6.16

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prop. Let A be a left H-module alg in $\mathcal{C}$. Then the functor $\underline{F : {}_A({}_H\mathcal{C}) \longrightarrow {}_{A\#H}\mathcal{C}}$

$$(\underline{(V,\lambda_H)},\lambda_A) \longmapsto (V,\ \underline{\lambda_A(id_A \otimes \lambda_H)})$$

where morphisms f are mapped onto f, is an isom.

The inverse functor is given by

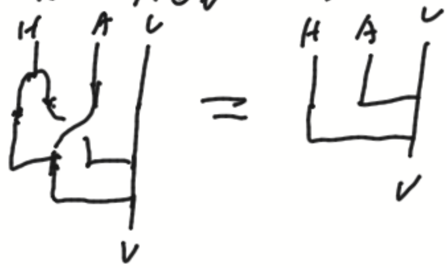
$$(V,\lambda_{A\#H}) \longmapsto ((V,\lambda_H),\lambda_A) \quad \text{where}$$

$$\lambda_H = \lambda_{A\#H}(\gamma \otimes id_V),\quad \lambda_A = \lambda_{A\#H}(\iota \otimes id_V)$$

$$\gamma = \eta_A \otimes id_H,\qquad \iota = id_A \otimes \eta_H$$

proof. $((V,\lambda_H),\lambda_A) \in {}_A(\underline{\underline{{}_H\mathcal{C}}})$

$\lambda_A : A \otimes V \longrightarrow V$ is H-linear. $a \otimes h \cdot (a \otimes v)$



$h \cdot (a \cdot v)$

$$\begin{tikzcd} A\otimes V \arrow[r, "\lambda_A"] & V \\ H\otimes A\otimes V \arrow[u, "\lambda_{A\otimes V}"] \arrow[r, "1\otimes\lambda_A"'] & H\otimes V \arrow[u, "\lambda_H"'] \end{tikzcd}$$

$h \otimes a \otimes v$ 　　　 h

To prove $(V,\ \underbrace{\lambda_A(id_A \otimes \lambda_H)}_{\lambda})$ is $A\#H$-module

$h.(a \otimes v)$

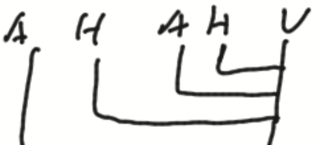
$\underline{(h_1 \cdot a)} \otimes (h_2 \cdot v)$

$= h \cdot (a \cdot v)$

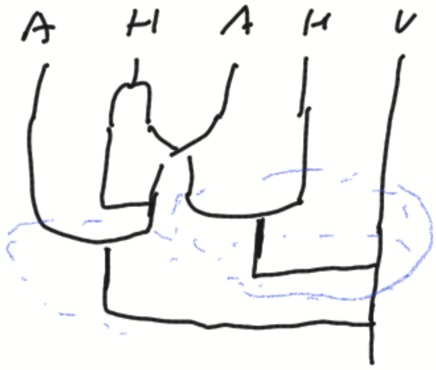
$$\lambda(1 \otimes \lambda) \overset{?}{=} \lambda(\mu_{A\#H} \otimes id_V)$$

$\mu_{A\#H} =$

$\lambda(1 \otimes \lambda) =$

,

$\lambda(\mu_{A\#H} \otimes id) =$



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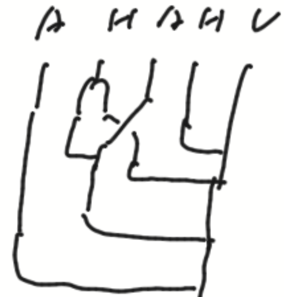

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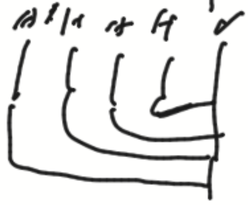


