We are going to characterize and classify finite connected Cartan graphs using their characteristic sequences.

<u>Def. 10.3.15</u>. Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a <u>SCG</u> of <u>rank 2</u> and let $X \in \mathcal{X}$ and $i \in I$. The characteristic seq. of $\mathcal{G}$ w.r.t <u>X and i</u> is the <u>infinite seq.</u> $(\underline{C_k^{X,i}})_{k\geq 1}$ of non-negative integers, where

$$C_{2k+1}^{X,i} = -a_{ij}^{(r_j r_i)^k(X)} \stackrel{\cdot}{=} -a_{ij}^{r_i(r_j r_i)^k(X)}$$

$$C_{2k+2}^{X,i} = -a_{ji}^{r_i(r_j r_i)^k(X)} \stackrel{\cdot}{=} -a_{ji}^{(r_j r_i)^{k+1}(X)}$$

for all $\underline{k \geq 0}$ and $j \in I \setminus \{i\}$. $\quad I = \{i, j\}$

<u>Lem. 10.3.16</u> $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ is SCG of rank 2. $X \in \mathcal{X}$, $i, j \in I$ $i \neq j$ (C_k) is the char seq of $\mathcal{G}$ w.r.t X, and i.

(1) The char. seq. of $\mathcal{G}$ w.r.t. $r_i(X)$ and j is $(C_k)_{k\geq 2}$

(2) Suppose that $(r_j r_i)^n(X) = X$ for some $n \geq 1$. Then $C_{2n+k} = C_k$ for all $k \geq 1$ and the char. seq. w.r.t X and j is $\underline{(C_{2n+1-k})_{k\geq 1}}$

proof. Both claims follow from the def.

proof. Both claims follows from the def.

(1) $$C_{2k+1}^{r_i(x),j} = -\underbrace{a_{ji}^{(r_i r_j)^k (r_i(x))}} = -a_{ji}^{r_i(r_j r_i)^k(x)} = C_{2k+2}^{x,i}$$

$$C_{2k+2}^{r_i(x),j} = C_{2k+2}^{x,i}$$

(2) $$C_k^{x,j} = C_{2n+1-k}^{x,i}$$

① $k = 2l+2$

② $k = 2l+1$

$-\alpha_s$

$\eta : \mathbb{Z} \longrightarrow SL_2(\mathbb{Z}) \quad a \longmapsto \begin{bmatrix} a & -1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} -a_{ij} \\ 1 \end{bmatrix}$

Calculate Δ^{xre} using η.

$\alpha_2 - a\alpha_1 = s_1(\alpha_2)$

$\tau \in \mathrm{Aut}(\mathbb{Z}^2) \quad (c_1, c_2) \longmapsto (c_2, c_1)$

Def. 10.3.17 Let $I = \{1,2\}$, Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a SCG of rank 2, $X \in \mathcal{X}$, $i \in I$. Let (c_k) be the char. seq. of $\mathcal{G}$ w.r.t X, i. The root seq. w.r.t X, i is the seq. $(\beta_k)_{k \geq 1}$ of elts of $\mathbb{Z}^2$, where

$$\beta_k = \eta(c_1) \cdots \eta(c_{k-1})(\alpha_1)$$

$$\beta_1 = \alpha_1$$

Remark 10.3.18 $I = \{1,2\}$, $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a SCG of rank 2. $X \in \mathcal{X}$. (β_k) the root seq. w.r.t X and 1 $(\gamma_k)_{k \geq 1}$ is the root seq. w.r.t X and 2. Then

$$\beta_{2k+1} = id_X (s_1 s_2)^k (\alpha_1), \quad \beta_{2k+2} = id_X (s_1 s_2)^k s_1(\alpha_2)$$

$$\tau\gamma_{2k+1} = id_X (s_2 s_1)^k (\alpha_2), \quad \tau\gamma_{2k+2} = id_X (s_2 s_1)^k s_2(\alpha_1)$$

Since $s_1^Y = \eta(-a_{12}^Y)\tau$, $s_2^Y = \tau\eta(-a_{21}^Y)$ $\forall Y \in \mathcal{X}$

Thus $\Delta^{xre} = \{\pm\beta_k, \pm\tau\gamma_k \mid k \geq 1\}$

proof. Induct on k.

