Thm. 9.4.7 Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a Cartan graph, $X \in \mathcal{X}$, $k \in \mathbb{N}$, and $i_1, \dots, i_k, i \in I$. Assume that

$$\mathrm{id}_X s_{i_1} \cdots s_{i_k}(\alpha_i) \in -\mathbb{N}_0^I .$$

Then there exist labels $j_1, j_2, \dots, j_k \in I$ such that $j_k = i$

$$\mathrm{id}_X s_{i_1} \cdots s_{i_k} = \mathrm{id}_X s_{j_1} \cdots s_{j_k} \quad \text{in } \mathrm{Cox}(\mathcal{G}).$$

Prop. 9.2.16 Assume that the semi-Cartan graph $\mathcal{G}$ satisfies (CG3') and (CG4'). Let $X \in \mathcal{X}$, $l \geq 1$, $\underline{k = (i_1, \dots, i_l) \in I^l}$ and $i \in I$ such that k is $\underline{X\text{-reduced}}$ and $\mathrm{id}_X s_{i_1} \cdots s_{i_l}(\alpha_i) \notin \mathbb{N}_0^I$. Then there exists an X-reduced sequence $(\underline{j}_1, \dots, j_l) \in I^l$, such that $\underline{j_l = i}$ and $\mathrm{id}_X s_{i_1} \cdots s_{i_l} = \mathrm{id}_X s_{j_1} \cdots s_{j_l}$.

小引理 $l = 1$

$< l$

$$k - p = \underline{m^r_{i i_l} = \overline{m}^r_{i i_l}}$$

proof. (1) Assume $(i_1, \dots, i_k)$ X-reduced

$\underline{i} \neq i_k$.

Choose $((j_1,\dots,j_k),p)$ s.t.

$id_x s_{i_1}\cdots s_{i_k} = id_x s_{j_1}\cdots s_{j_k}$ in $Wx(\mathcal{G})$

$0\le p<k$ $\quad j_n \in \{i, i_k\}$ $\quad p<n\le k$

$j_1 \cdots j_p \quad j_{p+1} \cdots j_k$

Choose p to be the least.

$u = id_x s_{j_1}\cdots s_{j_p}$

$\Lambda^x(j_1,\dots,j_k) \subseteq \mathbb{N}_0^I$

$(j_1,\dots,j_p)$ X-reduced

Let $j\in\{i,i_k\}$ and assume

$id_x s_{j_1}\cdots s_{j_p}(\alpha_j) \notin \mathbb{N}_0^I$ $\quad$ R 记

Then $p\ge 1$

By induction hypothesis there exist

$k_1',\dots,k_p'\in I$ such

$(k_1',\dots,k_p')$ is X-reduced

$\underline{k_p' = j}$

$id_X\, s_{j_1} \cdots s_{j_p} = id_X\, s_{k_1'} \cdots s_{k_p'}$

Let $K' = (k_1', \ldots, k_p', j_{p+1}, \ldots, j_k)$

Then $\underline{\Lambda^X(K')} = \underline{\Lambda^X(k_1', \ldots, k_p')}$

$\cup \{ \underline{\beta_n^{X,}(j_1, \ldots, j_k)} \mid p+1 < n \leq k \}$

$\subset N_0^?$ $\qquad \subseteq N_0^?$

and hence K' is X-reduced.

Thus $(K', \underline{\underline{p-1}}) \in M$

So $id_X\, s_{j_1} \cdots s_{j_p} (\alpha_j) \in N_0^?$

$\forall\ j \in \{ i, i_k \}$

$(*)$ $id_X\, s_{j_1} \cdots s_{j_p} (a\alpha_i + b\alpha_j) \in N_0^?$

Let $Y = r_{i_p} \cdots r_{i_1} (X)$

Then $(j_{p+1}, \ldots, j_k)$ is Y-reduced

$id_Y\, s_{j_{p+1}} \cdots s_{j_k} (\alpha_i) \in \mathbb{Z}\alpha_i + \mathbb{Z}\alpha_{i_k}|_{...}$

because of (*) and

$$N_0 \overset{?}{\ni} id \times s_{i_1} \cdots s_{i_k}(\alpha_i) = id \times s_{j_1} \cdots s_{j_k}(\alpha_i)$$

$$= id \times s_{j_1} \cdots s_{j_p}(id_Y\, s_{j_{p+1}} \cdots s_{j_k}(\alpha_i))$$

Thus $(j_{p+1}, \cdots j_k, i)$ is not Y-reduced by (CG3').

