Cor. 12.3.6 Let V be an object in ${}^{B}_{B}\mathcal{YD}_{rat}$

(1) $F_n \Omega(V) = F^n V$

$F^n \underline{\Omega(V)} = F_n V \quad \forall\ n \geq 0$

(2) If V is a $\mathbb{Z}$-graded object in ${}^{A}_{A}\mathcal{YD}_{rat}$, $\Rightarrow \Omega(V)$ is $\mathbb{Z}$-graded where $\Omega(V)(n) = V(-n)$

for all $n \in \mathbb{Z}$.

Pf. $F_n X = \{x \in X \mid R(i)x = 0 \ \forall\ i > n\}$

$F^n Y = \{y \in Y \mid \delta_Y(y) \in \bigoplus^{n} C(i) \otimes Y\}$

By Lem. 12.2.10 (1) $F_n \Omega(V) = F^n V$

$F^n (V, \delta_2) = F_n V$

$\underline{\Omega : (V, \lambda, \delta) \longmapsto (V, \lambda_1, \delta_1)}$

$$\bar{D}^L : {}^B\mathcal{C} \longrightarrow {}_{A^{op}}\bar{\mathcal{C}}_{rat}$$

$$(V,\delta) \longmapsto (V,\lambda)$$

$$\lambda = \left(A\otimes V \xrightarrow{id\otimes\delta} A\otimes B\otimes V \xrightarrow{\langle,\rangle\otimes id} V\right)$$

$$\lambda_1 = (\qquad\qquad)$$

$$\Omega(V,\delta) = (V,\lambda_1) = \bar{D}^L(V,\delta)$$

$$\Rightarrow F_n\,\Omega(V) = F^n V$$

$$\underline{(V,\lambda)} \longmapsto \underline{(V,\delta_1)}$$

$$\lambda = \left(B\otimes V \xrightarrow{id\otimes\delta_2} B\otimes A\otimes V \xrightarrow{\langle,\rangle^+\otimes id} V\right)$$

$\bar{D}^L$ 关于 $(B, A, \langle,\rangle^+)$ 的版本

By Lem 12.2.11

$$\underline{F^n(V,\delta_2) = F^n(V,\delta_1)} = F^n\,\Omega(V,\lambda)$$

$$[\ F^n V = F^n(V^m)\ ]$$

$$\delta^{(m)} = \left(V \xrightarrow{\delta} R\otimes V \xrightarrow{\delta_R^{2m}\otimes id} R\otimes V \xrightarrow{C^{2m}} R\otimes V\right)$$

$$\delta_1 = \delta_2^{(1)}$$

(ɪɪ) By 12.2.10

(V, δ) $\mathbb{Z}$-graded comodule

$\Rightarrow (\Omega(V), \lambda_1)$ is $\mathbb{Z}$-grade module.

(V, λ) $\mathbb{Z}$-graded module.

$\overset{?}{\Rightarrow}$ $(\Omega(V), \delta_1)$ $\mathbb{Z}$-grade comodule.

By $F^n V = F^n V^{(n)}$ δ_1 δ_2

$(\Omega(V), \delta_2)$ $\mathbb{Z}$-graded comodule

By 12.2.9. $(F^n V)_{n\geq}$ is a $\mathbb{N}_0$-filtr

$$\delta_2(\Omega(V)(n)) \subset \sum_{i+j=n} A(i) \otimes \Omega(V)(j) \ ?$$

$$\begin{array}{ccc} \| & & \| \\ \delta_2(V(-n)) & & \sum_{i+j=n} A(i) \otimes V(-j) \end{array}$$

$\forall\ \underline{x \in V(-n)}$ $\quad x \in F^m V$ for some m.

$$\delta_2(x) \in \bigoplus_{i=0}^{m} A(i) \otimes V$$

$$\delta_2(X) = \sum x_i \otimes v_i \qquad x_i \in \underline{\underline{A(i)}}$$

