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4.4 Rack type, infinite dimension

Any rack is assume to be isomorphic to a conjugacy class of a finite ~~a finite~~ group.

$x \triangleright y = xyx^{-1}$

X : rack $\quad q: X \times X \longrightarrow GL(W)$ 2-cocycle

$\underline{B(X,q) := B(V)} \qquad V = \underline{kX \otimes W}$

$C^q(e_x v \otimes e_y w) = \underset{\overset{\shortparallel}{e_y}}{\underline{e_{x \triangleright y}}}\, \underline{q_{x,y}(w)} \otimes e_x v \qquad x, y \in X$, $v, w \in W$

When $\dim B(X,q) < \infty$ or $GK\dim B(X,q) < \infty$?

Remark 6. Y is an abelian subrack of X

$U = kY \otimes W$ is of diagonal type. $\qquad x \triangleright y = y$

$\dim W = 1$ ✓

Def. (Collapse) A finite rack X collapse if $\dim B(X,q) = \infty$ $\forall\, q$.

Def. (type C) if $\exists$ decomposable subrack

$Y = R \amalg S$

$R = \mathcal{O}_r^{\mathrm{Inn}\,Y}, \quad S = \mathcal{O}_s^{\mathrm{Inn}\,Y}$

$\min\{|R|, |S|\} > 2$ or $\max\{|R|, |S|\} > 4$

[$\mathrm{Inn}\,Y$: the subgroup of S_Y generated by the image of $\phi: X \to S_Y$

$\underline{\mathrm{Inn}\,Y = \langle \phi_y : y \in Y \rangle}$

$\phi: X \times X \longrightarrow X \qquad y \triangleright -$]

$r \in R, s \in S$ satisfying $r \triangleright s \neq s \qquad (rs \neq sr$ in group$)$

Def. (type D) if $\exists$ decomposable subrack $Y = R \amalg S$

and $r \in R$, $s \in S$ s.t.

$r \triangleright (s \triangleright (r \triangleright s)) \neq s \qquad ((rs)^2 \neq (sr)^2)$

Def. (type F) if $\exists$ subracks $(R_a)_{a \in I_4}$, elts $r_a \in R_a$, $a \in I_4$, s.t.

$R_a \triangleright R_b = R_b \qquad a, b \in I_4$

$R_a \cap R_b = \emptyset \qquad a \neq b$

$r_a \triangleright r_b \neq r_b \qquad a \neq b$

Def. austere (a rack) if every subrack gen. by 2 elts is abelian or indecomposable

Sober if every subrack ....

Kthulhu if it is neither of C, D nor F

Cthulhu .... D. F

Thm A rack X of type D, F, C collapses.

We need a lemma

Lemm 2.3 Let X finite rack. Assume that

(B) For any finite group G, $M \in {}^{G}_{G}\mathcal{YD}$ such that $X \simeq$ a subrack of $\operatorname{Supp} M$, $\dim \mathfrak{B}(M) = \infty$

Then X collapses.

$$\operatorname{Supp} M = \{ g \in G : M_g \neq 0 \}$$

$$M = \oplus M_g \qquad M_g = \{ m \in M \mid \rho(m) = g \otimes m \}$$

Thm. 3.3 G finite group, $M(\mathcal{O}, \rho)$, $M(\mathcal{O}', \rho')$ irreducible objects in ${}^{G}_{G}\mathcal{YD}$ such that $\dim \mathfrak{B}(M(\mathcal{O}, \rho) \oplus M(\mathcal{O}', \rho')) < \infty$

Then $\forall r \in \mathcal{O}, s \in \mathcal{O}'$ $(rs)^2 = (sr)^2$

proof (type D) $\quad Kx \otimes W$

We will show X is of type B.

Let $Y \subset X$ $Y = R \amalg S$ as definition of D.

