Def 12.4.1

Hopf triple (P, π, γ) over Q

$\pi : P \to Q$

$\pi|_Q = \mathrm{id}$

Let $\mathcal{E}_r^+(P) = \{ E \mid E \subseteq P$ right coideal subalgebra in $\mathcal{C}$, $Q \subseteq E \}$

$\mathcal{E}_r(P, X) = \{ E \mid E \subseteq P$ ..... in $\mathcal{C}$, $E \subseteq X \}$

$X \subseteq P$ is a subobject in $\underline{{}^H_H\mathcal{YD}}$

$\mathcal{F}_r(P) = \{ F \mid F \subseteq R$ subalgebra in $\mathcal{C}$, $\Delta_R(F) \subseteq F \otimes R$, $F \subseteq R$ Q-submodule $\}$

Def 12.4.2

$\mathcal{E}_l^+(P) = \{ \;....\; \}$

$\mathcal{E}_l(P, X) = \{ \;---\; \}$

$\mathcal{F}_l(P) = \{ \widetilde{F} \mid F \subseteq R$, $\Delta_R(F) \subseteq \underline{R} \otimes F$, $F \subseteq R$ Q-subcomodule $\}$

Note that the sets in the previous definitions depend on the Hopf triple $\underline{(P, \pi, \gamma)}$

Lemma 12.4.3

(1) For all $E \in \mathcal{E}_r^+(P)$, the multiplication map $(E \cap R) \otimes Q \to E$ is an iso in $\boxed{{}^H_H\mathcal{YD}}$.

Proof:

$$R \otimes Q \xrightarrow{\cong} P$$

The map $E \to \underline{(E \cap R) \otimes Q}$

$$x \mapsto \underset{\underset{\in E}{=}}{x^1 S_Q(\pi(x^2))} \otimes \pi(x^3)$$

$$\| \qquad\qquad$$

$$\underline{\vartheta(x^1)} \otimes \pi(x^2)$$

check:

suppose $x \in E \cap R$, $y \in Q$,

$$x \otimes y \mapsto xy \mapsto (xy)^1 S_Q(\pi(xy)^2) \otimes \pi(xy)^3$$

$$\| \qquad$$

$$\vartheta((xy)^1) \otimes \pi((xy)^2)$$

$$\overset{?}{=} x \otimes y$$

$$\Delta(xy) = \Delta(x)\Delta(y)$$

$$= (x^1 \otimes x^2)(y^1 \otimes y^2)$$

$$= \underline{x^1 (x^2{}_{-1} \cdot y^1)} \otimes \underline{x^2{}_0 y^2}$$

$$\vartheta(x^1(x^2{}_{-1} \cdot y^1)) \otimes \underline{\pi(x^2{}_0 y^2)}$$

(12.4.7): $\vartheta(y\xi) = \vartheta(y)\varepsilon(\xi)$ for $y \in P$, $\xi \in Q$.

$$= \vartheta(x^1)\varepsilon(x^2{}_{-1} \cdot y^1) \otimes \pi(x^2{}_0) y^2$$

$$= \vartheta(x^1)\varepsilon(x^2{}_{-1})\varepsilon(y^1) \otimes \pi(x^2{}_0) y^2$$

$$= \vartheta(x^1) \otimes \pi(x^2) y$$

$$= \underset{x \in E \cap R}{\vartheta(x)} \otimes y = x \otimes y$$

第一方向 by 12.4.2 .

(2) The map $\mathcal{E}_r^+(P) \to \mathcal{F}_r(P)$

$E \mapsto E \cap R$

is bijective with inverse

given by $F \mapsto FQ$

well-defined:

$\rightarrow$:

Let $E \in \mathcal{E}_r^+(P)$

Then $E \cap R \subseteq R$ is a subalgebra

& $\forall x \in E \cap R$,

$$\Delta_R(x) \overset{12.4.1}{\underset{12.4.3}{=}} \underbrace{x^1 S_Q(\pi_1 x^2)}_{\vartheta(\cdots)} \otimes x^3$$

$$\in \underline{(E \otimes P) \cap (R \otimes R)}$$

$$\rotatebox{90}{$=$}$$

$$\underline{(E \cap R)} \otimes \underline{R}$$

再验证 $E \cap R$ is a Q-submodule of R.

