4.4.4 Sporadic groups

① 素数阶循环群
② A_n $n \geq 5$
③ Lie type
④ 26

Thm 13:

Let G be a Sporadic simple group, different from the Monster M, let $\mathcal{O}$ be a non-trivial conjugacy class in G or $\mathrm{Aut}\, G$, if $\mathcal{O}$ is not listed in Table 4 then it collapses.

Remark 10

M

- - - -

4.5 Rack type. f.d

If $B(V)$ is f.d.

then $\exists N$ s.t. $B^N(V) \neq 0$, $B^{N+1}(V) = 0$

N \ top degree

Note that $B^N(V)$ is the space of integral of $B(V)$

$$\forall x \in B(V),\ x \int = \varepsilon(x) \int$$

$\Rightarrow \dim B^N(V) = 1$

And $B(V)$ satisfies Poincaré duality:

$$\dim B^i(V) = \dim B^{N-i}(V)$$
$$\forall i \in I_{0,N} \quad N \text{ top degree}$$

~~Proof~~

Prop: Let $R = \bigoplus_{i=0}^{N} R(i)$ be a f.d graded Hopf algebra in ${}^H_H\mathcal{YD}$, Suppose that $R(N) \neq 0$ then $\dim R(i) = \dim R(N-i)$ $\forall i \in \ldots$.

Proof: Since R is graded, then R^* is also a graded Hopf - - -

Let $\lambda \in R^*$ be a nonzero left integral

we have $\underbrace{\lambda = \sum_i \lambda_i}$ $\lambda_i \in R^*(i)$

It's immediate that each λ_i is a left integral

$\swarrow$

$x \in I_\ell(R) \Leftrightarrow yx = \varepsilon(y)x$

Then by the 1 dimensionality of $\int$

$\Rightarrow \lambda = \lambda_j$ for some j

已有结论:

for $\lambda \in I_\ell(R^*)$.

$$(x.y) = \langle \lambda, (x,y) S(x_0) \rangle$$

is a nondegenerate bilinear form on R

Since $R(i)$ and $R(k)$ are orthogonal if $i+k \neq j$

this map induces a nondegenerate bilinear form between $R(i)$ and $R(j-i)$

Here in particular we have $R(i)=0$ $\forall i>j$

then $j=N$. □

解釋: $\boxed{(R(i),\ R(j-i))}$

取一組基 $x_1, \dots, x_k$

則 $(x_1, \_), \dots, (x_k, \_)$

is linearly independent in $R^*_{(j-i)}$

$\Rightarrow \dim R(i) = k \le_{\dim} R^*(j-i) =_{\dim} R^*(j-i)$

$\Rightarrow \dim R(i) \le \dim R(j-i)$

$\ge$

$=$

4.5.1

Fomin-Kirillov algebras

Let $m \ge 3$

Consider two Nichols algebras

associated to the conjugacy class

of transpositions in S_m $\quad \mathcal{O}_2^m$

(ij)

$\varepsilon \equiv -1 \qquad \chi(\sigma, \xi) = \begin{cases} 1 & \sigma(i)<\sigma(j) \\ -1 & \sigma(i)>\sigma(j) \end{cases}$

where $\xi = (ij)$ $i<j$

The braided v.s $(\Bbbk\mathcal{O}_2^m, c^{\varepsilon})$

$(\Bbbk\mathcal{O}_2^m, c^{\chi})$

Can be realized as YD modules M_1, M_2

over $\Bbbk S_m$

Further more, if $M \in {}^{\Bbbk S_m}_{\Bbbk S_m}\mathcal{YD}$,

$M \neq M_1, M_2$ and $m>6$.

then $\dim B(M) = \infty$

[11] Thm1.1

$\downarrow$

$\dim B(\mathcal{O}, \rho) < \infty$

所有可能的 $\mathcal{O}$, ρ 都列出了.

Def14:

Let FK_m be the algebra

- - - -.

This is the quadratic approximation

of $\underline{B(\mathcal{O}_2^m, c^{\chi})}$

(仅有证明 m=3.4.5 $B(\mathcal{O}_2^m, c^{\chi})$

$\cong FK_m$

$\underline{m \ge 6 \text{ unknown}}$

Prop9:

if m=3.4.5.

then the dimension, Hilbert series
and its top degree of $\boxed{FK_m}$
are given in Table 5

Proof: $m=3,4$ $\quad m=5$ 计算机

Def 15:

let B_m be the algebra - - -

——.

Cor 2 if $m=3,4,5$

then B_m has the same dimension
Hilbert series
top degree
as in Table 5

$\overset{FK_m}{\text{Prop 9}} \Leftrightarrow \overset{B_m}{\text{Cor 2}}$

Prop 10

[83]

$B(\Bbbk\mathcal{O}_2^m, c^{\varepsilon})$ and $B(\Bbbk\mathcal{O}_2^m, \overset{\chi}{(\,)})$

are twist-equivalent.

