P324 例9.1.26前两段

Recall: $\mathfrak{g}_U$

A – Cartan matrix

W – Weyl group $U \le W$

let $\mathcal{X} = W/U = \{wU \mid w \in W\}$

let $r_i : \mathcal{X} \to \mathcal{X}$ $wU \mapsto s_i wU$

let $A^x = A$ for all $x \in \mathcal{X}$.

给定 U 和 $U' \le W$

$\underline{\mathfrak{g}_U} \cong \underline{\mathfrak{g}_{U'}} \Leftrightarrow \exists$ a permutation β of I s.t $\underline{U'}$ is conjugate $\underline{\beta^*(U)}$ in W

$\Rightarrow$: Assume that $\mathfrak{g}_U = \mathfrak{g}_{U'}$

$\exists$ a permutation $\underline{\beta}$ of I,

s.t $\underline{a_{ij} = a_{\beta(i)\beta(j)}}$

Moreover, $\exists$ bijection γ :

$W/U \to W/U'$ of left cosets

s.t $\gamma(s_{i_1} \cdots s_{i_k} U) = s_{\beta(i_1)} \cdots s_{\beta(i_k)} \gamma(U)$

(9.1.2) for all ≥ 0, $i_1, \dots, i_k \in I$

β^*:

Since W is Coxeter group and the coxeter relations are obtained from the entries of A,

the permutation β induceds a group iso

$\beta^* : W \to W$

$\beta^*(s_i) = s_{\beta(i)}$

Therefore, by (9.1.2) there exists $\boxed{w' \in W}$

s.t $\boxed{\beta^*(w) w' U' = w' U'}$ for all $w \in U$

that is $\underline{U' \text{ is conjugate to } \beta^*(U)}$.

细节: Suppose $U \mapsto w'U'$

Since $w \in U$

By (9.1.2)

$\gamma(\underline{w}U) = \gamma(U) = w'U'$

$\shortparallel$

$\beta^*(w) w' U'$ $\Rightarrow$

$\forall U$

$\Rightarrow w'^{-1} \beta^*(w) w' U' = U'$

$\Rightarrow w'^{-1} \beta^*(U) w' \boxed{\subseteq} U'$

考虑 γ^{-1}:

let $\underline{\gamma^{-1}(U') = w'' U}$

则同上，我们有

$w'^{-1} {\beta^*}^{-1}(U') w'' \subseteq U$

两边同时作用 β^*

$$\Rightarrow \boxed{\beta^*(w''^{-1})\, U'\, \beta^*(w'') \subseteq \beta^*(U)}$$

$$\underline{U'} \xmapsto{r^{-1}} w''U \xmapsto{r} \beta^*(w'')\, r(u)$$

$$\| $$

$$\underline{\beta^*(w'')\, w' U'}$$

同一个陪集

$$\Rightarrow \underline{\beta^*(w'')\, w' = u'} \text{ for some } u' \in U'$$

$$\Rightarrow w' u'^{-1}\, U'\, u' w'^{-1} \subseteq \beta^*(U)$$

$$\|$$

$$w' U' w'^{-1} \subseteq \beta^*(U) \quad \square$$

$\underline{\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, (A))}$ Semi-Cartan graph

§5.2 Reduced sequences and positivity of roots

Def 5.2.1

Let $X \in \mathcal{X}$, $l \geq 0$, $K = (i_1, \dots, i_l) \in I^l$

(1) For all $1 \leq k \leq l$.

let $\boxed{\beta_k^{X,K}} = \mathrm{id}_X\, s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$ $\Delta_+^{X\,re}(w)$

and let $\Lambda^X(K) = \{\beta_k^{X,K} \mid 1 \leq k \leq l\}$

(2) We say that K is $\boxed{X}$-reduced if for any $1 \leq k < l$

$$\underline{\alpha_{i_k} \notin \Lambda^{r_{i_k} \cdots r_{i_1}(X)}(i_{k+1}, \dots, i_l)}$$

l is called the length of K

The def implies immediately the following lemma:

Lemma 5.2.2

Let $X \in \mathcal{X}$, $l \geq 1$, $i_1, \dots, i_l \in I$

(1) $\Lambda^X(i_1, \dots, i_l) = \underline{\{\alpha_{i_1}\}} \cup \underline{s_{i_1}^{r_{i_1}(X)}} \underline{\left(\Lambda^{r_{i_1}(X)}(i_2, \dots, i_l)\right)}$

(2) TFAE:

(a) $(i_1, \cdots, i_l)$ is X-reduced

(b) $\underline{(i_2, \cdots, i_l) \text{ is } r_{i_1}(x)\text{-reduced}}$ $1<k<l$

and $\underline{\alpha_{i_1} \notin \Lambda^{r_{i_1}(x)}(i_2, \cdots, i_l)}$

$k=1$

Lemma 5.2.4

Let $x \in X$, $l \geq 1$, $K = (i_1, \cdots, i_l) \in I^l$

$\underline{\boxed{w} = \mathrm{id}_x s_{i_1} \cdots s_{i_l}}$,

$y = r_{i_l} \cdots r_{i_1}(x)$, $K' = (i_l, \cdots, i_1)$

Then $\underline{\beta^{y,K'}_{k}} = \boxed{-w^{-1}}(\underline{\beta^{x,K}_{l+1-k}})$ for all $1 \leq k \leq l$

Proof:

Let $1 \leq k \leq l$, then

$$w(\beta^{y,K'}_k) = \underbrace{\mathrm{id}_x s_{i_1} \cdots \cancel{s_{i_l}}}_{w} \underbrace{\cancel{s_{i_l}} \cdots s_{i_{l+2-k}}(\alpha_{i_{l+1-k}})}_{\beta^{y,K'}_k}$$

$$= \mathrm{id}_x s_{i_1} \cdots s_{i_{l+1-k}}(\alpha_{i_{l+1-k}})$$

$$= - \underline{id_X s_{i_1} \cdots s_{i_{l-k}} (\alpha_{i_{l+1-k}})}$$

$$= - \underline{\beta^{X,K}_{l+1-k}}$$

Lemma 9.2.5

X, l, K

TFAE:

(1) K is X-reduced

* $\boxed{(2)}$ $\underline{\beta^{X,K}_p \neq -\beta^{X,K}_q}$ for all $\underline{1 \le p < q \le l}$

(3) $(i_l, \dots, i_1)$ is $r_{i_l} \cdots r_{i_1}(X)$-reduced

In particular, if $\underline{\Lambda^X(K) \subseteq \mathbb{N}_0^I}$

then K is X-reduced. (By (2))

Proof: Let $1 \le p < q \le l$.

$$\beta^{X,K}_p = id_X s_{i_1} \cdots s_{i_{p-1}} (\alpha_{i_p}) = \underline{-id_X s_{i_1} \cdots s_{i_{p-1}} \boxed{s_{i_p}} (\alpha_{i_p})}$$

$$\beta^{X,K}_q = id_X s_{i_1} \cdots s_{i_p} s_{i_{p+1}} \cdots s_{i_{q-1}} (\alpha_{i_q})$$

Hence $\underline{\beta_p^{x,K} = -\beta_q^{x,K}} \Leftrightarrow \alpha_{i_p} = \boxed{s_{i_{p+1}} \cdots s_{i_{q-1}}(\alpha_{i_q})}$

$\in \Lambda^{r_{i_p} \cdots r_1(x)}(i_{p+1}, \dots, i_q)$

$\Rightarrow (1) \Leftrightarrow (2)$

$(2) \Leftrightarrow (3)$ is obvious

Remark 9.2.6

(1) Let $x \in X$, $l \geq 0$, $K = (i_1, \dots, i_l)$

If $i_j = i_{j+1}$, for some $1 \leq j < l$

then $\underline{\alpha_{i_j}} \in \Lambda^{r_{i_j} \cdots r_{i_1}(x)}(\underline{i_{j+1}}, \dots, i_l)$

$\alpha_{i_{j+1}} = \alpha_{i_j}$

hence K is not reduced

(2) - - ....

Then $\underbrace{\beta_j^{X,K} = \beta_{j+1}^{X,K}}$ $\iff$ $\underset{Y = r_{i_{j-1}} \cdots r_{i_1}(X)}{\underline{s_{i_j}^{Y}(a_{i_j}) = \boxed{\alpha_{i_{j+1}}}}}$

This is impossible, since $s_{i_j}^{Y}(a_{i_j}) = \boxed{-a_{i_j} \neq \alpha_{i_{j+1}}}$

$$\beta_j^{X,K} = -\beta_{j+1}^{X,K} \iff i_r = i_{j+1}$$

Example 4.2.3

Let $I = \{1, 2\}$, $X = \{x_1, x_2, x_3\}$

$\underset{(12)}{r_1}, \underset{(23)}{r_2} : X \to X$

Let $A^{x_1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$, $A^{x_2} \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$

$A^{x_3} = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$

Figure 9.2.1

Direct calculation:

$$\Delta^{X_1, re} = \{ \cdots \}$$

$$\Delta^{X_2, re} = \{ \cdots \}$$

$$\Delta^{X_3, re} = \{ \cdots \}$$

In particular, $\mathcal{G}$ is finite but it doesn't satisfies (CG3) since $\underline{\alpha_1 - \alpha_2} \in \Delta^{X_3, re}$

Thus $\mathcal{G}$ is not a Cartan graph.

~~By direct~~

$$\Lambda^{X_1} \underline{(1,2,1,2,1,2,1,2,1)}$$

$$= \underline{\{ \cdots \}} \neq \Delta_+^{X_1, re}$$

$\alpha_2, \alpha_1+\alpha_2 \subsetneq F$

but $\sigma(\alpha_1) = \boxed{-\alpha_2}$ $\sigma(\alpha_2) = \boxed{-\alpha_1-\alpha_2}$

for $\sigma = id_{X_1}, s_1s_2\, s_1s_2\, s_1s_2\, s_1s_2s_1$

Hence $\underbrace{(1,2,1,2,1,2,1,2,1,i)}$ is not

X-reduced for any $i \in \{1,2\}$ by Lem 9.2.5 ✓

Further, one can show that there is no

X_1-reduced sequence of length 10

$(1,2,1,2,1,2,1,2,1,2)$ ✗

$\wedge^{X_1}$ $(2,1,2,1,2,1,2,1,2,1)$ ✗

One can also check that the longest

X_2-reduced sequence starting with 2

has length $\underline{8}$

$$\left(\begin{array}{c}(2)\\(2,1)\\(2,1,2)\\(2,1,2,1)\\\vdots\end{array}\right)$$

$$\Lambda^{X_3}(1,2,1,2,1,2,1,2,1) = \{\cdots\}$$

Note that there's a root with $\notin N_3^I \cup -N_3^I$

but the sequence is $\underline{X_3\text{-reduced}}$.

Thm 9.3.5: Cartan graph:
a $\underline{\text{sequence } (i_1, \dots, i_l)}$ is X-reduced
$\Leftrightarrow$ it's a reduced ~~sequence~~ decomposition
of $\underline{\mathrm{id}_X s_{i_1} \cdots s_{i_l}}$