$(V, \lambda_{A\#H})$ is $A\#H$ – module

$\not\Rightarrow$ $((V, \lambda_H), \lambda_A) \in {}_A({}_H\mathcal{C})$

$\lambda_H = \lambda_{A\#H}(\gamma \otimes id_V)$, $\gamma = \eta_A \otimes id_H$

$\lambda_A = \lambda_{A\#H}(\iota \otimes id_V)$ $\quad \iota = id_A \otimes \eta_H$

① $(V, \lambda_H) \in {}_H\mathcal{C}$ $\quad \lambda_H(1 \otimes \lambda_H) = \lambda_H(\mu_H \otimes 1)$

$h \cdot (\underline{g \cdot v}) \overset{?}{=} (hg) \cdot v$

$= h \cdot (\gamma(g) \cdot v)$

$= \gamma(h) \cdot (\gamma(g) \cdot v)$

$= [\gamma(h)\gamma(g)] \cdot v$ $\quad$ γ is alg map.

$= \gamma(hg) \cdot v$

$= (hg) \cdot v$

② (V, λ_A) is an A – module

$a, b \in A$, $\quad a \cdot (b \cdot v) = a \cdot (\underline{(id \otimes \eta)(b)} \cdot v)$

$= a \cdot ((b \# 1) \cdot v)$

$= [(a \# 1) \cdot (b \# 1)] \cdot v$

$= (ab \# 1) \cdot v$

$= (ab) \cdot v$

③ λ_A is H-linear.

$a \cdot (h \cdot v) \stackrel{?}{=} h \cdot (a \cdot v)$

$= (a \# 1) \cdot ((1 \# h) \cdot v) = [(a \# 1)(1 \# h)] \cdot v = (a \# h) \cdot v$

$h \cdot (a \cdot v) = [(1 \# h)(a \# 1)] \cdot v = (a \# h) \cdot v$ //

Lem 12.2.9. R : $\mathbb{N}_0$-graded alg, C : $\mathbb{N}_0$-graded coalg in $\mathcal{C} = {}^H_H\mathcal{YD}$. $(X, \lambda_X) \in {}_R\mathcal{C}$, $(Y, \delta_Y) \in {}^C\mathcal{C}$

(1) $(F_n X)_{n \geq 0}$ is $\mathbb{N}_0$-filtration in $\mathcal{C}$ of the largest rational R-submodule of X

(2) $(F^n Y)_{n \geq 0}$ is $\mathbb{N}_0$-filtration in $\mathcal{C}$ of Y.

proof (1) Let $x \in X$, $i \in \mathbb{N}_0$. $R(i) x = 0$. Then

$\forall r \in R(i)$

(a) $0 = h_{(2)} [(\underline{S^{-1}(h_{(1)}) \cdot r) x}] = \underline{r (h \cdot x)}$

λ_X is H-linear $\quad R(i) \subset R$ is H-submodule.

[$h_{(2)} = 1 \# h_{(2)}$, $S^{-1}(h_{(1)}) \cdot r = S^{-1}(h_1) \cdot r \# 1$.

$(1 \# h_2)(S^{-1}(h_1) \cdot r \# 1)$

$= \underline{h_{21} \cdot (S^{-1}(h_1) \cdot r)} \# h_{22}$ $\qquad$ ~~$(r \# 1)(1 \# h)$~~ $(r \# h) \cdot x$.