$$\beta_1 = \alpha_1, \quad \beta_2 = \eta(C_1^{x,1})(\alpha_1) = \begin{bmatrix} -a_{12}^x & 1 \\ * & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$= \alpha_2 - a_{12}^x \alpha_1$$
$$= \mathrm{id}_x s_1{}^x(\alpha_2)$$

$$\beta_{2k+3} = \eta(C_1)\cdots\eta(C_{2k+1})\underline{\eta(C_{2k+2})(\alpha_1)}$$
$$= \eta(C_1)\cdots\eta(C_{2k+1})\left(\alpha_2 - a_{21}^{r_1(r_2r_1)^k(x)}\alpha_1\right)$$
$$= \eta(C_1)\cdots\underline{\eta(C_{2k+1})(\alpha_2)} - a_{21}\,\underline{\eta(C_1)\cdots\eta(C_{2k+1})(\alpha_1)}$$
$$= \underline{\eta(C_1)\cdots\eta(C_{2k})(-\alpha_1)} - a_{21}\beta_{2k+2}$$
$$= -\mathrm{id}_x(s_1s_2)^k(\alpha_1) - a_{21}\,\mathrm{id}_x(s_1s_2)^k s_1(\alpha_2)$$
$$= \mathrm{id}_x(s_1s_2)^k s_1(\alpha_1 - a_{21}\alpha_2)$$
$$= \mathrm{id}_x(s_1s_2)^k s_1 s_2(\alpha_1) = \mathrm{id}_x(s_1s_2)^{k+1}(\alpha_1)$$
$$\beta_{2k+4} = \mathrm{id}_x(s_1s_2)^{k+1}s_1(\alpha_2)$$

$$s_2 s_1 = \tau\,\eta(-a_{21})\,\eta(-a_{12})\,\tau$$
$$(s_2 s_1)^k(\alpha_2) = \tau\,\underline{\eta(-a_{21})\eta(-a_{12})\cdots\eta(-a_{21})\eta(-a_{12})\overset{\alpha_1}{\tau(\alpha_2)}}$$
$$= \tau\,\gamma_{2k+1}$$

$$\{\pm\beta_k, \pm\tau\gamma_k \mid k \geq 1\} \subset \Delta^{x\,re}$$
$$r_2r_1(x) \xrightarrow{s_2} r_1(x) \xrightarrow{s_1} x$$

$\mathrm{Hom}(W(G), x)$

$\mathrm{id}_x(s_1s_2)^k, \quad \mathrm{id}_x(s_1s_2)^k s_1, \quad \mathrm{id}_x(s_2s_1)^k, \quad \mathrm{id}_x(s_2s_1)^k s_2$

Example 10.3.19 Let $I = \{1,2\}$. $G = G(I, X, r, A)$ is a connected SCG of rank 2. $x \in X$. Assume $a_{12}^x = 0$

Since A^x is Cartan Matrix $a_{12}^{r_1(x)} = a_{12}^x$, $a_{21}^{r_2(x)} = a_{21}^x$

We conclude $a_{12}^x = a_{21}^x = 0$ $\quad a_{12}^{r_1(x)} = a_{21}^{r_2(x)} = 0$

Since $\mathcal{G}$ is connected, $a_{12}^{Y} = a_{21}^{Y} = 0 \quad \forall Y \in X$.

$\eta(0)^2 = -\mathrm{id}$, and hence <u>root seq.</u> w.r.t X and 1 is $(\alpha_1, \alpha_2, -\alpha_1, -\alpha_2)^\infty$

$\Big[\quad \beta_1 = \alpha_1, \quad \beta_2 = \eta(C_1)(\alpha_1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \alpha_2$

$\beta_3 = -\alpha_1, \quad \beta_4 = -\alpha_2, \quad \beta_5 = -\beta_1 \cdots \quad \Big]$

In particular, $\underline{m_{12}^{X}} = 2$ by Remark 10.3.18

$\Delta^{X\mathrm{re}} = \{ \pm \beta_k, \underline{\tau \gamma_k} \mid k \geq 1 \} \subset \mathbb{N}_0^2 \cup -\mathbb{N}_0^2 \quad (CG)$

