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P321/Lem 9.1.19 Let $X \in \mathcal{X}$, $i \in I$,
and assume that
$\Delta^{X re}, \Delta^{r_i(X) re} \subseteq N_0^I \cup -N_0^I$

($\Delta^{X re} = \{ w(\alpha_i) \in \mathbb{Z}^I \mid w \in Hom(\mathcal{W}(G), X), i \in I \}$)

(1) The map s_i^X map $\pm\alpha_i$ to $\mp\alpha_i$
and it induces bijections
$s_i^X : \Delta_+^{X re} \setminus \{\alpha_i\} \longrightarrow \Delta_+^{r_i(X) re} \setminus \{\alpha_i\}$
$s_i^X : \Delta_-^{re} \setminus \{-\alpha_i\} \rightarrow \Delta_-^{r_i(X) re} \setminus \{-\alpha_i\}$

$Hom(X, Y) = \{ (Y, f, X) \mid f \in M \}$ $X, Y \in \mathcal{X}$

(2) $m_{ij}^X = m_{ij}^{r_i(X)}$

Pf: (1) Note that $s_i^X(\alpha) \in \alpha + \mathbb{Z}\alpha_i$, for all $\alpha \in \mathbb{Z}^I$
by Rmk 9.1.16 (2),
(P321/Rmk 9.1.16 (2) Let $X \in \mathcal{X}$, $\alpha \in \Delta^{X re}$
Then the only multiples of X ...... $\pm\alpha$)

$m\alpha_i \notin \Delta^{X re}$ for any $m \in \mathbb{Z} \setminus \{1, -1\}$ $(\setminus \{\alpha_i\})$

moreover, $\Delta^{r_i(X) re} \subseteq N_0^I \cup -N_0^I$ by assumption
Hence both maps in the claim are well-defined

良定义的含义在于,从左侧能映射到右侧
$\forall \alpha_j \neq \alpha_i$, $s_i^X(\alpha_j) \in \alpha_j + \mathbb{Z}\alpha_i$
$\because \Delta^{r_i(X) re} \subseteq N_0^I \cup -N_0^I$
$\therefore s_i^X(\alpha_j) \in \alpha_j + N_0^I \alpha_j$
$\therefore s_i^X(\alpha_j) \in \Delta_+^{r_i(X) re} \setminus \{\alpha_i\}$
$\therefore$ well-defined

Their inverses are induced by $s_i^{r_i(X)}$
and they are welled-defined, by Rmk 9.1.16(2)
$\Delta^{X re} \subseteq N_0^I \cup -N_0^I$

[by (CG2) $s_i^{r_i(X)} s_i^X(\alpha_j) = \alpha_j$ → P316/line 3
$\therefore s_i^{r_i(X)}, s_i^X$ — bijective

$s_i^X(\alpha_j) = \alpha_j - a_{ij}^X \alpha_i$
$s_i^{r_i(X)}(\alpha_j - a_{ij}^X \alpha_i)$
$= \alpha_j - a_{ij}^{r_i(X)} \alpha_i + a_{ij}^X \alpha_i$
$= \alpha_j$

(2) $m_{ij}^X = |\Delta^{X re} \cap (N_0\alpha_i + N_0\alpha_j)|$
$m_{ij}^X = m_{ij}^{r_i(X)}$

[Recall Humphreys, Lie algebra P51
$\sigma \in \mathcal{W}$, $\sigma = \sigma_{\alpha_1} \cdots \sigma_{\alpha_{l(\sigma)}}$, reduced,
$n(\sigma) = $ ~~ $\sigma(\alpha) \prec 0$
Lemma: $l(\sigma) = n(\sigma)$,

As for Weyl groups, we associated an

$n(\sigma)$:= number of positive roots α for which $\sigma(\alpha) \prec 0$

Defn 9.1.20 Let $X, Y \in \mathcal{X}$, and $w \in Hom(Y, X)$
we define $\Delta^{X re}(w) = \{ \alpha \in \Delta_+^{X re} \mid w^{-1}(\alpha) \in -N_0^I \}$
$N(w) = |\Delta^{X re}(w)|$