Let G be a finite group, $M \in {}^{G}_{G}\mathcal{YD}$

$X \simeq$ a subrack of $\operatorname{Supp} M$

identify X to this subrack. take M_R, M_S to be non-trivial obj in ${}^{G}_{G}\mathcal{YD}$. $K = \langle Y \rangle < G$

We may assume M_R, M_S are irreducible.

Now $\dim \mathfrak{B}(M_R \oplus M_S) = \infty$ by Thm. 3.3

$\Rightarrow \dim \mathfrak{B}(M) = \infty$

(type C)

$X \simeq$ a subrack of $\mathrm{supp}\, M$

We'll show $\dim \mathcal{B}(M) = \infty$

$Y = R \amalg S \qquad K = \langle Y \rangle \leq G$

$M_Y := \bigoplus_{y \in Y} M_y \quad G\ {}^{K}_{K}\mathcal{YD} \qquad \underline{M_R := \bigoplus_{x \in R} M_x}$

$\underline{M_S := \bigoplus_{z \in S} M_z}$ being YD-submodule of M_Y

$R = \mathcal{O}_r^{\mathrm{Inn}\, Y}, \quad S = \mathcal{O}_s^{\mathrm{Inn}\, Y}$

$\quad\ \ \| \qquad\qquad\quad \|$

$\quad \mathcal{O}_r^K \qquad\qquad \mathcal{O}_s^K$

Let $\underline{V, W}$ be simple YD-Submodule $\underline{\text{of } M_R,\ M_S}$

$\mathrm{Supp}\, V = R \qquad \mathrm{Supp}\, W = S \qquad$ (since supp V is stable under $\cdot$, $g \cdot M_r = M_{grg^{-1}}$

$\mathrm{Supp}(V \oplus W) = Y$ gen. K.

$\underline{(\mathrm{Id} - c_{W,V} c_{V,W})(V \otimes W) \neq 0}$ since $rs \neq s$.

$\dim V \leq \dim W$

$\dim V \geq |R| > 2$ or $\dim W \geq |S| > 4$

Hence $(\dim V, \dim W)$ does not belong to the set

$\underline{\{(1,3), (1,4), (2,2), (2,3), (2,4)\}}$

Thus $\dim \mathcal{B}(V \oplus W) = \infty \implies \dim \mathcal{B}(M) = \infty$

[8] Thm 2.2.

G non-abelian $\quad V, W$ simple YD-mod over G.

G is gen. by $\mathrm{Supp}(V \oplus W) \qquad (\mathrm{Id} - c_{V,W} c_{W,V})(\cancel{V} \otimes W) \neq 0$

~~TFAE~~ ~~Then~~ $\dim \mathcal{B}(V \oplus W) < \infty$.

①

② (V, W) admit all reflections and $\underline{\text{the Weyl groupoid of } (V, W)}$ is finite.

In particular. $(\dim V, \dim W)$ belongs to the set

$\{ \qquad\qquad \}$

Prop. 8. Let $X \xrightarrow{ka} \mathcal{Y}$ Surjective morphism of racks

$\mathcal{Y}$ is $\underline{\text{not kthulhu}} \Rightarrow X$ is $\underline{\text{not kthulhu}}$.

$\cancel{X}$ C, D, F C, D, F

$$X \hookrightarrow Y$$

proof. (~~type~~ C) $\pi : Z \to X$ surj.

$Y = R \amalg S \subset X$ be as in def. of type C.

$|R| \le |S|$. $|R| > 2$ or $|S| > 4$

Fix $\tilde{r}, \tilde{s} \in Z$ such that $\pi(\tilde{r}) = r$, $\pi(\tilde{s}) = s$

Def.

$\underline{R_1 = \pi^{-1}(R)}$ $S_1 = \pi^{-1}(S)$, $Y_1 = \pi^{-1}(Y)$

$K_1 = \langle \varphi_y : y \in Y_1 \rangle \le \mathrm{Inn}\, Z$

$R_2 = \mathcal{O}_{\tilde{r}}^{K_1}$, $S_2 = \mathcal{O}_{\tilde{s}}^{K_1}$, $Y_2 = R_2 \amalg S_2$

$K_2 = \langle \varphi_y : y \in Y_2 \rangle \le \mathrm{Inn}\, Z$

...

