We next deal with the following situation:

$V_\circ = V_1 \oplus V_2$

V_1 has dim 2 of diagonal type

V_2 has dim 1

but the braiding between them is not diagonal

$(x_i)_{i \in I_3}$ for some ε, $q_{ij} \in \Bbbk^\times$
$i, j \in I_2$

$$c\left(c(x_i \otimes x_j)\right)_{i,j \in I_3} = \begin{pmatrix} \varepsilon x_1 \otimes x_1 & \varepsilon x_2 \otimes x_1 & q_{12}\, x_3 \otimes x_1 \\ \varepsilon x_1 \otimes x_2 & \varepsilon x_2 \otimes x_2 & q_{12}\, x_3 \otimes x_2 \\ q_{21}\, x_1 \otimes x_3 & q_{21}(x_2 + x_1) \otimes x_3 & q_{22}\, x_3 \otimes x_2 \end{pmatrix}$$

Thm 9

[4] Thm 8.3:

Let V be as above

Since V_1 is of diagonal type

we may assume $\varepsilon \in G_2 \cup G_3$

If $\varepsilon \in G'_\infty$, $\mathcal{B}(V_1) \cong T(V_1)$ $\mathrm{GK} = \infty$

(a) If $\varepsilon \in G_3$, then $\mathrm{GKdim}\, \mathcal{B}(V) = \infty$

(b) If $\varepsilon = -1$ and $\mathrm{GKdim}\, \mathcal{B}(V) < \infty$

then either of the following holds:

(i) $\tilde{q}_{12} = 1$ and $q_{22} = \pm 1$ $\mathrm{GK} = 1$

(ii) $q_{22} = -1 = \tilde{q}_{12}$ $\mathrm{GK} = 2$

Proof:

(a) If $\varepsilon = 1$, then this is $\boxed{\text{Prop } 8.5}$

If $\varepsilon \in G'_3$, § 8.1.5

(b) Assume $\tilde{q}_{12} = 1$

If $q_{22} = \pm 1$, $\boxed{\text{Prop } 8.6 \;\; 8.7}$

If $q_{22} \neq \pm 1$ § 8.1.4

Assume $\tilde{q}_{12} \neq 1$

Prop 8.8. Lemma 8.9 8.10, 8.11

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Consider $K = \mathcal{B}(V)^{\mathrm{co}\, \mathcal{B}(V_1)}$

$K = \bigoplus_{n\geq 0} K^n$ inherits the grading of $\mathcal{B}(V)$

$$\begin{tikzcd} & \mathcal{B}(V_1) \arrow[dl, bend right] \arrow[d, equal] \\ \mathcal{B}(V) \arrow[r, "\pi"'] & \mathcal{B}(V_1) \end{tikzcd}$$

$$\underline{\mathcal{B}(V)^{co\,\mathcal{B}(V_1)}} = \{ a \in \mathcal{B}(V) \mid (id \otimes \pi)\Delta(a) = a \otimes 1 \}$$

$\mathcal{B}(V) \cong K \# \mathcal{B}(V_1)$

K is the Nichols algebra of

$$\underline{K' = ad_c\, \mathcal{B}(V_1)(V_2)}$$

$K' \in \boxed{{}^{\mathcal{B}(V_1)\#\Bbbk\Gamma}_{\mathcal{B}(V_1)\#\Bbbk\Gamma}\mathcal{YD}}$ with the

adjoint action: $\underline{\mathcal{B}(V)\#\Bbbk\Gamma}$

$$\mathcal{B}(V_1)\#\boxed{\Bbbk\Gamma} \otimes K \xrightarrow{a} K$$

$$a \otimes x \mapsto \underset{\substack{\shortparallel \\ a_1 x S a_2}}{(ad\, a)x}$$

coaction $\delta = (\pi_{\mathcal{B}(V_1)\#\Bbbk\Gamma} \otimes id)\Delta_{\mathcal{B}(V)\#\Bbbk\Gamma}$

For later use, we introduce:

$$\underline{\text{ш}}_{m,n} = (ad_c x_1)^{\boxed{m}} (ad_c x_2)^{\boxed{n}} x_3$$