Goal: $N(w) = l(w)$
$\leq$

Lem 9.1.21 Let $X, Y \in \mathcal{X}$, $i \in I$, and $w \in Hom(Y, X)$,
(1) $N(w) = N(w^{-1})$
(2) Assume that $\Delta^{Y re}, \Delta^{r_i(Y) re} \subseteq N_0^I \cup -N_0^I$
(a) If $w(\alpha_i) \in N_0^I$, then $N(w s_i) = N(w) + 1$
and $\Delta^{X re}(w s_i) = \Delta^{X re}(w) \cup \{ w(\alpha_i) \}$
(b) If $w(\alpha_i) \in -N_0^I$, then $N(w s_i) = N(w) - 1$
and $\Delta^{X re}(w s_i) = \Delta^{X re}(w) \setminus \{ -w(\alpha_i) \}$
(3) If $\Delta^{Z re} \subseteq N_0^I \cup -N_0^I$, for all $Z \in \mathcal{X}$
then $N(w) \leq l(w)$

Pf. (1) $\Delta^{X re}(w) \to \Delta^{Y re}(w^{-1})$
$\alpha \mapsto -w^{-1}(\alpha)$ is bijective

$(w(-w^{-1}(\alpha)) = -\alpha)$

(2) [$w(\alpha_i) \in N_0^I$ $\therefore \alpha_i \notin \Delta^{X re}(w)$
$w s_i(\alpha_i) \in -N_0^I$ $\therefore \alpha_i \in \Delta^{X re}(w s_i)$
$\therefore N(w s_i) = N(w) + 1$

$\Delta^{X re}(w) = \{ \alpha \in \Delta_+^{X re} \mid w^{-1}(\alpha) \in -N_0^I \}$
$w(-w^{-1}(\alpha)) = -\alpha \in -N_0^I$
well-defined

Note that $\Delta^{X re}(w s_i) = \{ \alpha \in \Delta_+^{X re} \mid s_i^Y(w^{-1}(\alpha)) \in -N_0^I \}$
Hence (a) and (b) follow from Lemma 9.1.19 (1)

(3) $\because N(id_X) = 0$
assume $l-1$ hold
$w = id_X s_{i_1} \cdots s_{i_l}$ reduced
$N(id_X s_{i_1} \cdots s_{i_{l-1}}) \leq l(id_X s_{i_1} \cdots s_{i_{l-1}})$
by (2) $N(id_X s_{i_1} \cdots s_{i_l}) \leq N(id_X s_{i_1} \cdots s_{i_{l-1}}) + 1$
$\leq l(id_X s_{i_1} \cdots s_{i_{l-1}}) + 1 = l(id_X s_{i_1} \cdots s_{i_l})$

characteristic property

Thm 9.1.22 Assume that (CG3) holds
((CG3): $\Delta^{X re}$ consists ... positive and negative)
then for any $X \in \mathcal{X}$ and
any finite subset R of $\Delta^{X re}$,
the following are equivalent
(1) There exists $w \in Hom(\mathcal{W}(G), X)$,
s.t. $R = \Delta^{X re}(w)$
(2) For any $k, l \geq 0$, and any $\beta_1, \ldots, \beta_k \in \Delta_+^{X re} \setminus R$,
and $\gamma_1, \ldots, \gamma_l \in R$, $\sum_{i=1}^k \beta_i - \sum_{j=1}^l \gamma_j \in \mathbb{Z}^I \setminus R$ 不在R?