then $l - p = \overline{m}^Y_{i,i_k} = m^Y_{i,i_k} = m$

$$\cancel{id (s_i s_{i_k})^m = id}$$

$$s_{j_{p+1}} \cdots s_{j_k} = s_{j_{p+2}} \cdots s_{j_k} s_{j_{p+1}}$$

$k - l(id \cdot s_{i_k})$

$s_{i_1} \cdots s_{i_k} = s_{j_1} \cdots s_{j_k}$ $\quad j_k = i$

$N^{\#}_0$

Thm. 9.4.8. Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, B)$ be a CG. Then the functor

$$W : \mathrm{Cox}(\mathcal{G}) \to \mathcal{W}(\mathcal{G})$$
$$X \longmapsto X$$
$$s_i^X \longmapsto s_i^X$$

is an equivalence of categories.

Proof. By Cor. 9.2.22.

$$\underline{id_X (s_i s_j)^{m_{ij}^X} = id_X} \text{ in } \mathcal{W}(\mathcal{G})$$

Hence W is a <u>well defined</u>.

Next prove

$$l(W(w)) = l(w) \quad \forall\ w \text{ in } \mathrm{Cox}(\mathcal{G})$$

Let $X \in \mathcal{X}$, $l \geq 0$ $i_1, \dots, i_l \in I$

$$\underline{w = id_X s_{i_1} \cdots s_{i_l}} \text{ in } \mathrm{Cox}(\mathcal{G})$$

$\underline{l(w) = l}$. Then $l(W(w)) \leq l$.

$\forall\ 2 \le n \le l$, there is no

$\underline{(j_1, \ldots, j_{n-1}) \in I^{n-1}}$ s.t.

$$id_x s_{i_1} \cdots s_{i_{n-1}} = id_x s_{j_1} \cdots s_{j_{n-1}}$$

in $Cox(G)$ $\quad \underline{j_{n-1} = i_n}$

$$\left[\ w = \underbrace{id_x s_{i_1} \cdots s_{i_{n-1}}}_{id_x s_{j_1} \cdots s_{j_{n-1}}} \cancel{s_{i_n}} \cdots s_{i_l}\ \right]$$

Therefore $\Lambda^x(i_1, \ldots, i_l)$ consists of

positive roots by Thm 9.4.7.

$\underline{l(W(w))} = N(W(w)) = |\Lambda^x(i_1, \ldots, i_l)|$

$= l$

By defin of the morphism,

W is surjective on the set of

morphism

To prove injectivity.

$X, Y \in \mathcal{G}$ $v, w : X \to Y$ be morphism in $\mathrm{Cox}(\mathcal{G})$, $\underline{W(v) = W(w)}$

Then $W(w^{-1}v) = id_X$

$$\underline{0} = l(id_X) = l(W(w^{-1}v)) = \underline{l(w^{-1}v)}$$

$w^{-1}v = id$ $w = v$ #

Def. 9.4.9. $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$: SCG

$\phi \neq J \subset I$. Then quadruple

$$\mathcal{G}|_J = \mathcal{G}(J, \mathcal{X}, r|_{J \times \mathcal{X}}, A|_{(J \times J \times \mathcal{X})})$$

is called the restriction to J.

$\mathcal{G}$. . $I \times I$

$J \times J$

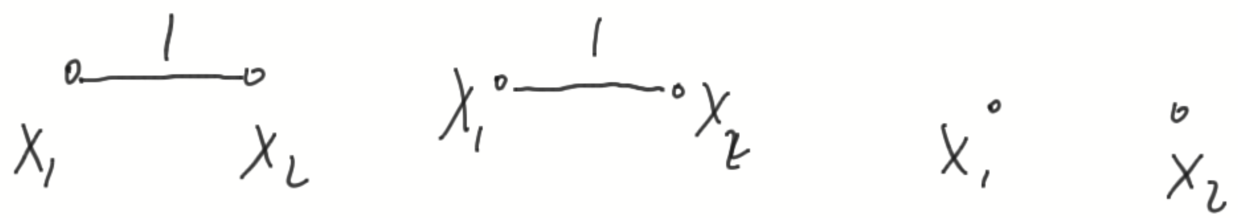
Lem. 9.4.10. Any restriction of a SCG (CG) is a SCG. (CG)

$r_{i}(x)$

proof. $\underline{r_i^2 = \mathrm{id}}$ $\quad a_{ij}^{X} = a_{ij}^{r_i(X)}$

Seq. of labels are X-reduced in the restriction $\Leftrightarrow$ they are X-red. in the SCG.