$R_j = \mathcal{O}_{\tilde{r}}^{K_{j-1}}$, $S_j = \mathcal{O}_{\tilde{s}}^{K_{j-1}}$, $Y_j = R_j \amalg S_j$

$K_j = \langle \varphi_y : y \in Y_j \rangle \le \mathrm{Inn}\, Z$

$R_1 \supset R_2 \supset \cdots$

$S_1 \supset S_2 \supset \cdots$ $Y_i = R_i \amalg S_i$

$\tilde{R} := R_i = R_{i-1} = \mathcal{O}_{\tilde{r}}^{K_{i-1}}$

$\tilde{S} := S_i = S_{i-1} = \mathcal{O}_{\tilde{s}}^{K_{i-1}}$

$\tilde{Y} = \tilde{R} \amalg \tilde{S}$ is a subrack of Z

$\tilde{r} \triangleright \tilde{s} \ne \tilde{s}$ $\tilde{R} = \mathcal{O}_{\tilde{r}}^{\mathrm{Inn}\, Y}$, $\tilde{S} = \mathcal{O}_{\tilde{s}}^{\mathrm{Inn}\, Y}$

We claim $\underline{\pi(Y_j) = Y}$ $\forall j \in \mathbb{N}$.

$|R_j| \geq |R| > 2$ or $|S_j| \geq |S| > 4$

$\pi(R_j) = R$

Indeed $\pi(R_j) = R$

Hence, $\pi(R_j) = R$

Assume $\pi(Y_j) = Y$, hence $\cancel{\pi(R)}$ $\longrightarrow$ $\pi(S_j) = S$

Let $\underline{t \in R} = \mathcal{O}_r^{\mathrm{Inn}\,Y} \Rightarrow \exists\ y_1, \dots, y_h \in Y$

$y_1 \triangleright (y_2 \triangleright \dots (y_h \triangleright r) \dots) = t$

pick $\tilde{y}_1, \dots \tilde{y}_h \in Y_j$ such $\pi(\tilde{y}_i) = y_i$

$$\begin{tikzcd}
\tilde{y}_1 \triangleright (\tilde{y}_2 \triangleright \dots (\tilde{y}_h \triangleright \tilde{r}) \dots) \arrow[d, "\pi"] \arrow[r, phantom, "\in"] & \mathcal{O}_{\tilde{r}}^{K_j} = R_{j+1} \arrow[d, "\pi"] \\
\pi(\tilde{y}_1) \triangleright (\pi(\tilde{y}_2) \triangleright \cdots) & \\
y_1 \triangleright (y_2 \triangleright \cdots (y_h \triangleright r) \dots) = t \arrow[u, phantom, "\parallel"] \arrow[r, phantom, "\in"] & \underline{\pi(R_{j+1})}
\end{tikzcd}$$

$R = \pi(R_j)$ $\forall j$

$S = \pi(S_j)$

$Y = \pi(Y_j)$ $\forall j$

(type D) $R_1 = \pi^{-1}(R)$, $S_1 = \pi^{-1}(S)$

fix $\tilde{r} \in R_1$ $\tilde{s} \in S_1$ s.t.

$\pi(\tilde{r}) = r$, $\pi(\tilde{s}) = s$

$\tilde{r} \triangleright \tilde{s} \neq \tilde{s}$ $Y_1 = R_1 \amalg S_1$ $\Rightarrow X$ is of type D.