$= [h_{21} S^{-1}(h_1)] \cdot r \# h_{22}$

$= S^{-1}(\underline{h_1 S(h_2)}) \cdot r \# h_3$

$= \underline{S^{-1}(\varepsilon(h_1) 1)} \cdot r \# h_2$

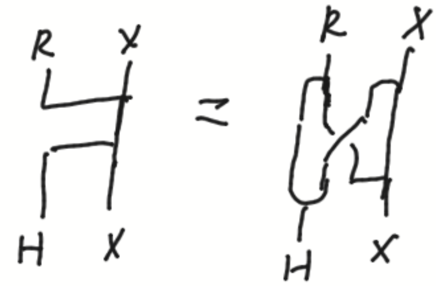
$= r \# h$

(b) $0 = r_{(-1)} \otimes \delta(r_{(0)} x) = r_{(-2)} \otimes r_{(-1)} x_{(-1)} \otimes r_{(0)} x_{(0)}$

Since λ_X is H-Colinear, $R(i)$ is H-subcomodule

Hence $0 = x_{(-1)} \otimes r x_{(0)}$

[$\lambda_X : R \otimes X \to X$ H-Colinear.

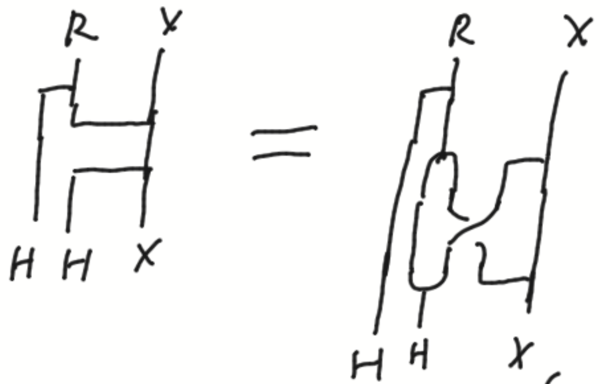


$$\begin{tikzcd} R\otimes X \arrow[r, "\lambda_X"] \arrow[d, "\delta_{R\otimes X}"'] & X \arrow[d, "\delta_X"] \\ H\otimes R \otimes X \arrow[r, "1\otimes\lambda_X"'] & H\otimes X \end{tikzcd}$$

$r_{(-1)} \otimes \delta(r_{(0)} x) =$ ${}^H X$

$= (1 \otimes \delta_X)(1 \otimes \lambda_X)(\delta_R \otimes 1)(r \otimes x)$

$r_{(-2)} \otimes r_{(-1)} x_{(-1)} \otimes r_{(0)} x_{(0)} = (1 \otimes \lambda_{R\otimes X})(\delta_R \otimes 1)(r \otimes x)$



$0 = r_{(-2)} \otimes r_{(-1)} x_{(-1)} \otimes r_{(0)} x_{(0)} \xmapsto{(m\otimes 1)(S\otimes 1\otimes 1)} S(r_{(-2)}) r_{(-1)} x_{(-1)} \otimes r_{(0)} x_{(0)}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad ||$

$r_{(-1)} x_{(-1)} \otimes r_0 x_0 = 0 \qquad\qquad x_{(-1)} \otimes r x_{(0)}$]

By (a) (b) $F_n X \subset X$ is a subobj in $\mathcal{C}$.

$x \in F_n X$ $\quad rx = 0$, $\quad r(h.x) = 0$ $\quad F_n X$ submodule

$\delta(x) = x_{(-1)} \otimes x_{(0)}$ $\quad x_{(0)} \notin F_n X$. $\quad r x_0 \neq 0$

$0 = x_{(-1)} \otimes r x_{(0)} \quad \not\Rightarrow \quad r x_{(0)} = 0$

$M \subset X$ rational R-module $\quad M \subset \bigcup F_n X$

$R(i) x = 0 \Rightarrow x \in F_n X$

(2) δ_Y is map in $\mathcal{C}$ $\quad \oplus C(i) \otimes Y \subset C \otimes Y$ is a sub obj. in $\mathcal{C}$.

$$F^n Y = \delta_Y^{-1} (\oplus C(i) \otimes Y)$$

Lem 12.2.10. Let $(A, B, \langle , \rangle)$ be a dual pair ...

$V \in {}^B\mathcal{C}$

(1) $F_n D^l(V) = F^n V$

(2) We view A, B as $\mathbb{Z}$-graded Hopf alg. in $\mathcal{C}$.