Let $\underline{b \in B(i)}$

$$\underline{b\, X = \quad \langle b, x_i \rangle^{+} v_i \in V(i-n)}$$

$v_i \in V(i-n)$

$$\delta_2(X) \in \sum A(i) \otimes V(i-n)$$

#

Def. 12.3.7 Let $V \in \mathcal{C} = {}^H_H\mathcal{YD}$ be f.dim. We denote

$$(\Omega_V, \omega_V) : {}^{B(V)}_{B(V)}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}} \to {}^{B(V^*)}_{B(V^*)}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$$

the braided monoidal isom. of Thm 12.3.2 for $(B(V^*), B(V), \underline{\langle\ \rangle})$ with Hopf pairing $\langle\ \rangle : B(V^*) \otimes B(V) \to \mathbb{k}$.

Remark. If R is $\mathbb{N}_0$-graded Hopf alg in C. S. bijective

${}^{R\#H}_{R\#H}YD_{rat}$ denote the full subcat of Yetter Drinfeld module in ${}^{R\#H}_{R\#H}YD$.

$$\begin{tikzcd}
{}^{B\#H}_{B\#H}YD_{rat} \arrow[r, phantom, "\underset{3.8.7}{\cong}"] \arrow[d, "{(\Omega,\omega)}"] & {}^{B}_{B}YD(\underline{C})_{rat} \arrow[d, "\Omega"] \\
{}^{A\#H}_{A\#H}YD_{rat} \arrow[r, phantom, "\overset{3.8.7}{\cong}"] & {}^{A}_{A}YD(C)_{rat}
\end{tikzcd}$$

(3.8.7) $\quad {}^{R}_{R}YD\left({}^{H}_{H}YD(\underline{C})\right) \cong {}^{R\#H}_{R\#H}YD(C)$

take $C = M_k$.

For $Q \in {}^{B\#H}_{B\#H}YD_{rat}$.

$B(Q)$ is $\mathbb{N}_0$-graded Hopf alg in ${}^{B\#H}_{B\#H}YD$.

By Lem 12.2.4 (3) $\quad B(Q) \in {}^{B\#H}_{B\#}YD_{rat}$

Cor. 12.3.7. Let $Q \in {}^{B\#H}_{B\#H}\mathcal{YD}_{rat}$. $B(Q)$ its Nichols alg. Then

$$(\Omega, \omega)(B(Q)) \cong B((\Omega,\omega)(Q))$$

as $\mathbb{N}_0$-graded Hopf alg. in ${}^{A\#H}_{A\#H}\mathcal{YD}_{rat}$

Pf. It is easy $(\Omega,\omega)(B(Q))$ is connected $\mathbb{N}_0$-graded Hopf alg.

as $C(0)$ is 1-dim

$$\begin{tikzcd}
\Omega(B(Q)) \otimes \Omega(B(Q)) \arrow[r, "\omega"] \arrow[d, equal] & \Omega(B(Q) \otimes B(Q)) \arrow[d, "\Omega(\mu)"] \\
B(Q) \otimes B(Q) & \Omega(B(Q)) \arrow[d, equal] \\
 & B(Q)
\end{tikzcd}$$

$$\omega = C^{{}^B_B\mathcal{YD}(C)}_{Y,1,X} \quad \bar{C}_{X,Y}$$

generated as an alg by $\Omega(Q)$

Any homogeneous primitive of deg ≥ 2

is zero.

$\underline{\omega^{\#}\Delta}$

$x \in P(B(Q)) \Longleftrightarrow x \in P(\Omega(B(Q)))$

Hence $(\Omega, \omega)(B(Q))$ is a Nichols alg of $\Omega(Q)$. and the claim follows from Thm 7.1.14.

12.4. one-sided coideal

Subalg of braided Hopf alg.

Let H be a Hopf alg with bij. antipode.

