Def. 10.1.13 Let $G=G(I, X, r, A)$, $G'=G(J, Y, t, B)$ be Semi Cartan graphs. Assume that $I \cap J = \phi$. The product SCG $G \times G'$ is the quadruple

$$G(I \cup J,\ X \times Y,\ q = r \times t,\ C)$$

where $q_i(x,y) = (r_i(x), y)$, $q_j(x,y) = (x, t_j(y))$.

for all $i \in I$, $j \in J$ $x \in X$, $y \in Y$. and

$$\underline{c_{kl}^{(x,y)}} = \begin{cases} a_{kl}^{x} & \text{if } k,l \in I \\ b_{kl}^{y} & \text{if } k,l \in J \\ 0 & \text{otherwise} \end{cases}$$

$$C^{(x,y)} = \begin{bmatrix} A^x & 0 \\ 0 & B^y \end{bmatrix}$$

Lem. 10.1.14. The product of two CG is a CG.

proof. Let $G_1 = G(I_1, X_1, r, A)$, $G_2 = G(I_2, X_2, t, B)$ be CG. $I_1 \cap I_2 = \phi$. we prove (CG3)(CG4) for $G_1 \times G_2$. $I = I_1 \cup I_2$.

$\Delta^{(x,y)re} \subset \mathbb{Z}^I$ $\quad \Delta^{xre} \subset \mathbb{Z}^{I_1}$, $x \in X_1$ $\quad \Delta^{yre} \subset \mathbb{Z}^{I_2}$, $y \in X_2$

Regard $\mathbb{Z}^{I_1}$, $\mathbb{Z}^{I_2}$ as subgroups of $\mathbb{Z}^I$ via the identification of $\alpha_i \in \mathbb{Z}^{I_k}$ with $\alpha_i \in \mathbb{Z}^I$ $\underline{k \in \{1,2\}}$

$\forall\ x_1 \in X_1$, $x_2 \in X_2$ $j \in I_k$, $j' \in I \setminus I_k$ $\underline{\alpha \in \mathbb{Z}^{I_k}}$

the def of $G_1 * G_1 \Rightarrow$ $\underline{s_j^{(x_1,x_2)}(\alpha) = s_j^{x_k}(\alpha)}$

$\underline{c_{\cdots}^{(x_1,x_2)}(\alpha) = \alpha}$

$$\left[j \in I_1,\ j' \in I_1,\ \alpha_i \in \mathbb{Z}^{I_1} \quad (\alpha_i, \alpha_i) \right.$$

$$s_j^{(x_1,x_2)}(\alpha_i) = \alpha_i - c_{ij}^{(x_1,x_2)} \alpha_j = \alpha_i - a_{ij}^{x_1} \alpha_j$$

$$= s_j^{x_1}(\alpha_i) \quad \left. \right]$$

$\alpha_i \in \mathbb{Z}^I \qquad \Delta^{(x_1,x_2)re} = \Delta^{x_1 re} \cup \Delta^{x_2 re} \subseteq \mathbb{N}_0^I \cup -\mathbb{N}_0^I$

(CG3) is proved!

(CG4) $\quad (r_i r_j)^{m_{ij}^x}(x) = x.$

$$\left((r \times t)_k (r \times t)_l\right)^{m_{kl}^{(x,y)}}(x,y) = \left((r_k r_l)^{m_{kl}^x}(x), y\right) = (x,y)$$

$m_{kl}^{(x,y)} = m_{kl}^x$ for all $k, l \in I_1$

$$\left| \Delta^{(x,y)re} \cap \mathbb{N}_0 \alpha_k + \mathbb{N}_0 \alpha_l \right| = \left| \Delta^{xre} \cap \mathbb{N}_0 \alpha_k + \mathbb{N}_0 \alpha_l \right| = m_{kl}^x$$

If $(k,l) \in I_1 \times I_2$, then $m_{kl}^{(x,y)} = 2 \quad x \in X_1, y \in X_2$

$$m_{kl}^{(x,y)} = \left| \Delta^{(x,y)re} \cap \mathbb{N}_0 \alpha_k + \mathbb{N}_0 \alpha_l \right| = \left| \Delta^{xre} \cap \mathbb{N}_0 \alpha_k \cup \Delta^{yre} \cap \mathbb{N}_0 \alpha_l \right|$$

$$= 2$$

$$\left((r \times t)_k (r \times t)_l\right)^2 = (r_k \times t_l)^2 = (r_k^2 \times t_l^2) = id$$