(type F) $\tilde{R}_a = \pi^{-1}(R_a)$

$\tilde{R}_a \cap \tilde{R}_b = \phi$

$\tilde{r}_a \triangleright \tilde{r}_b \neq \tilde{r}_b$

we'll show $\underline{\tilde{R}_a \triangleright \tilde{R}_b \overset{\supseteq}{\underset{\subseteq}{=}} \tilde{R}_b}$

$\subseteq$: $\forall\ \tilde{r}_a \in \tilde{R}_a$, $\tilde{r}_b \in \tilde{R}_b$ $\tilde{r}_a \triangleright \tilde{r}_b \in \pi^{-1}(R_b)$

$\pi(\quad) = r_a \triangleright r_b \in R_b$

$\supseteq$: $\forall\ \tilde{r}_b \in \pi^{-1}(R_b)$ $\tilde{r}_a \in \pi^{-1}(R_a)$

由 $\phi_{\tilde{r}_a} : X \longrightarrow X$ bijective $\Rightarrow \exists!\ x \in X$

$\tilde{r}_a \triangleright x = \tilde{r}_b$

$\pi(\tilde{r}_a) \triangleright \pi(x) = \pi(\tilde{r}_b)$

$r_a \triangleright \pi(x) = r_b$

$R_a \triangleright R_b = R_a \Rightarrow \exists\, r \in R_b$ s.t. $r_a \triangleright r = r_b$

ϕ_{r_a} bij. $\Rightarrow \pi(x) = r \Rightarrow x \in \pi^{-1}(R_b) = \tilde{R}_b$

$\Rightarrow \tilde{R}_b = \tilde{R}_a \triangleright \tilde{R}_b$. #

Cor. $\underline{X \longrightarrow Y} \longrightarrow 0$

simple

Y not ~~[illegible]~~ $\Rightarrow X$ collapses.

$\Downarrow \qquad\qquad \Uparrow$

$C, D, F \Rightarrow X$ ~~is~~ C, D, F

<u>Simple rack</u>

$X \to Y$

$\cong$

Y

4.4.2 S_n, A_n $(n \geq 5)$

$\sigma \in S_m$ $(1^{n_1}, 2^{n_2}, \ldots m^{n_m})$

n_1 fixed points

$\underline{(12)(34)} \in S_4$ $(1, 2, 3, 4)$ [illegible]

(2^2)

$(\underline{23})(456) \in S_6$

$(1, 2, 3)$

Thm 11 Let $\mathcal{O} = \begin{cases} \mathcal{O}_\sigma^{S_m}, & \sigma \notin A_m \\ \mathcal{O}_\sigma^{A_m}, & \sigma \in A_m \end{cases}$

If $\mathcal{O}$ is not listed in Table 2, then it is of type D

$\Downarrow$

collapses.

proof. step. 1 $m = p + q$ $\mu \in S_p$, $\tau \in S_q$ $\sigma = \mu \perp \tau \in S_m$

If $\mathcal{O}_\mu^p$ is type D $\Rightarrow$ $\mathcal{O}_\sigma^m$ also is

Step 2. σ is (m) $m \geq 6$ not prime
then σ is of type D

Step 3. $\underline{(n, p)}$ in A_{n+p} is type D
n, p odd $n \geq 3$, $p \geq 5$

Step 4. $(1^2, j)$ $j > 5$ odd $\Rightarrow$ type D

Step 5. $\underline{(1, 2, \sigma_0)}$ $\underline{\sigma_0 \neq id}$
$(2^3, \sigma_0)$ $\sigma_0 \neq id$
$(4, \sigma_0)$ $\sigma_0 \neq id$ $\Rightarrow$ type D
(4^2)

Step 6. $(2, j)$ $j > 3$ odd type D

Step 7 $\underline{(2, 3^2)}$

Step 8 $\underline{(1, 4)}$

Step 9 $\underline{(2, 4)}$

<u>Step 10</u> $\underline{(1^3, 2^2)}$
$(1, 3^2)$
(3^3)
(2^5)
$(1, 2^3)$

Final step. Assume σ is not type D. We use step 1.
By step 2. $n_j = 0$ if $j \geq 6$ not prime.
$(1^{n_1}, 2^{n_2}, 3^{n_3}, \ldots 5^{n_5}, 6^0, 7^{n_7}, 8^0 \cdots)$