- particularly, $w_m = \underline{\text{ш}}_{m,0}$

$z_n = \underline{\text{ш}}_{0,n}$

then (8.5) $\boxed{g_1 \cdot \underline{\text{ш}}_{m,n} = q_{12}\, \varepsilon^{m+n}\, \underline{\text{ш}}_{m,n}}$

$g_2 \cdot w_m = q_{21}^m q_{22} w_m$

(8.6) $z_{n+1} = x_2 z_n - q_{12}\varepsilon^n z_n x_2$

$\underline{\text{ш}}_{m+1,n} = x_1 \underline{\text{ш}}_{m,n} - q_{12}\varepsilon^{m+n} \underline{\text{ш}}_{m,n} x_1$

(8.7) $\partial_1(\underline{\text{ш}}_{m,n}) = 0$ $\partial_2(\underline{\text{ш}}_{m,n}) = 0$

(8.8) $\partial_3(w_m) = \prod_{0\leq j\leq m-1}(1 - \varepsilon^j \tilde{q}_{12}) x_1^m$

(8.7)第一个

$\partial_1(\underline{\text{ш}}_{m,n}) = 0$:

$\underline{\text{ш}}_{m+1,n} = x_1 \underline{\text{ш}}_{m,n} - q_{12}\varepsilon^{m+n} \underline{\text{ш}}_{m,n} x_1$

$\underline{\partial_1(\underline{\text{ш}}_{0,n})} = \partial_1((ad_c x_2)^n x_3) = 0$

$$\underline{\partial_1(\underline{w}_{m+1,n})} = \underline{\partial_1(x_1\underline{w}_{m,n})} - \varepsilon_{12}\varepsilon^{m+n}\underline{\partial_1(\underline{w}_{m,n}x_1)}$$

$$= x_1\boxed{\partial_1(\underline{w}_{m,n})} + g_1\cancel{\underline{w}_{m,n}}$$

$$-\varepsilon_{12}\varepsilon^{m+n}\left(\cancel{\underline{w}_{m,n}} + \boxed{(\partial_1(\underline{w}_{m,n})}g_1\cdot x_1\right)$$

$\partial_3(w_m)$

$m=1.$ $\partial_3(w_1) = \partial_3(x_1x_3 - \varepsilon_{12}x_3x_1)$

$$= x_1 - \varepsilon_{12}\varepsilon_{21}x_1$$

Suppose m ✓

$$\partial_3(w_{m+1}) = \partial_3(x_1w_m - \varepsilon_{12}\varepsilon^m w_m x_1)$$

$$= x_1\underline{\partial_3(w_m)} - \varepsilon_{12}\varepsilon^m\,\underline{\partial_3(w_m)}\underbrace{g_2\cdot x_1}_{\varepsilon_{21}}$$

$= \cdots$

∎

8.13 ($\varepsilon=1$, prop 8.5)

$B(V_1) \cong S(V_1)$ x_1, x_2 commute

$$(\mathrm{ad}_c x_2)^p\,\underline{w}_{m,n} = \underline{w}_{m,n+p} \quad (8.9)$$

Consequently $\boxed{\underline{w}_{m,n}}$ $m,n \in \mathbb{N}_0$ generates K'

We claim that

$$g_2\cdot\underline{w}_{m,n} = \cdots \quad (8.10)$$

Now we may assume $\tilde{\varepsilon}_{12} = 1$ by lemma 2.8

Then $w_m = 0$ $\forall m>0.$ $\langle x_1, x_3\rangle$

by (8.7), (8.8)

$$\boxed{(ii)_{m,n} = 0 \quad \forall m > 0 \quad \text{by } (8.9)}$$

Here $g_2 \cdot z_n = q_{21}^n q_{22} z_n$ (8.11)

Lemma 8.4.

TFAE:

(i) $\operatorname{GKdim} K < \infty$

(ii) $q_{22} = -1$

If this happens, $K \cong \Lambda(K')$

$\operatorname{GKdim} K = 0$

Proof:

First we claim that $\forall n \in \mathbb{N}_0$

$\partial_3(z_n) = (-1)^n x_1^n$ (8.12)

$\delta(z_n) = \sum_{0 \le j \le n} (-1)^{n+j} \binom{n}{j} x_1^{n-j} g_1^j g_2 \otimes z_j$

(8.13)