($r_k^2 = id$, $t_l^2 = id$)

Thus $G_1 \times G_2$ is a CG. #

Def. 10.1.15 A matrix $A = (a_{ij})_{i,j \in I} \in R^{I \times I}$, R ring is called decomposable, if $\exists\ I_1, I_2 \subset I$ $I_1 \cap I_2 = \emptyset$

$I_1 \cup I_2 = I$, $\quad a_{ij} = a_{ji} = 0 \quad \forall\ i \in I_1, j \in I_2$.

A SCG $G = G(I, X, r, A)$ is called if $\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$

A^x is decomposable for all $x \in X$.

- Indecomposable.

Remark 10.1.16 Let $A = (a_{ij})_{i,j\in I} \in R^{I\times I}$, R domain

Assume $\forall\ i,j\in I \quad a_{ij}=0 \Rightarrow a_{ji}=0$. Then.

(1) For all $i,j\in I$ def. $i\sim j$, if $i=j$ or there are $k>0$ $i_1,\dots,i_k\in I$ with $a_{ii_1}a_{i_1i_2}\cdots a_{i_kj}\neq 0$

Then $\sim$ is an equivalance relation on I.

$i\sim i$

$i\sim j \quad i\neq j \Rightarrow a_{ii_1}\cdots a_{i_kj}\neq 0 \Rightarrow a_{ji_k}a_{i_{k-1}i_{k-2}}\cdots a_{i_1i}\neq 0$

$i\sim j \quad j\sim k \Rightarrow i\sim k$ $\qquad\qquad \Downarrow$

$j\sim i$

(2) Let $I=\bigcup_{1\le l\le m} I_l$ be the decomposition of I into pairwise disjoint equivalance class by $\sim$. as in (1)

Then $\underline{(a_{ij})_{i,j\in I_l}}$ $1\le l\le m$ are indecomposable, and

$a_{ij}=0=a_{ji}$ $\forall\ 1\le k\neq l\le m \quad i\in I_k,\ j\in I_l$

[If $(a_{ij})_{i,j\in I_l}$ is decomposable, then $I_l=I'\cup I''$

$a_{ij}=a_{ji}=0 \quad \forall\ i\in I'',\ j\in I'$

$\underline{a_{ii_1}\cdots a_{i_kj}=0} \quad \forall\ i_1,\dots,i_k\in \underline{I_l}$

Contradict to $i\sim j$! $\Rightarrow (a_{ij})_{i,j\in I_l}$ is indecomp.

To prove $a_{ij}=0=a_{ji}$ $\forall\ i\in I_k,\ j\in I_l$

if $\exists\ i\in I_k,\ j\in I_l$ s.t. $a_{ij}\neq 0$ $a_{ij}a_{ji}a_{ij}\neq 0$

$\Rightarrow i\sim j$ ——×

Suppose $I=\bigcup_{1\le q\le r} J_q$ is the union of pairwise disjoint subsets, $J_q\subset I$, $1\le q\le r$. s.t. $(a_{ij})_{i,j\in J_q}$ are indecom. and that $i\in J_p$, $j\in J_q$ $1\le p<q\le r$ implies $a_{ij}=0=a_{ji}$ Then $r=m$ and there is

a permutation $w \in S_m$ $J_l = I_{w(l)}$

[if $J_2 \cap I_1 \neq \phi$ $J_2 \not\subset I_1$ $J_2 = (J_2 \cap I_1) \cup (J_2 \cap (I \setminus I_1))$

$(a_{ij})_{i,j \in J_2}$ is decomposable. $J_2 \subset I_1$

$\bigcup J_a \subsetneq \bigcup I_k$

prop. 1.0, 1.19 Let $G = G(I, X, r, A)$ be a connected CG rank at least 2. $X \in \mathcal{X}$. $I = I_1 \cup I_2$ a decomp. $I_1 \cap I_2 = \phi$. Then TFAE.