$\mathcal{C} = ({}^{H}_{H}\mathcal{YD}, c)$

c could be Yetter-Drinfeld braiding

$c^{{}^{H}_{H}\mathcal{YD}}$ or its inverse $\bar{c}^{{}^{H}_{H}\mathcal{YD}}$.

X bialg in $\mathcal{C}$.

A left coideal subalg of in $\mathcal{C}$ is a subobj. $E \subset X$ and an alg in $\mathcal{C}$ s.t. $E \subset X$ is an alg morphism in $\mathcal{C}$. $\Delta_*(E) \subset X \otimes E$

Let P be Hopf alg in $\mathcal{C}$.

$Q \subset P$ Hopf subalg,

$\pi : P \longrightarrow Q$ Hopf alg map in $\mathcal{C}$.

$\pi|_Q = \mathrm{id}_Q$.

$R = P^{\mathrm{co}Q}$ be the space of right coinvariant elt of P w.r.t. π.

$R = \{ a \mid (\mathrm{id} \otimes \pi)\Delta(a) = a \otimes 1,\ a \in P \}$

We are in Thm 3.10.4.

(P, π, γ)

$$\begin{tikzcd} R \arrow[r, hook] \arrow[d, "="'] & P \arrow[r, "\pi"'] \arrow[dl, "\theta"] & Q \\ R & & \end{tikzcd}$$

* (A, π, γ) $\quad \begin{array}{l} \pi : A \to H \\ \gamma : H \to A \end{array}$ are

morphm of Hopf alg, $\pi\gamma = id_H$.

Thm. 3.10.4, (A, π, γ) Hopf triple over H.

$$\delta_A = (id_A \otimes \pi)\Delta_A : A \to A \otimes H$$

$$\lambda_A = \mu_A(id_A \otimes \gamma) : A \otimes H \to A$$

$$\underline{\Theta_A} = \mu_A(id_A \otimes \gamma\pi S_A)\Delta_A : A \to A$$

$$\Sigma_A = \mu_A(\gamma\pi \otimes S_A)\Delta_A : A \to A,$$

Assume an equalizer $(R, \iota : R \to A)$

of the pair $(\delta_A, id_A \otimes \eta_H)$

exists.

There is unique ϑ

$\Theta_A = \iota\vartheta$, $\vartheta\iota = id_R$.

$(A, \vartheta: A \to R)$ coequalizer

of $(\lambda_A, id_A \otimes \varepsilon_H)$

λ_R. δ_R

$\iota\lambda_R = ad_A(\gamma \otimes \iota)$

$\delta_R\vartheta = (\pi \otimes \vartheta) \circ ad_A$

(2) $\Phi = (R\#H \xrightarrow{\iota \otimes \gamma} A \otimes A \xrightarrow{\mu_A} A)$

$\psi = (A \xrightarrow{\Delta_A} H \otimes A \xrightarrow{\vartheta \otimes \pi} R\#H)$

give isom.

$\Delta_P : x \mapsto x^{(1)} \otimes^{(2)}$

$\vartheta(x) = x^{(1)} S_\theta(\pi(x^{(2)}))$

$\forall\, x \in P$

$R \subset P$ is a subalg. in C.

$$\Delta_P(R) \subset P \otimes R$$

[$r \in R$. $\quad \Delta_P(r) = r^{(1)} \otimes \underline{r^{(2)}}$

$$r^{(1)} \otimes \tau(r^{(2)}) = r \otimes 1$$

$$\Delta(r^{(1)}) \otimes S\tau(r^{(2)}) = \Delta(r) \otimes S(1)$$

$$r^{(1)} \otimes \underline{r^{(2)} \otimes S\tau(r^{(3)})} = \underline{r^{(1)}} \otimes \underline{r^{(2)} \otimes 1}$$

$$\|$$

$$r^{(1)} \otimes \underbrace{\vartheta(r^{(2)})}_{\in R} = r^{(1)} \otimes r^{(2)}$$