$n < 0$ $A(n) = 0 = B(n)$. If V is a $\mathbb{Z}$-graded B-comodule, then $\bar{D}^l(V)$ is $\mathbb{Z}$-graded A-module

$\bar{D}^l(V)(n) = V(-n) \quad \forall n \in \mathbb{Z}$.

proof. $\forall i \geq 0$ the kernel of the induced map.

$$f: B \otimes V \longrightarrow \mathrm{Hom}(A(i), V)$$
$$b \otimes v \longmapsto (a \mapsto \langle a, b \rangle v)$$

is $\bigoplus_{j \neq i} B(j) \otimes V$ by (12.1.1) and non-degeneracy of $\langle , \rangle$

[$f(\bigoplus_{j \neq i} B(j) \otimes V) = 0$

Let $f(b \otimes v) = 0$, $\forall a \in A(i)$

$0 = f(b \otimes v)(a) = \langle a, b \rangle v \Rightarrow \langle a, b \rangle = 0$

$\Rightarrow \langle A(i), b \rangle = 0$

If $b \in B(i)$ then $\langle A, b \rangle = 0 \Rightarrow b = 0$ *

$\Rightarrow b \in \bigoplus_{j \neq i} B(j)$]

Let $y \in F^n V$. $\delta(y) \in \bigoplus_{i=0}^{n} B(i) \otimes Y$

$\forall a \in A(j), j > n$

$a \cdot y = (\langle , \rangle \otimes id)(id \otimes \delta)(a \otimes y)$ $(V, \delta) \mapsto (V, \lambda)$

$= \langle a, y_{(-1)} \rangle y_{(0)} = 0$ $\lambda = (A \otimes V \xrightarrow{id \otimes \delta} A \otimes B \otimes V \xrightarrow{\langle , \rangle \otimes id} V)$

$y_{(-1)} \in \bigoplus_{i=0}^{n} B(i)$

$\Rightarrow y \in F_n \bar{D}^L(V)$

$\bar{D}^L : {}^B\mathcal{C} \to {}_{A^{cop}}\bar{\mathcal{C}}$

if $y \in F_n \bar{D}^L(V)$

$\langle a, y_{(-1)} \rangle y_{(0)} = 0$ $\forall a \in A(i)$ $i > n$

$f(y_{(-1)} \otimes y_{(0)}) = 0 \Rightarrow y_{(-1)} \otimes y_{(0)} \in \ker f = \bigoplus_{j \neq i} B(j) \otimes V$

$y_{(-1)} \in \bigcap_{i > n} \bigoplus_{j \neq i} B(j) = \bigoplus_{i=0}^{n} B(i) \Rightarrow y \in F^n V$.

(2) $m, n \in \mathbb{Z}$

$A(m) \bar{D}^L(V)(n) = A(m) V(-n) \subseteq V(-m-n) = \bar{D}^L(V)(m+n)$

[Let $a \in A(m)$, $x \in V(-n)$

$ax = \sum \langle a, x_{(-1)} \rangle x_{(0)}$ $x_{(-1)} \otimes x_{(0)} \in \bigoplus B(i) \otimes V(j)$

if $x_{(-1)} \notin B(m)$ then $\langle a, x_{(-1)}\rangle = 0$

if $x_{(-1)} \in B(m)$ then $x_{(0)} \in V(-n-m)$

$\Rightarrow ax \in V(-n-m) = \bar{D}^c(V)(m+n)$.]

$= \bigoplus_{i=0}^{\infty} B(i)V(-n-i)$

$i+j=-n$

Lem 12.2.11 Let R be an $\mathbb{N}_0$-graded Hopf alg. in $\mathcal{C}$

$V = (V, \delta) \in {}^R\mathcal{C}$ $m \in \mathbb{Z}$. Define

$$\delta^{(m)} = (V \xrightarrow{\delta} R\otimes V \xrightarrow{S_R^{2m}\otimes id} R\otimes V \xrightarrow{c^{2m}_{R,V}} R\otimes V)$$

(1) $V^{(m)} = (V, \delta^{(m)})$ is an obj in ${}^R\mathcal{C}$.