(1) $\forall\ i \in I_1$ $j \in I_2$ $a_{ij}^X = 0 = a_{ji}^\lambda$

(2) $\forall\ Y \in \mathcal{X}$, $a_{ij}^Y = 0 = a_{ji}^Y$ $\forall\ i \in I_1$, $j \in I_2$

(3) $\Delta^{Xre} = (\Delta^{Xre} \cap \sum_{i \in I_1} \mathbb{Z}\alpha_i) \cup (\Delta^{Xre} \cap \sum_{i \in I_2} \mathbb{Z}\alpha_i)$

(4) $\forall\ Y \in \mathcal{X}$

$\Delta^{Yre} = (\Delta^{Yre} \cap \sum_{i \in I_1} \mathbb{Z}\alpha_i) \cup (\Delta^{Yre} \cap \sum_{i \in I_2} \mathbb{Z}\alpha_i)$

(5) There is a covering $G_1 \times G_2 \longrightarrow G$, where G_l, $l = 1, 2$ is the connected component of G/I_l containing X.

proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (1)

$\Updownarrow$

(5)

(1) $\Rightarrow$ (2) By Lem 9.3.3 ($a_{ij}^X = 0 \Rightarrow a_{jl}^X = a_{jl}^{r_l(X)}$ $\forall\ l \in I$.)

$\forall\ i, l \in I_1$, $j \in I_2$ $0 = a_{jl}^X = a_{jl}^{r_i(X)} \Rightarrow a_{ji}^{r_i(X)} = 0$

By Symmetry, $a_{jl}^{r_i(X)} = 0$ $\forall\ i, l \in I_2$, $j \in I_1$

We proved $\forall\ l \in I_1$, $j \in I_2$ $k \in I_1 \cup I_2 = I$

$a_{jk}^{r_k(X)} = 0 = a_{jk}^{r_l(X)}$

$a_{ij} = a_{jk}$

$\gamma = r_{i_1} \cdots r_{i_k}(x)$

now (2) follows since $\mathcal{G}$ is connected.

(2) $\Rightarrow$ (4) by the def. of Δ^{re}.

[$i \in I_1$, $k \in I_1$, $j \in I_2$ $\quad s_i^r(\alpha_k) = \alpha_k - a_{ik}^r \alpha_i \in \sum_{l \in I_1} \mathbb{Z}\alpha_l$

$s_j^r(\alpha_k) = \alpha_k$]

(4) $\Rightarrow$ (3) trivial.

(3) $\Rightarrow$ (1) by Lem W. 1.9. $\quad -a_{ij} = \max\{m \mid \alpha_j + m\alpha_i \in \Delta^{re}\}$

[if $a_{ij}^r \neq 0$ for some $r \in I_1$, $j \in I_2$

$\Delta^{re} \ni s_i^r(\alpha_j) = \alpha_j - a_{ij}^r \alpha_i \notin \sum_{i \in I_1} \mathbb{Z}\alpha_i$

$\notin \sum_{i \in I_2} \mathbb{Z}\alpha_i$ $\quad \notin \Delta^{re}$]

(5) $\Rightarrow$ (2) because of Def. 1.1.18.

$\mathcal{G}_1 \times \mathcal{G}_2 = \mathcal{G}(I_1 \cup I_2, X_1 \times X_2, r, B)$

$\mathcal{G}_1 \times \mathcal{G} \xrightarrow{\pi} \mathcal{G}$

$0 = b_{ij}^{(\gamma,z)} = a_{ij}^{\pi(\gamma,z)} = a_{ij}^{x}$

π surjective $\Rightarrow$ $\forall x \exists (\gamma, z)$

$\pi(\gamma, z) = x$

(2) $\Rightarrow$ (5)

Let $X_l = \{ r_{i_1} r_{i_2} \cdots r_{i_k}(x) \mid k \geq 0, i_1, \ldots, i_k \in I_l \}$. $l = 1, 2$

By def. $\mathcal{G}_l = \mathcal{G}(I_l, X_l, r|(I_l \times X_l)}, A|(I_l \times I_l \times X_l))$

Def. $\pi : X_1 \times X_2 \to X$ by

$\pi(a(x), b(x)) = ab(x) = ba(x)$

$a = r_{i_1} \cdots r_{i_p}$, $b = r_{j_1} \cdots r_{j_q}$ $i_1 \cdots i_p \in I_1$, $j_1 \cdots j_q \in I_2$ $p, q \geq 0$

~~π~~ is well defined $a(x) = a'(x)$, $b(x) = b'(x)$, $ab(x) = a'b'(x)$

$m_{ij}^x = 2$ $\forall i \in I_1, j \in I_2$ $x \in X$.

by (2) and Lem 9.5.1

[G is $CG \Rightarrow$ $a_{ij}^x = a_{ji}^x = 0 \in \mathbb{Z}_2 \Rightarrow m_{ij}^x = 2$].