Twist-equivalent:
Let X be a rack
q, q' be 2-cocycles on X

We say that q and q' are twist-eq- if $\exists\, \phi : X\times X \to \Bbbk^*$ s.t $q' = q^\phi$

$$q^\phi_{xy} = \phi(xy)\,\phi^{-1}(x\triangleright y, x)\, q_{xy}$$

$B(X,q)$ $B(X,q')$ has the same Hilbert series

Proof of prop 10:

Thm: Given $n\geq 4$ define a group $\boxed{T_n}$

$$T_n = \left\langle z, t_1, \dots, t_{n-1} \;\middle|\; \begin{array}{l} t_i^2 = (t_j t_{j+1})^3 = 1 \\ (t_k t_l)^2 = z \\ z^2 = 1 = [z, t_i] \end{array} \right\rangle$$

$1\leq i\leq n-1$
$1\leq j\leq n-2$ $\quad k\leq l-2$

Then $\exists$ a central extension

$$0 \to A \xrightarrow{i} T_n \xrightarrow{p} S_n \to 1$$

$A = \langle z\rangle$

$z \mapsto 1$

$t_i \mapsto \tau_i$
$\tau_i = (i\ i{+}1) \in S_n$

Lemma 3.6

Let A be ~~ea~~ an abelian group.
let $\psi : A \to \mathbb{C}^\times$ be a group homo-
for $\phi \in Z^2(S_n, A)$

define $\phi_\psi(x,y) = \psi(\phi(x,y))$

Then $\phi_\psi \in Z^2(S_n, \mathbb{C}^\times)$

lem 3.7

$\exists$ a section $s: S_n \to T_n$

s.t. if $\tau = (i\,j)$ $i<j$

~~then~~ $s(\sigma) \triangleright s(\tau) = \begin{cases} s(\sigma\triangleright\tau)z & \text{if } \sigma(i)<\sigma(j) \\ s(\sigma\triangleright\tau) & \text{if } \sigma(i)>\sigma(j) \end{cases}$

Proof:

by lemma 3.1 $\exists$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, phantom, "\rotatebox{90}{$\cong$}"] & T_n \arrow[r, "p"] & S_n \arrow[r] & 1 \\ & \langle z \rangle & & & \end{tikzcd}$$

let $\underline{s}: S_n \to T_n$ in lem 3.7

let $\phi(x,y) \in A$ defined by

$s(x)s(y) = i(\phi(x,y))\,s(xy)$

Then $\phi \in Z^2(S_n, A)$

解釋:

在 lem 3.7 的证明里, $ps = id$

$p(s(x)s(y)s(xy)^{-1}) = xy(xy)^{-1}$

$= 1$

$\Rightarrow s(x)s(y)s(xy)^{-1} \in \underline{\ker p} = \underline{\operatorname{im} i}$

$\Rightarrow s(x)s(y) = i(\phi(x,y))\,s(xy)$

Then $\phi \in Z^2(S_n, A)$

Take $\psi: A \to \mathbb{C}^\times$

$\psi = \operatorname{sgn}$ where $\psi(z) = 1$

$\Rightarrow \underline{\underline{\phi_\psi}} \in Z^2(S_n, \mathbb{C}^\times)$

claim

$\underset{\varepsilon}{\overset{\cup\!|}{\Theta}} = \phi_\psi(\sigma, \tau)\,\phi_\psi(\sigma \triangleright \tau, \sigma)^{-1} \boxed{\chi(\sigma, \tau)}$

In fact

$\phi_\psi(\sigma, \tau)\,\phi_\psi(\sigma \triangleright \tau, \sigma)^{-1}$

$= - - - -$

Thm 14:

If $m = 3, 4, 5$ then

$FK_m \cong \mathcal{B}(\mathcal{O}_2^m, \mathbb{C}^\chi)$

Proof:

The proof appears in $\underline{\underline{[70]}}$

for $\mathcal{B}_m$ and $m \leq 4$.

猜: $\boxed{\mathcal{B}_m \cong \mathcal{B}(\mathcal{O}_2^m, \mathbb{C}^\varepsilon)}$ 3,4,5 ✓

verified by Graña $\underline{m = 5}$

by prop 10, it translates to FK_m

$FK_m \cong \mathcal{B}_m \cong \mathcal{B}(\mathcal{O}_2^m, \mathbb{C}^\varepsilon)$

as V.S

$\cong \mathcal{B}(\mathcal{O}_2^m, \mathbb{C}^\chi)$

as V.S.

$\Rightarrow FK_m \cong \underline{\mathcal{B}(\mathcal{O}_2^m, \mathbb{C}^\chi)}$

as V.S

但 FK_m 是 $\mathcal{B}(\mathcal{O}_2^m, \mathbb{C}^\chi)$ 的二次逼近

$\Rightarrow FK_m \cong \mathcal{B}(\mathcal{O}_2^m, \mathbb{C}^\chi)$

Open question:

Let $m \geq 6$

① $FK_m \simeq \mathcal{B}(\mathcal{O}_2^m, \mathbb{C}^\chi)$ is a Nichols algebra

② $\dim FK_m$ is finite

③ $\dim \mathcal{B}(\mathcal{O}_2^m, \mathbb{C}^\chi)$ is finite $\Big\}$ 有人说不对

Example 3.2 [18]

The Nichols algebra of the conjugacy class $\mathcal{O}_4^4$ of 4-cycles in S_4 with $\varepsilon = -1$, is quadratic and has the same dimension and Hilbert series as those of FK_4, in Table 3.

4.5.2