(2) For all $n \geq 0$ $F^n V = F^n V^{(m)}$

(3) If V is $\mathbb{Z}$-graded obj in ${}^R\mathcal{C}$, then $V^{(m)}$ with the grading of V is $\mathbb{Z}$-graded obj in ${}^R\mathcal{C}$.

proof. (1) follow Cor. 3.3.6, Since

$$(F_+^{rl} F_+^{lr})^m (V) = V^{(m)}$$

$$(F_-^{rl} F_-^{lr})^m (V) = V^{(-m)} \quad \forall\, m \geq 0.$$

Def. 3.3.4. H Hopf alg S isom.

$(V, \delta) \in {}^H\mathcal{C}$

$$\delta_\pm = (V \xrightarrow{\delta} H\otimes V \xrightarrow{(c^\pm)_{H,V}} V\otimes H \xrightarrow{id\otimes S^{\mp}} V\otimes H$$

$(V, \delta) \in \mathcal{C}^H$

$$\delta_\pm = (V \xrightarrow{\delta} V\otimes H \xrightarrow{(c^\pm)_{H,V}} H\otimes V \xrightarrow{S^{\mp}\otimes id} H\otimes V)$$

Cor. 3.3.6

$${}^H\mathcal{C} \xrightarrow{F_-^{lr}} \bar{\mathcal{C}}^{H^{op}} \qquad {}^{H^{op}}\bar{\mathcal{C}} \xrightarrow{F_+^{lr}} \mathcal{C}^H$$

$$\mathcal{C}^H \xrightarrow{F_-^{rl}} {}^{H^{op}}\bar{\mathcal{C}} \qquad \bar{\mathcal{C}}^{H^{op}} \xrightarrow{F_+^{rl}} {}^H\mathcal{C}$$

$F_{\pm}^{lr}(V,\delta) = (V, \delta_{\pm}) \qquad (V,\delta) \in {}^{H}C$

$F_{\pm}^{rl}(V,\delta) = (V, \delta_{\pm}) \qquad (V,\delta) \in C^{H}.$

are strict monoidal isomorphisms.

For $(V,\delta) \in {}^{R}C$

$(F_{+}^{rl} F_{+}^{lr})^{*}(V) = (V, \delta_{++})$

$\delta_{+} =$  $\qquad (\delta_{+})_{+} =$  $=$  $= \delta^{(1)}$

(2), (3) Since

$$C^{2m}_{R,V}(S_R^{2m} \otimes id)(R(n) \otimes V') = R(n) \otimes V'$$

$V' \subset V$ is subobj. in $\mathcal{C}$.

(2) $y \in F^{n}V \Longrightarrow \delta_{Y}(y) \in \bigoplus_{i=0}^{n} R(i) \otimes Y$

$$\delta^{(m)}(y) = C^{2m}(S^{2m} \otimes id)\,\delta(y)$$
$$\in C^{2m}(S^{2m} \otimes id)\left(\bigoplus_{i=0}^{n} R(i) \otimes Y\right)$$
$$= \bigoplus_{i=0}^{n} R(i) \otimes Y$$

$y \in F^{n}V^{(m)}$

S_R is graded, $\quad r \in R(n),\ x \in V'$

$C(r \otimes x) = r_{(-1)} \cdot x \otimes r_{(0)}$

(3) $(V,\delta) \in {}^{R}C \Longrightarrow (V, \delta^{(m)}) \in {}^{R}C.$

$$\delta(V(n)) \subset \sum_{i+j=n} R(i) \otimes V(j)$$

$$\delta^{(m)}(V(n)) \subset$$

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