Assume $n_4 \neq 0 \Rightarrow \sigma$ is of type $(1^{n_1}, 2^{n_2}, 4)$ by

(∵ (3, 4) is ~~type~~ D)

(3, 4) step 5

$(4, \sigma_0) \in D$ $(4^2) \in D$

~~n_3~~ $(1, 4) \in D \Rightarrow n_1 = 0$ (step 8

$(2, 4) \in D \Rightarrow n_2 = 0$

$(4) \in S_4$ ✗

$n_4 = 0$

$n_3 \neq 0$ $(\underline{1^{n_1}, 2^{n_2}}, 3^{n_3})$

$\Rightarrow n_1 = 0$ or $n_2 = 0$ by step 5

[if $n_1 > 0$ & $n_2 > 0$ $\Rightarrow$ ~~type~~ D $(1, 2, \sigma_0) \in D$.]

if $n_1 = 0$ then $\underline{n_3 = 1}$ by $\underline{\text{step 7}}$

$(2^{n_2}, 3^{n_3})$ since $(2, 3^2) \in D$

$n_2 \leq 2$ by step 5

since $(2^3, \sigma_0) \in D$

$(2, 3)$, $(2^2, 3)$ remain.

if $n_2 = 0$ $(1^{n_1}, 3^{n_3})$

① $n_1 = 0$ $(3^{n_3}) \Rightarrow (3^2)$ remain by step 10

since $(3^3) \in D$

② $n_1 \neq 0$ $(1^{n_1}, 3)$ remains

$n_3 = 0$ bf. $(1^{n_1}, 2^{n_2}, \underline{j}^{n_j})$ by step 3

[$(\underline{1^{n_1}, 2^{n_2}, \underline{j}^{n_j}, k^{n_k}})$ $\underline{(j^{\bullet}, k) \in D}$]

$j \geq 3$, $k \geq 5$

$n_j = 0$ or 1

$(j^2) \in D$ $\quad j^{n_1} \perp j$ step 3

≥ 5 $\quad \geq 5$

Further $n_2 \leq 4$, by $\underline{\text{step}}$ 10

$(2^5) \in D$

① $n_j = 0$ bf. $n_1 \neq 0$ $n_2 \leq 2$. by step 10

$(1, 2^3) \in D$

$$n_2 = \begin{cases} 2 \\ 1 \\ 0 \end{cases}$$

$n_2 = 2 \Rightarrow n_1 \le 2$ by step 6

since $(1^3, 2^2) \in D$,

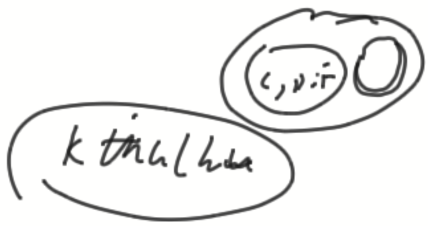
$(1, 2^2)$, $(1^2, 2^2)$ remains

$n_2 = 1$ ng. $(1^n, 2)$ remains.

$n_1 = 0$ ng (2^3), (2^4) 1

② $n_j = 1$ ng $\Rightarrow n_1 = 0$ or $n_2 = 0$ by step 5

C, D, F



$(3, 4) \in D$

$\sigma_1 = (123)(4567)$

$\sigma_2 = (132)(4657)$

$\sigma_1\sigma_2 \neq \sigma_2\sigma_1$

$(\sigma_1\sigma_2)^2 \neq (\sigma_2\sigma_1)^2$

$\sigma_1 \notin O_{\sigma_2}^{\langle \sigma_1, \sigma_2 \rangle}$

PermutationReplace[_, _]