]

$$\underline{r^{(2)} \otimes \tau(r^{(3)})}$$

$$\vartheta(qx) = (\mathrm{ad}\, q)(x), \quad \underline{\vartheta(yq) = \vartheta(y)\varepsilon(q)}$$

$x \in R$, $y \in P$, $q \in Q$.

$$\vartheta(yq) = (yq)^{(1)} S_0 \tau((yq)^{(2)})$$

$$= y^{(1)} \xi^{(1)} S_Q \pi (y^{(2)} \xi^{(2)})$$

$$= y^{(1)} \xi^{(1)} S_Q [\pi (y^{(2)}) \underline{\pi (\xi^{(2)})}]$$

$$= y^{(1)} \xi^{(1)} S_Q [\pi (y^{(2)}) \xi^{(2)}]$$

$$= \underline{y^{(1)} \xi^{(1)} S_Q (\xi^{(2)})} S_Q \pi (y^{(2)})$$

$$= \varepsilon(\xi) \vartheta (y)$$

$$\vartheta (\xi x) = (\xi x)^{(1)} S_Q [\pi ((\xi x)^{(2)})]$$

$$= \xi^{(1)} \underline{x^{(1)}} S_Q [\underline{\pi (\xi^{(2)})} \underline{\pi (x^{(2)})}]$$

$$= \xi^{(1)} x S_Q [\xi^{(2)}]$$

$$= (\mathrm{ad}\, \xi)(x)$$

$$\Delta_P (x) = \vartheta (x^{(1)}) \pi (x^{(2)}) \otimes x^{(3)}$$

[By Thm. 3.10.4 (2)

$$\mu_P (1 \otimes \cancel{\pi}) (\vartheta \otimes \pi) \underline{\Delta_P (x)} = x$$

$$\vartheta(x^{(1)})\,\pi(x^{(2)}) = x \in P$$

$$\Delta_P(x) = x^{(1)} \otimes x^{(2)}$$
$$= \vartheta(x^{(1)})\,\pi(x^{(2)}) \otimes x^{(3)} \quad]$$

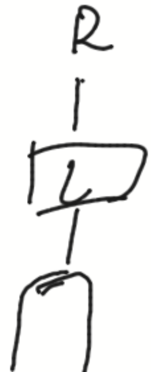
$$\Delta_R(x) = \vartheta(x^{(1)}) \otimes x^{(2)} \qquad x \in R.$$

$$[\quad \Delta_P(x) = x^{(1)} \otimes x^{(2)} \in P \otimes R$$

$$\Delta_R \vartheta = (\vartheta \otimes \vartheta)\,\Delta_P \qquad \vartheta\iota = id_R$$

$$\Delta_R \vartheta(x) = (\vartheta \otimes \vartheta)\,\Delta_P(x)$$
$$\| \qquad = (\vartheta \otimes \vartheta)(x^{(1)} \otimes x^{(2)})$$
$$\Delta_R(x) \qquad = \vartheta(x^{(1)}) \otimes x^{(2)} \quad]$$