Rem. 9.4.11. A restrict of a connected SCG is not necessarily connected.



$\mathcal{G}$ $\qquad\qquad$ $\mathcal{G}|_{\{1\}}$ $\qquad\qquad$ $\mathcal{G}|_{\{2\}}$

$$\underline{(r_j r_i)^m (X) = X} \qquad \mathcal{G}|_J$$

$$\underline{\mathrm{id}_X (s_j s_i)^m (\alpha_k) = \alpha_k}$$

$\forall\, k \in I \setminus \underline{\{i,j\}} \qquad m = m_{ij}^X$.

Cor. 9.4.12. $\mathcal{G}$: CG. $\emptyset \neq J \subset I$

Then $\exists!$ faithful functor

$$
\begin{aligned}
W(\mathcal{G}|J) &\longrightarrow W(\mathcal{G}) \\
X &\longmapsto X \qquad \forall X \in \mathcal{G} \\
s_j^X &\longmapsto s_j^X \qquad j \in J
\end{aligned}
$$

proof. $\overline{m}_{ij}^X = m_{ij}^X \quad \forall\ i, j \in J$

$\overline{m}_{ij}^X$ is the same in $\mathcal{G}$ and $\mathcal{G}|J$.

<u>Thus</u>, Thm 9.4.8 $\Rightarrow$ $\exists!$

$$
\begin{aligned}
T_J : W(\mathcal{G}|J) &\longrightarrow W(\mathcal{G}) \\
X &\longmapsto X \\
s_i^Y &\longmapsto s_i^Y
\end{aligned}
$$

$$
\begin{tikzcd}
W(\mathcal{G}|J) \arrow[r, dashed, "T_J"] \arrow[d] & W(\mathcal{G}) \\
\mathrm{Cox}(\mathcal{G}|J) \arrow[r] & \mathrm{Cox}(\mathcal{G}) \arrow[u]
\end{tikzcd}
$$

$X, Y \in \mathcal{G}$, $\quad w, w' \in \mathrm{Hom}_{W(\mathcal{G}|J)}(X, Y)$

$Tr_J(w) = Tr_J(w')$ Then

$F(w) = F(w')$ $\mathbb{Z}^I$

$\Rightarrow w = w'$ by Cor. 9.3.8(3).

#

Def. 9.4.13. $\mathcal{G} = \mathcal{G}(I, X, r, \mathcal{A})$: SCG

$\phi \neq J \subset I$. let $W_J(\mathcal{G})$ be the subcategory of $W(\mathcal{G})$

objs $= X$.

morphism $s_{i_1} \cdots s_{i_k}^{X} \in \mathrm{Hom}_{W(\mathcal{G})}(X, Y)$

$i_1, \cdots, i_k \in J$ s.t. $r_{i_1} \cdots r_{i_k}(X) = Y$

Then $W_J(\mathcal{G})$ is a groupoid. and is call a <u>parabolic subgroupoid</u> of $W(\mathcal{G})$

prop. 9.4.14. Let $\mathcal{G}$ be a CG.

$\phi \neq J \subset I$. Then there exist an equivalence of cat.

$$\mathcal{W}(\mathcal{G}|J) \longrightarrow \mathcal{W}_J(\mathcal{G})$$
$$X \longmapsto X$$
$$s_j^x \longmapsto s_j^x \quad j \in J$$
$$x \in X.$$

Proof. F_J is faithful and has its image in $\mathcal{W}_J(\mathcal{G})$.

$$s_{j_1} \cdots s_{j_n} \quad j_1 \ldots j_n \in J$$

It is full by def of $\mathcal{W}_J(\mathcal{G})$.

#

Prop. 9.4.15. $\mathcal{G} = \mathcal{G}(I, X, r, H)$: CG $J \subset I$. Then any reduced decomp. of a morphism in $\mathcal{W}_J(\mathcal{G})$ is in $\mathcal{W}_J(\mathcal{G})$. In particular, any $w \in W_J(\mathcal{G})$. can be written as a product of $l(w)$ simple reflections s_j, $j \in J$.

proof. $X \in \mathcal{X}$. $w = \mathrm{id}_X s_{i_1} \cdots s_{i_l}$ is a reduced decomposition of w.

Assume to the contrary that $\{i_1, \dots, i_l\} \not\subset J$. Let $1 \le k \le l$ be minimal, with $i_k \notin J$.