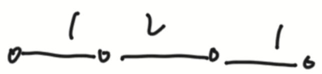
$|\Delta^{X\mathrm{re}} \cap \mathbb{N}_0 \alpha_1 + \mathbb{N}_0 \alpha_2| = m_{12}^{X} = 2$

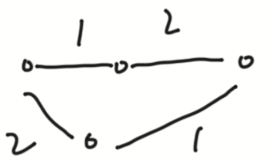
$(r_2 r_1)^{m_{12}^X}(X) = X \iff \mathcal{G}$ is a CG.

up to isomorphism there exist precisely 4 such CG:

one obj. 1 4 two obj 2 4

four obj. 1 4.

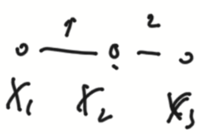




∘ $\quad r_1 = r_2 = \mathrm{id}$.

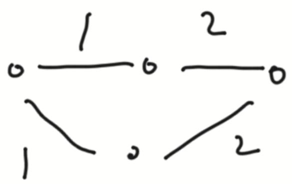
$r_1 = r_2 = (12) \qquad r_1 = (12) \quad r_2 = \mathrm{id}$

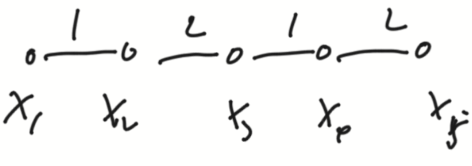
$\cancel{X}$



$r_2 r_1 \quad r_2 r_1 (\underline{X_1}) \overset{r_2}{=} r_1(X_3) = r_2(X_3) = \underline{X_2}$

$r_1(X_1) = X_2, \quad r_1(X_2) = X_1 \quad r_1(X_3) = X_3$



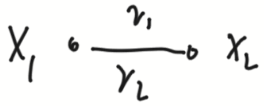


$X_2 \xrightarrow{r_1} X_1 \xrightarrow{r_2}$

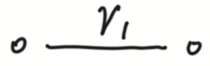
All of them are isomorphic to products of CG of rank one. $\quad \underline{\mathcal{G}(\{1\}, X, r, A)} \quad r = id$

$\underline{\mathcal{G}(\{1\}, \{X_1, X_2\}, r, A)} \quad r = (12)$

$\quad (2)$



$\mathcal{G}(\{1\}, \{X_1, X_2\}, r_1, A) \times \mathcal{G}(\{2\}, \{X_1, X_2\}, r_2, A)$



$r_1(X_1) = X_2$
$r_2 = id$

$\mathcal{G}(\{1\}, \{X_1, X_2\}, r, (2)) \times \mathcal{G}(\{2\}, \{X_1\}, r_2 = id, (1))$

$\mathcal{G}(\{1\}, \underline{\{X_1, X_2\}}, r_1, (2)) \times \mathcal{G}(\{2\}, \underline{\{X_3, X_4\}}, r_2, (2))$

?

Example 10.3.20. Let $I = \{1, 2\}$, $\mathcal{G} = \mathcal{G}(I, X, r, A)$ a Connected SCG of rank 2. Assume $a_{12}^Y, a_{21}^Y \leq -2$ $\forall Y \in X$. Let $X \in X$ and $(c_k)_{k \geq}$ and $(\beta_k)_{k \geq 1}$ be the char. seq. and root seq. w.r.t X and 1.

Then $c_k \geq 2 \quad \forall k \geq 1$

Thus $\underline{\beta_k \in \mathbb{N}_0^2} \quad \forall k \geq 1$ and $\underline{\beta_k \neq \beta_l} \quad \forall 1 \leq k < l$ by $\underline{\text{Lem } 10.3.12(3)}$. $\left[c_1 \geq 1, c_k \geq 2 \Rightarrow \beta_k \in \mathbb{N}_0^2 \quad \beta_k - \beta_{k-1} \in \mathbb{N}_0^2 \setminus \{0\} \right]$

By Rem 10.3.18, $\Delta^{X re}$ is infinite and is contained in $\mathbb{N}_0^2 \cup -\mathbb{N}_0^2$.

$\quad \supseteq \{\pm\beta_k, \pm \tau \beta_k \mid k \geq 1\}$

Hence $\mathcal{G}$ is a CG.

