22}\, x_2 \otimes x_2}$$
$$= (k_{12} - k_{21})\, x_1 \otimes x_2 + (k_{21} - k_{12})\, x_2 \otimes x_1$$
$$= 0$$

$\Rightarrow k_{12} = k_{21}$

$$y = k_{11}\, x_1 \otimes x_1 + k_{12}\,(x_1 \otimes x_1 + x_2 \otimes x_1) + k_{22}\, x_2 \otimes x_2$$

$$\in \langle x \otimes y - y \otimes x \,.\ x, y \in V \rangle .$$

$z \in \langle \mathrm{Ker}(c+id) \rangle \cap T^n(V)$. Similar to (1).

$B(V) \simeq \Lambda(V)$.

(3) if $q \in G_N'$ then V is of Cartan type $\begin{pmatrix} 2 & 2-N \\ 2-N & 2 \end{pmatrix}$

Then if $N=3$, V is of A_2 type. $\underline{\dim B(V) = 27}$.

$q^N = 1$. $q^m \neq 1$ $m < N$.

$q_{12} = q_{21} = q$ $\quad q_{12}\, q_{21} = q_{11}^{a_{12}} \Rightarrow q^2 = q^{a_{12}}$

$\Rightarrow q^{2-a_{12}} = 1$ $\quad 2 - a_{12} = N$ $\quad a_{12} = 2-N$

$a_{21} = 2-N$.

$N=3$. $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

By [3, section 4] P397 $\dim B = N^{\binom{\theta+1}{2}}$

if V is of A_θ type.

(4) if $N > 3$ then $\mathrm{GK}\text{-}\dim B(V) = \infty = \dim B(V)$

$N = 4$ $\quad A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ is of affine type.

by Thm 6. $\mathrm{GKdim}\, B(V) = \infty$.

$N > 4$. $|A| = 4 - (2-N)^2 < 0 \Rightarrow A$ is of indef. type.

by prop 4.9 [Kac] $|W| = \infty$.

$\underline{W(V) = W \times B}$ (半直积.) $\therefore W(V)$ is infinite.

by Thm 6. $\mathrm{GKdim}\, B(V) = \infty$

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