Then $\qquad i_1, \dots, i_{k-1} \in J$

$$\alpha = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \in \Delta^{X re}(w)$$

$$\| \qquad\qquad\qquad \| \ 9.3.5$$

$$\beta_k^{X,(i_1,\dots,i_k)} \in \Lambda^X(i_1,\dots,i_k)$$

$$w^{-1}(\alpha) \in -\mathbb{N}_0 I \qquad \alpha \in \Delta_+^{X re}$$

Moreover

$$\underline{\alpha \in \alpha_{i_k} + \underline{\sum_{j \in J}} \mathbb{N}_0 \alpha_j}$$

$$\alpha_{i_k} + \sum \mathbb{Z} \alpha_j$$

$$w^{-1}(\alpha) \in \alpha + \sum_{j \in J} \mathbb{Z} \alpha_j \qquad i_k \notin J$$

$$= \underline{\alpha_{i_k}} + \sum_{j \in J} \mathbb{Z} \alpha_j$$

$$\not\subset -\mathbb{N}_0^I$$

$$\Rightarrow w^{-1}(\alpha) \in \mathbb{N}_0^I \qquad \cancel{\times}$$

<u>Cor. 9.4.16</u> $\mathcal{G}(I, \mathcal{X}, r, A)$: CG

$J \subset I$, $X \in \mathcal{X}$, $w \in \mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$

If $w(\alpha_j) \in \mathbb{N}_0^I$ $\forall j \in J$.

then $l(wv) = l(w) + l(v)$ $\forall v \in \mathcal{W}_J(\mathcal{G})$

proof. Let $v = s_{i_1} \cdots s_{i_l}$, $l = l(v)$.

Then $i_1, \cdots, i_l \in J$ by Prop 9.4.15.

Since $\underline{w(\alpha_j) \in \mathbb{N}_0^I}$ $\forall j \in J$

$\underline{w s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \in \mathbb{N}_0^I}$ $\underline{1 \le k \le l}$

Thus $l(wv) = l(w) + l$ by Cor 9.3.6(1)

$\left[w(\alpha_i) \in \mathbb{N}_0^I \iff l(ws_i) = l(w) + 1 \right]$

$l(ws_{i_1}) = l(w) + 1$

#

<u>Cor. 9.4.17.</u> (Kostant's decomp.)

Let $G = G(I, X, r, A)$ be a CG.

$X \in \mathcal{X}$, $w \in \mathrm{Hom}(W(G), X)$, $J \subset I$

Then $\exists!$ determined $Y \in \mathcal{X}$ $u \in \mathrm{Hom}(Y, X)$

$v \in \mathrm{Hom}(W(G), Y)$ such that $w = uv$

$l(w) = l(u) + l(v)$ $v \in W_J(G)$

$u(\alpha_j) \in \mathbb{N}_0^I$ $\forall j \in J$ Moreover

$w = u'v'$ with $l(w) = l(u') + l(v')$

$v' \in W_J(\mathcal{G}) \Rightarrow l(u) \le l(u')$

proof. (existence)

Let M denote the set of all pairs (u', v') of morphisms in $W(\mathcal{G})$

s.t. $w = u'v'$, $l(w) = l(u') + l(v')$

$v' \in W_J(\mathcal{G})$. $(w, id) \in M$.

Let $(u, v) \in M$ be such that

$l(u) \le l(u')$ for all $(u', v') \in M$

Then $\underline{u(\alpha_j) \in \mathbb{N}_0^I}$ $\forall j \in J$.

Indeed. assume $u(\alpha_j) \underline{\in -\mathbb{N}_0^I}$ for some $j \in J$. Then

$w = (u s_j)(s_j v)$ $\quad \underline{\underline{l(u s_j) = l(u) - 1}}$

by Cor. 9.3.6.(2)

Then $(\underline{u s_j}, s_j v) \in M$. $\qquad \frac{\quad}{(u,v)}$ ~~x~~

(uniqueness) Let $(u_1, v_1) \in M$

$u_1(\alpha_j) \in \mathbb{N}_0 1 \quad \forall j \in J$.