$(r_i r_j)^{m}(X) = X$
if $m < \infty$

$m_{ij}^X = |\Delta^{X re} \cap \mathbb{N}_0 \alpha_i + \mathbb{N}_0 \alpha_j| = \infty$

Affine

Thm. 10.3.21. Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ with $I = \{i, j\}$ be a connected SCG of rank 2, such that $|\mathcal{X}| < \infty$. Let $X \in \mathcal{X}$ and $n > 0$ be ~~the~~ integer with $(r_j r_i)^n(X) = X$

$(r_j r_i)^k(X) \neq X \quad \forall\ 1 \leq k < n$. Let $(c_k)_{k \geq 1}$ be the char. seq. w.r.t X and i, and

$$\overline{\omega} = 6n - \sum_{k=1}^{2n} c_k . \qquad \text{TFAE}.$$

(1) $\mathcal{G}$ is a finite CG

(2) $\overline{\omega} > 0$, $\overline{\omega} / 12$, $(c_1, \dots, c_{12n/\overline{\omega}}) \in \mathcal{A}^+$, and

$$(c_k)_{k \geq 1} = (c_1, \dots, c_{12n/\overline{\omega}})$$

In this case $12n/\overline{\omega} = |\Delta_+^{X\,re}| = m_{ij}^X$.

proof. Up to isomorphism, we assume $I = \{1, 2\}$, $i = 1$, $j = 2$.

(1) $\Rightarrow$ (2) Let $q = m_{ij}^X = \overline{m}_{ij}^X$. See Cor. 9.2.20.

Then Rem. 10.3.18 and Lem 9.2.7 imply that

$\beta_k \in \mathbb{N}_0^2 \quad \forall\ 1 \leq k \leq q \qquad$ and $\beta_{q+1} = -\beta_l$

[K reduced $\Leftrightarrow$ $\beta_1^{X,K}, \dots, \beta_l^{X,K}$ are distinct in $\mathbb{N}_0^I$

$\Leftrightarrow$ $\Lambda(K) \subset \mathbb{N}_0^I$

$K = (\underbrace{1\ 2\ 1\ 2 \dots}_{q}) \qquad \Lambda(K) = \{\beta_1, \dots, \beta_q\}$

K -

$K' = (K, 1)$ or $(K, 2)$ is not reduced

by Lem 9.2.5 (2) $\quad \beta_{q+1} = -\beta_l \quad 1 \leq l \leq q$]

Since $(c_k)_{k \geq 2}$ is the char. seq of w.r.t $r_i(X)$ and j

by Lem 10.3.16 (1), and $\overline{m}_{ji}^{r_i(X)} = q$ by Prop 9.2.14.

it follows that $c = 1$.

[$\gamma_k = \eta(c_2)\eta(c_3)\cdots\eta(c_k)(\alpha_1)$

$\gamma_1, \gamma_2, \ldots, \gamma_q \in \mathbb{N}_0^2$ $\qquad \gamma_q = \begin{bmatrix} a \\ b \end{bmatrix} \geq 0$

$\beta_1, \beta_2, \ldots, \beta_q \in \mathbb{N}_0^2$

$\beta_{q+1} = \eta(c_1)\cdots\eta(c_q)(\alpha_1) = \eta(c_1)\gamma_q$

$= \begin{bmatrix} -a_{12}a - b \\ a \end{bmatrix} \leq 0$

$\Rightarrow a = 0 \Rightarrow \beta_{q+1} = \begin{bmatrix} -b \\ 0 \end{bmatrix} = -b\alpha_1$ $\quad \begin{bmatrix} -b \\ 0 \end{bmatrix}$

$\Rightarrow \beta_{q+1} = -\alpha_1 = -\beta_1$. $\qquad \Rightarrow b = 1$]

that is $-\eta(c_1)\cdots\eta(c_q)(\alpha_1) = -\beta_{q+1} = \alpha_1$

Thus $-\eta(c_1)\cdots\eta(c_q) = id$ by Lem 9.2.19

[w $\quad w(\alpha_k) \in \alpha_k + \mathbb{Z}\alpha_i + \mathbb{Z}\alpha_j$ $\quad w^{-1}(\alpha_j), w(\alpha_j) \in \mathbb{N}_0^2 \cup \mathbb{N}_0^2$]

$\det(w) = 1$ $\quad w(\alpha_i) = \alpha_i$ $\quad w(\alpha_k), w^{-1}(\alpha_k) \in \mathbb{N}_0^2 \cup \mathbb{N}_0^2$.

