4.5.2

f.d Nichols algebras of some affine racks

$\mathrm{Aff}(\mathbb{F}_q, T)$

给定有限域

Affine rack:

$\mathrm{Aff}(A, T)$ — A abelian, $T \in \mathrm{Aut}\, A$

$x \triangleright y = (1-T)x + Ty \quad x, y \in A$

$\mathbb{F}_q$: 元素个数只能是 p^n, p 素.
且由 $|\mathbb{F}_q|$ 唯一决定.

$\mathbb{F}_4$: $\mathbb{F}_2[x]/x^2+x+1$

$\{0, 1, \omega, 1+\omega\}$

$(1+1=0 \quad \omega^2+\omega+1=0)$

Consider the constant cocycle $q \equiv -1$

Set $\mathfrak{B}(\mathbb{F}_q, T) := \mathfrak{B}(\mathrm{Aff}(\mathbb{F}_q, T), q)$

Notice that $\mathrm{Aff}(\mathbb{F}_3, 2) \cong \mathcal{O}_2^3$ ($\times 2$)

$\mathbb{F}_3 \quad \mathcal{O}_2^3$

$0 \mapsto (12)$

$1 \mapsto (23)$ rack iso

$2 \mapsto (13)$

As we have seen

$\dim \mathfrak{B}(\mathbb{F}_3, 2) = 12$ the top degree $= 4$

$\mathfrak{B}(\mathcal{O}_2^3, c^{q\equiv -1})$

$\mathcal{O}_2^m$ $m=3$ 时,

example 33, 34, 35, 36

知识储备:

we have proved $\dim \mathfrak{B}^r(V) = \dim \mathfrak{B}^{n-r}(V)$
n – top degree

we have $\mathfrak{B}^m(V) = 0 \quad m > n$.

$\mathfrak{B}^0(V) = 1 = \dim \mathfrak{B}^n(V)$

and $\dim \mathfrak{B}^r(V) \neq 0 \quad 0 \leq r \leq n$

Since $\mathfrak{B}(V)$ is generated by $\mathfrak{B}^1(V)$

If $\mathfrak{B}^{i}(V) = 0$

Choose a non-zero integral $\int$

$\exists$ a nondegenerate bilinear form

$(x|y) = \lambda$

if $xy = \lambda \int +$ terms of degree $< n$

Thm:

Let $J \subseteq \ker(T(V) \to \mathfrak{B}(V))$ be an ideal which is also a coideal and is compatible with the braiding. if $c(V \otimes J) = J \otimes V$, $c(J \otimes V) = V \otimes J$.

Suppose that $T(V)/J$ is f.d. it has top degree n, and $\dim \mathfrak{B}^n(V) \neq 0$

then $T(V)/J = \mathfrak{B}(V)$.

Proof:

This is so thanks to the non-degenerate bilinear form of $T(V)/J$:

Let $\int$ be an integral in $T(V)/J$

If $0 \neq x \in \ker(T(V)/J \to \mathfrak{B}(V))$
then $\exists\, y \in T(V)/J$ s.t $xy = \int$.

解释: 由 $T(V)/J$ 上的 $\exists$ 非退化…
Suppose x 的 i 阶, $n-i$ 正交,
则 $\exists\, y \in n-i$ 阶

s.t $(x|y) = 1 \Rightarrow xy = \int$

Then $\int \in \ker(T(V)/J \to \mathfrak{B}(V))$
since $x \in \ker(\cdots)$

$\Rightarrow \mathrm{Im}(T^n(V)/J \to \mathfrak{B}(V)) = 0$

But $\mathfrak{B}^n(V) \neq 0$ (条件), $\Rightarrow$ to 满射 $T(V)/J \to \mathfrak{B}(V)$ □

Example 35 [15]

Consider $\mathrm{Aff}(\mathbb{F}_5, 2)$, $\mathrm{Aff}(\mathbb{F}_5, 3)$

$\mathfrak{B}(\mathbb{F}_5, 2)$ is generated by $(x_i)_{i \in \mathbb{F}_5}$

with relations

$$\begin{cases} x_i^2 = 0 \\ \cdots \\ \cdots \end{cases} \quad (*)$$

Also $\dim \mathfrak{B}(\mathbb{F}_5, 2) = 1280$
top degree $= 16$
Hilbert series $(\cdots)$

解释: [15] 思路.

Let J be the ideal generated by $(*)$

It's easy to prove J is a homogeneous coideal compatible with braiding

Using Gröbner bases

it can be seen $(*)$ yields the stated dimensions in each degree.

Then use the previous theorem, it sufficient to see that

0101201020303124 doesn't vanish in $\mathfrak{B}(V)$
16个

known: $T(V)/J$ is f.d top degree $= 16$.

则说明 $\mathfrak{B}^{16}(V) \neq 0$ 即证 $T(V)/J = \mathfrak{B}(V)$

It's straight forward to compute that

$\partial_{1041423424323434}(\cdots) = 160$

$\Rightarrow (\cdots) \neq 0$ in $\mathfrak{B}(V)$

□