Pf: ($\Rightarrow$) assume (1)
and let $\beta_1, \ldots, \beta_k \in \Delta_+^{X re} \setminus R$,
$\gamma_1, \ldots, \gamma_l \in R$
Then $w^{-1}(\beta_i)$ and $w^{-1}(-\gamma_j)$ are positive (Definition)
for any $1 \leq i \leq k$ and any $1 \leq j \leq l$
Hence $w^{-1}(\beta)$ $(\beta = \sum_{i=1}^k \beta_i - \sum_{j=1}^l \gamma_j)$
is a sum of positive roots
$\beta \notin R$, $\therefore$ (2) holds

($\Leftarrow$) Assume now (2), we prove (1) by induction on $|R|$
For $R = \phi$, $\Delta^{X re}(id_X) = \phi$, $\therefore$ claim holds
Let $X \in \mathcal{X}$, $R \subseteq \Delta^{X re}$, and $m = |R|$
Assume that $m \geq 1$, and that the claim holds
for subsets of real roots with $m-1$ element
$\because R \neq \phi$, and any element of R is a sum of simple roots
(2) with $l = 0$ and $\beta_1, \ldots, \beta_i$ simple
imples that there exists $i_0 \in I$, s.t. $\alpha_{i_0} \in R$ important

If there's no $\alpha_{i_0} \in R$
Then $\forall \alpha_i \in \Delta_+^{X re} \setminus R$
$\therefore \{\alpha_i\}$ generalize $\Delta_+^{X re} \setminus R$
$\therefore R = \phi$ $\therefore$ conflict

Let $Y = r_{i_0}(X)$, and $R' = s_i^X(R \setminus \{\alpha_{i_0}\})$
Then $|R'| = m-1$, and $R' \subseteq \Delta_+^{Y re}$ by (Lem 9.1.19 (1))
By assumption
$\sum_{i=1}^k \beta_i - \sum_{j=1}^l \gamma_j - n\alpha_{i_0} \in \mathbb{Z}^I \setminus R$
for any $k, l, n \geq 0$, $\beta_1, \ldots, \beta_k \in \Delta_+^{X re} \setminus R$
and $\gamma_1, \ldots, \gamma_l \in R \setminus \{\alpha_{i_0}\}$,
Thus $\sum_{i=1}^k s_{i_0}^X(\beta_i) + n\alpha_{i_0} - \sum_{j=1}^l s_i^X(\gamma_j) \in \mathbb{Z}^I \setminus s_i^X(R)$ $m-1$
$w' \in Hom(\mathcal{W}(G), Y)$ with $R' = \Delta^{Y re}(w')$
$\therefore R = \Delta^{X re}(s_i^Y w')$, and the proof is completed.
$\Delta^{X re}(s_i^Y w') = \{ \alpha \in \Delta_+^{X re} \mid w'^{-1}(s_i^Y)^{-1}(\alpha) \in -N_0^I \}$
$\Delta^{Y re}(w') = \{ \alpha \in \Delta_+^{Y re} \mid w'^{-1}(\alpha) \in -N_0^I \}$

Lem 9.1.23 Let I be a non-empty finite set
and $J \subseteq I$. Let $w \in Aut(\mathbb{Z}^I)$,
s.t. $w(\alpha_j) \in -\sum_{k \in J} N_0 \alpha_k$ for any $j \in J$
Assume that $w^{-1}(\alpha_j) \in N_0^I \cup -N_0^I$ $\forall j \in J$.
Then there is permutation σ of J
s.t. $w(\alpha_j) = -\alpha_{\sigma(j)}$ for $\forall j \in J$

Pf: Since $w \in Aut(\mathbb{Z}^I)$, $w(\alpha_j) \in \sum_{k \in J} \mathbb{Z}\alpha_k$
$w^{-1}(\alpha_j)$ with $j \in J$ form a basis of $\sum_{k \in J} \mathbb{Z}\alpha_k$
Let $j \in J$
$\exists (\lambda_k)_{k \in J} \in \mathbb{Z}^I$
s.t. $w^{-1}(\alpha_j) = \sum_{k \in J} \lambda_k \alpha_k$
all $\lambda_k \geq 0$ or all $\lambda_k \leq 0$
$\alpha_j = w w^{-1}(\alpha_j) = \sum_{k \in J} \lambda_k w(\alpha_k)$
$\because w(\alpha_k) \in -N_0^I$, $\forall k \in J$
$\therefore \lambda_k = -1$, $w(\alpha_k) = -\alpha_j$, for some $k \in J$
$\lambda_l = 0$, for $l \in J \setminus \{k\}$
$\therefore$ implies the claim $\square$

$$\begin{pmatrix} & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$