$(r_i r_j)^2(y) = y$ by (CG4) for G. $r_i r_j = r_j r_i$

$\Rightarrow \pi(a(x), b(x)) = ba(x)$. π is well-def

π is surjective. Since G is connected.

$\pi(X_1 \times X_2)$ is invariant under all r_i $i \in I$.

[$r_i \pi(a(x), b(x)) = r_i ab(x) = \pi(r_i a(x), b(x)) \in \pi(X_1 \times X_2)$

G is connected $\Rightarrow \forall x \in X$ $\exists y$, $x = r_{i_1} \cdots r_{i_n}(y)$

$= \underbrace{r_{i_1} \cdots r_{i_p}}_{a} \underbrace{r_{j_1} \cdots r_{j_q}}_{b}(y)$

$= \pi(a(y), b(y))$

We prove (id, π) is $G_1 \times G_2 \to G$ is a morphism.

Let $a = r_{i_1} \cdots r_{i_p}$, $b = r_{j_1} \cdots r_{j_q}$, $i_1 \cdots i_p \in I_1$, $j_1 \cdots j_q \in I_2$

Then $a(x) \in X_1$, $b(x) \in X_2$

$\forall i \in I_1$, $j \in I_2$

$\pi r_i(a(z), b(z)) = \pi(r_i a(z), b(z)) = r_i ab(z)$

$= r_i \pi(a(z), b(z))$

$\pi r_j(a(z), b(z)) = \pi(a(z), r_j b(z)) = a r_j b(z)$

$= r_j ab(z)$

$= r_j \pi(a(z), b(z))$

By def. the entries of the GCM of $G_1 \times G_2$ at $(a(x), b(x))$ are

$$C_{kl}^{(a(x),b(x))} = \begin{cases} a_{kl}^{a(x)}, & k,l \in I_1 \\ b_{kl}^{b(x)}, & k,l \in I_2 \\ 0, & \text{otherwise} \end{cases}$$

Since $r_i r_j = r_j r_i$ $\forall i \in I_1, j \in I_2$ it follows by Lem 9.3.3 that.

$$\underline{C_{kl}^{(a(x),b(x))}} = a_{kl}^{\underline{a(b(x))}} = \underline{a_{kl}^{\pi(a(x),b(x))}}$$

[$k,l \in I_1$, $C_{kl}^{(a(x),b(x))} = a_{kl}^{a(x)} = a_{kl}^{r_{i_1}\cdots r_{i_p}(x)}$

$a_{kl}^{a(x)} \overset{?}{=} a_{kl}^{a b(x)} = a_{kl}^{b a(x)}$

$a_{kl}^2 = a_{kl}^{b(z)} = a_{kl}^{r_{j_1}\cdots r_{j_q}(z)} = a_{kl}^{r_{j_2}\cdots r_{j_q}(z)}$

$\cdots = a_{kl}^2$

Thus $(G_1 \times G_2, G, \tau_C)$ is a covering. #

Cor. 10.1.18. Let G be connected CG. Then $\exists!$ decomposition $I = \bigcup_{1 \le l \le m} I_l$ $m \ge 1$ $I_k \cap I_l = \emptyset$ $\forall 1 \le k \ne l \le m$ s.t the following hold.

(1) $\forall x \in X$, $(a_{ij}^x)_{i,j \in I_l}$ $1 \le l \le m$ are indecomposable $a_{ij} = 0$ $\forall i \in I_k, j \in I_l$ $k \ne l$.

(2) $\forall x \in X$

$$\Delta^{re} = \bigcup_{1 \le l \le m} (\Delta^{re} \cap \sum_{i \in I_l} \mathbb{Z} \alpha_i)$$

(3) Let $X \subset X$, $\exists$ covering $G_1 \times G_2 \cdots \times G_m \to G$

G_i is the connected component of G/I_i containing X.

G_i is an indecomposable CG.

$\pi_i(a_1(x), \cdots, a_m(x)) = a_1 a_2 \cdots a_m(x)$