$\Rightarrow w = id$.]

[$\eta(c_1)\cdots\eta(c_n) = -id$ $\quad \beta_n \in \mathbb{N}_0^2 \Leftrightarrow (c_1, \ldots, c_n) \in A^+$]

Hence $(c_1, \ldots, c_q) \in A^+$ by Thm 10.3.14.

Then $\sum_{i=1}^{q} c_i = 3q - 6$ by Lem 10.3.2

Because of Lem 10.3.16(1) $(c_k)_{k\geq 2}$ is the char. seq w.r.t $r_i(X)$ and j. $\quad (c_2, \ldots, c_{q+1}) \in A^+$

$\sum_{i=2}^{q+1} = 3q - 6$

Hence $c_{q+1} = c_1$, and $(c_k)_{k\geq 1} = (c_1, \ldots, c_q)^\infty$

by induction.

$r_i r_j(X)$ $\quad i$ $\qquad (c_3, \ldots, c_{q+2}) \in A^+$

$$C_{q+2} = C_2$$

$\cdots$

In particular, $\underline{q \sum_{i=1}^{2n} C_i} = \sum_{i=1}^{2qn} C_i = 2n \sum_{i=1}^{q} C_i$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \underline{2n(3q-6)}$

$$\underline{\sum_{i=1}^{2n} C_i} = \frac{6nq - 12n}{q} = 6n - \frac{12n}{q}$$

$\Rightarrow q \mid 12n$

$\underline{\varpi} = 6n - \sum_{k=1}^{m} C_k = \underline{\frac{12n}{q}}$

$(r_j r_i)^q (X) = X$ by (CG4)

and hence $n \mid q$. Therefore $\varpi \mid 12$

(2) $\Rightarrow$ (1) Let $q = \frac{12n}{\varpi}$. Then $(C_1, \cdots, C_q) \in A^+$

and hence $\beta_k \in \mathbb{N}_0^2$ $\forall 1 \le k \le q$, and $\eta(C_1) \cdots \eta(C_q) = -\mathrm{id}$

by Thm 10.3.14. Therefore, $(C_n)_{n \in \mathbb{Z}} = (C_1, \ldots, C_q)^\infty$

in the root seq. of $\mathcal{G}$ w.r.t X and i, only q elts of $\mathbb{N}_0^2$ and q elts of $-\mathbb{N}_0^2$ appear.

[$\underline{\beta_1 = \alpha_1, \ \beta_2 = \eta(C_1)(\alpha_1), \ldots, \ \beta_q} \in \mathbb{N}_0^2$

$\beta_{q+1} = \underline{\eta(C_1) \cdots \eta(C_q)}\, (\alpha_1) = -\alpha_1$

$\beta_{q+2} = -\beta_2 \qquad \cdots$]

Since $(C_q, \cdots, C_1) \in A^+$ by Cor. 10.3.8 (1)

Lem. 10.3.16 (2) $\Rightarrow$ the same holds for the root seq of $\mathcal{G}$ w.r.t X and j.

$\Delta^{X\,re} \subseteq \mathbb{N}_0^2 \cup -\mathbb{N}_0^2$ by Rem 10.3.18 and $\underline{\mathcal{G} \text{ is finite}}$

Because of Lem 10.3.16 (1),

$\Delta^{Y\,re} \subset \mathbb{N}_0^2 \cup -\mathbb{N}_0^2 \quad \forall\, Y \in \mathcal{X}.$

$|\Delta_+^{X\,re}| = \underline{m_{ij}^X = \overline{m}_{ij} = q}$ by Cor. 9.2.20

Further $\underline{n \mid q \text{ by assumption}}$? $\quad \underline{(r_2 r_1)^q (X) = X}$

$q = \frac{12n}{\omega} \quad \omega \mid 12$ $\quad$ (CG4)

Thus $\mathcal{G}$ is CG. $\quad$ #