$$\delta(x) = \pi(x^{(1)}) \otimes x^{(2)}$$

$\delta_R =$ 

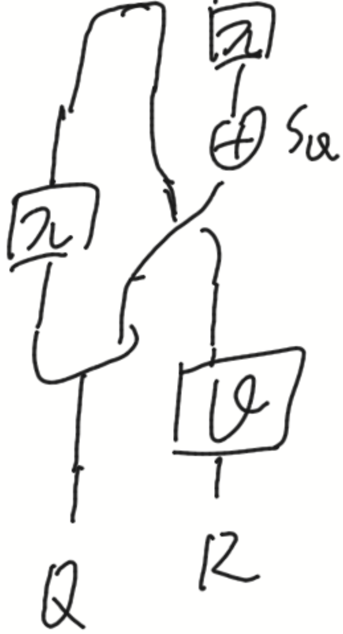
$x \in R$

$\downarrow$

x

$\downarrow$

$x^{(1)} \otimes x^{2}$



$x^1 \otimes x$ =

$\downarrow$

$x^1 \otimes x^2 \otimes \pi(x^3)$

$\|$

$x^1 \otimes x^2 \otimes 1$

$\downarrow S_a$

$x^1 \otimes x^2 \otimes 1$ =

$\downarrow \pi \otimes c$

$\pi(x^1) \otimes 1 \otimes x^2$

$\downarrow \mu \otimes \vartheta$

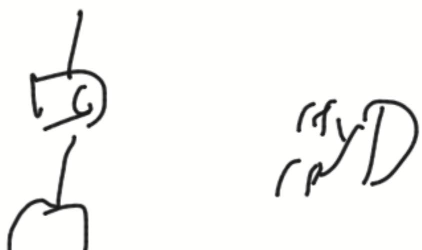
$\pi(x^1) \otimes \vartheta(x^2)$

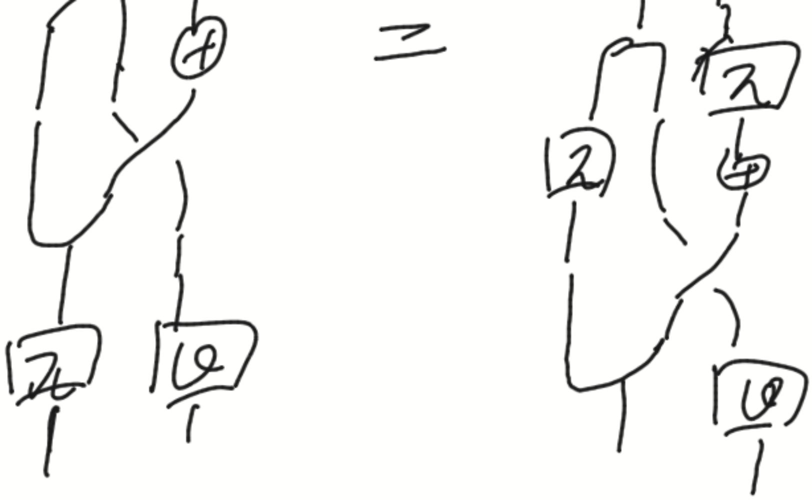
$\delta_R = (\pi \otimes \vartheta) \circ \mathrm{coad}_A\, \iota$

coad:

$V \xrightarrow{\delta_r} V \otimes H \xrightarrow{\delta_l \otimes S_H} H \otimes V \otimes H$

$\xrightarrow{id \otimes c} H \otimes H \otimes V \xrightarrow{\mu \otimes id} H \otimes V$





C

x_r, C flip

Def.

$\mathcal{E}_r^+(P) = \{E \mid E \subset P$ right coideal subalg. in C, $\Omega \subset E\}$

$\mathcal{E}_r(P, X) = \{E \mid E \subset P$ right coideal subalg. $E \subset X\}$

$X \subset P$ subobj. in ${}^H_H\mathcal{YD}$.

$\underline{\mathcal{F}_r(P)} = \{F \mid F \subset R$ subalg in C, $\delta_R(F) \subset F \otimes R$, $F \subset R$, Ω-$\underline{\text{submod}}\}$

Def. 12.4.2..

$\mathcal{F}_C(P) = \{ F \mid F \subset R$ subalg in C.

$\overline{} \Delta_R(F) \subset R \otimes F \quad F \subset R$

Q - $\underline{\text{subcomodule}}$ $\}$

Lem 12.4.2. $\forall E \in \Sigma_\gamma^+(D)$, the mult. map $(E \cap R) \otimes Q \longrightarrow E$ is an isom. in ${}^H_{\;\;}\mathcal{M}/D$.

(2) $\Sigma_\gamma^+(D) \longrightarrow \mathcal{F}_\gamma(D)$

$E \longmapsto E \cap R$

is bij. inverse: $F \longmapsto FQ$.

Pf. $E \longrightarrow E \cap$