(8.12): $n = 0$ ✓

n ✓

$$\partial_3(z_{n+1}) = \partial_3(x_2 z_n - q_{12} z_n x_2)$$

$$= x_2 \partial_3(z_n) - q_{12}(\partial_3(z_n) \underbrace{g_2 \cdot x_2}_{= q_{21}(x_2 + x_1)})$$

$$= \cdots = (-1)^{n+1} x_1^{n+1}$$

(8.13)

for $n = 0$.

$$\delta(x_3) = \left(\underline{\pi_{B(V_1)\# k_{\Gamma}}} \otimes id\right)\left(\boxed{x_3 \otimes 1} + g_2 \otimes x_3\right)$$

$$= g_2 \otimes x_3 \checkmark$$

Suppose n ✓

$$\delta(z_{n+1}) = \boxed{\delta(x_2 z_n - \underset{\overset{\shortparallel}{q_{12} z_n}}{(g_1 \cdot z_n)} x_2)}$$

$$\delta(x_2) = \boxed{x_2 \otimes 1} + g_1 \otimes x_2$$

$$\delta(z_n) = \sum_{0 \le j \le n} (-1)^{n+j} \binom{n}{j} x_1^{n-j} g_1^j g_2 \otimes z_j$$

$$= \sum_{0 \le j \le n} (-1)^{n+j} \binom{n}{j}$$

$$\Bigg(\underset{\alpha}{x_2 x_1^{n-j} g_1^j g_2 \otimes z_j} + \underset{\beta}{g_1 x_1^{n-j} g_1^j g_2 \otimes x_2 z_j}$$

$$- q_{12} \underset{\gamma}{\boxed{x_1^{n-j} g_1^j g_2 x_2}} \otimes z_j - \underset{\eta}{q_{12} x_1^{n-j} g_1^j g_2 g_1 \otimes z_j x_2} \Bigg)$$

$$\underline{\alpha + \gamma} = x_1^{n-j} x_2 g_1^j g_2 - \boxed{\overset{1}{\overset{\shortparallel}{q_{12} q_{21}}}} x_1^{n-j} x_2 g_1^j g_2$$

$$= \underline{q_{12} q_{21} x_1^{n-j} x_1 g_1^j g_2} \otimes z_j$$

$$= -x_1^{n+1-j} g_1^j g_2 \otimes z_j$$

$$\underline{\beta + \eta = x_1}^{n-j} g_1^j g_2 g_1 \otimes z_{j+1}$$

$$C_n^m = C_{n-1}^{m-1} + C_{n-1}^m$$

~~[illegible]~~ Therefore by (8.11), (8.13)

we have $\forall p, n \in \mathbb{N}_0$,

$$L(z_n \otimes z_p)$$

$$= \boxed{\sum_{0 \le j \le n}} (-1)^{n+j} \binom{n}{j} \underline{(x_1^{n-j} g_1^j g_2) \cdot z_p} \otimes z_n$$

$$g_1^j g_2 \cdot z_p = g_1^j (q_{21}^p q_{22} z_p)$$

$$= q_{12}^j q_{21}^p q_{22} z_p$$

$$\underline{W}_{m,n} = 0, \forall m > 0$$

$$= \underline{q_{12}^n q_{21}^p q_{22}} z_p \otimes z_n$$

In conclusion K' is of diagonal

$(q_{21}^n q_{22})_{n, \underline{p} \in \mathbb{N}_0}$ with respect

to the basis

$q_{12}^n q_{21}^p q_{22}$ $(z_n)_{n \in \mathbb{N}_0}$

Thus locally $\overset{q_{22}}{\circ}\overset{q_{22}^{2}}{\rule{3cm}{0.4pt}}\overset{q_{22}}{\circ}$

Then GKdim $B(K') = \infty$ (unless $q_{22}=-1$)

in this case $B(K') = \Lambda(K')$

then GKdim $B(K') = 0$

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Prop 8.5 GK $B(V) = \infty$

Proof: by lem 8.4, it remains to consider $q_{22}=-1$

Now $B(V)$ is a graded connected f.g algebra $\cong B(V_1) \# \Lambda(W)$

W has basis $(z_n)_{n\in\mathbb{N}_0}$

Since $z_{n_1}\cdots z_{n_\ell}$ are linearly independent when $n_1<\cdots<n_\ell$

then GKdim $B(V) = \infty$ by lem 2.5.