$l(u) \le l(u') \quad l(u') \le l(u)$

$\Rightarrow \underline{l(u) = l(u')}$

$\underline{u = w v^{-1}} = \underline{u_1 (v_1 v^{-1})} \quad (\underline{w = uv})$

$l(u) = l(u_1(v_1 v^{-1})) = l(u_1) + \underbrace{l(v_1 v^{-1})}_{=0}$

by Cor. 9.4.16

$\Rightarrow v = v_1 \quad u = u_1$

<u>Prop. 9.4.18</u> Let $x \in \mathcal{X}$, $\phi \neq J \subset I$

Assume (CG3) in the connected component of x. If $\alpha \in \underline{\Delta}^{x\,re} \cap \sum_{j \in J} \mathbb{N}_0 \alpha_j$

Then $\exists\, k \in \mathbb{N}_0 \quad j \quad j \quad l \in J$

$$\alpha = \mathrm{id}_X\, s_{i_1} \cdots s_{i_k} (\alpha_l)$$

proof. It enough to prove the following, $\mathcal{Y}$ is the connected component of X.

(*) $Y \in \mathcal{Y}$, $w \in \mathrm{Hom}(W(\mathfrak{g}), Y)$,

$\alpha \in \Delta^{Y re}(\underline{\underline{w}}) \cap \sum_{j \in J} \mathbb{N}_0 \alpha_j$

Then $\exists$ $k \in \mathbb{N}_0$, $i_1 \cdots i_k$, $l \in J$ s.t.

$$\alpha = \mathrm{id}_Y\, s_{i_1} \cdots s_{i_k} (\alpha_l)$$

Indeed if $\alpha \in \Delta^{X re} \cap \sum_{j \in J} \mathbb{N}_0 \alpha_j$

then there $w \in \mathrm{Hom}(W(\mathfrak{g}), X)$

$w(\alpha_i) = \alpha$

hence $\underline{\alpha \in \Delta^{X re}(\underline{\underline{w}}\, s_i)}$

By assumption in (*), $\underline{w^{-1}(\alpha) \in -\mathbb{N}_0^I}$

$\alpha \in \sum_{j \in J} \mathbb{N}_0 \alpha_j$. Hence $\exists\, n \in J$

$\underline{w^{-1}(\alpha_n) \in -\mathbb{N}_0^I}$ $\alpha_n \in \Delta^{\Psi re}(w)$

[Remind, $w^{-1}(\alpha_j) \in \mathbb{N}_0^I$]

If $\alpha = \alpha_n$, then the claim (*) is obvious. $k = 0$, $l = n$

We prove (*) by induction on $N(w) \geq 1$.

Assume $N(w) = 1$, done.

Assume $\alpha \neq \alpha_n$. Then we know from Lem 9.1.19(1) that

$$s_n^{\Psi}(\alpha) \in \Delta_+^{s_n(\Psi) re}$$

Since $(S_n w)^{-1}(S_n^{\gamma}(\alpha)) = w^{-1}(\alpha) \in -\mathbb{N}_0^{I}$

$$S_n^{\gamma}(\alpha) \in \Delta^{r_n(\gamma)\,re}(S_n w)$$
$$\cap \sum_{j \in J} \mathbb{N}_0 \alpha_j$$

On the other hand, $w^{-1}(\alpha_n) \in -\mathbb{N}_0^{I}$

T2. Thus. $\underline{N(S_n w)} = N(w^{-1} S_n)$
$$= \underline{N(w) - 1}$$

by Lem 9.1.21.

By induction hypothesis $\exists\, k \geq 1$

$\underline{i_1, \ldots, i_k,\ l \in J}$. s.t.

$$S_n^{\gamma}(\alpha) = id_{r_n(\gamma)} S_{i_1} \cdots S_{i_k}(\alpha_l)$$

$$\alpha = id_{\gamma} S_n S_{i_1} \cdots S_{i_k}(\alpha_l)$$

#

Cor. 9.1.[illegible] ...

Cor. 1.4.17. $\mathcal{G}$: CG. I label.

$\emptyset \neq J \subset I$. Then $\forall X \in \mathcal{G}$,

$\Delta^{X re} \cap \sum_{j \in J} \mathbb{Z} \alpha_j$ is the set

of real roots of $\mathcal{G}|_J$ at X.

proof. Let $X \in \mathcal{G}$. A real root

of $\mathcal{G}|_J$ at X is of the form

$\underline{id_X s_{i_1} \cdots s_{i_k} (\alpha_j)}$, $i_1, \dots, i_k \in J$

Since the entries of the GCM

come from the entries of the

GCM of $\mathcal{G}$, these roots are

in $\underline{\Delta}^{X re} \cap \sum_{j \in J} \mathbb{Z} \alpha_j$

Conversely, any root in ~~the set~~

$\Delta^{X re} \cap \sum_{j \in J} \mathbb{Z} \alpha_j$ is of the form

$\underline{id \times s_{i_2} \cdots s_{i_k}(\alpha_l)}$ by prop 9.4.18

#