Let $x \in E \cap R$, $q \in Q$.

$$(\mathrm{ad}_q)(x) = \vartheta(qx) \in R$$

$$\overset{12.4.6}{=} (qx)^1 S_Q \underline{\pi_1(qx)^2}$$

$$\overset{?}{\in} \underline{E \cap R}$$

$$\underline{\Delta_P(qx)} = \underline{(q^1 \otimes q^2)} \otimes \underline{(x^1 \otimes x^2)}$$

$$\in QE \otimes QP$$

$$\underline{\subseteq E \otimes P}$$

$\Rightarrow \mathrm{ad}_q(x) \in E \cap R$

well-defined

$\leftarrow$:

Let $F \in \mathcal{F}_r(P)$, $FQ \overset{?}{\in} \mathcal{E}_r^+(P)$

By (12.4.3), (12.4.5) $\boxed{\Delta_P(F) \subseteq FQ \otimes P}$

$\Delta_R(x) = \vartheta(x^1) \otimes x^2$

$\Delta_P(x) = (\vartheta(x^1)\pi_1(x^2)) \otimes x^3$

To ~~see~~

Have for all $x \in F$, $q \in Q$,

$$\Delta_P(xq) = \underline{\Delta_P(x)}\, \Delta_P(q)$$

$$\in FQ \otimes P \Rightarrow FQ \cdots$$

To see that $FQ \subseteq P$ is a subalgebra,

we have prove that $\boxed{QF \subseteq FQ}$

$\forall q \in Q$, $x \in F$

$$\Delta_P(qx) = \underset{\in Q \otimes Q}{\underline{\Delta_P(q)}}\ \underset{\in FQ \otimes P}{\underline{\Delta_P(x)}}$$

$$\in \underline{QFQ \otimes P}$$

$$qx = \vartheta((qx)^1)\pi_1((qx)^2)$$

$$\in \vartheta(QF\cancel{Q})Q$$

$$\rotatebox{90}{$=$}$$

$$\underline{(\mathrm{ad}Q)(F)Q} \subseteq FQ$$

$$\underline{\vartheta(qx) = (\mathrm{ad}_q)x}$$

$$\underline{\vartheta(yq) = \vartheta(y)\varepsilon(q)}$$

Finally, ---

If $E \in \mathcal{E}_r^+(P)$,

then $E = (E \cap R)Q$ by (1)

另一方面: if $F \in \mathcal{F}_r(P)$

then $\underline{(FQ \cap R)Q = FQ}$

$\subseteq$: LHS $\subseteq FQQ \subseteq FQ$

$\supseteq$: 取Q里的1

By (12.4.2) maps

$$(FQ \cap R) \otimes Q \rightarrow (FQ \cap R)Q$$

$$\rotatebox{90}{$=$}$$

$$F \otimes Q \longrightarrow \underline{FQ} \xrightarrow{PQ}$$

$$\Rightarrow FQ \cap R = F$$

Lem 12.4.4

$$\mathcal{E}_l(P,R) = \mathcal{F}_l(P)$$

Proof: $\mathcal{E}_l(P,R) \subseteq \boxed{\mathcal{F}_l(P)}$

follows from (12.4.3) $-\ \underline{\delta_P(x) = \theta(x^1) \otimes x^2}$

~~(12.4.4)~~

Suppose $x \in E \in \mathcal{E}_l(P,R)$

then $\delta_P(x) \in P \otimes E$

$$\Rightarrow \delta_R(x) \in R \otimes E$$

(12.4.4) $\quad \delta(x) = \pi(x^1) \otimes \underbrace{x^2}_{\in E}$

$\supseteq$ follows from (12.4.5)

(12.4.5) $\quad \delta_P(x) = \theta(x^1)\pi(x^2) \otimes \boxed{x^3}$

要证 $\delta_P(x) \in \underline{P} \otimes \boxed{F}$ (suppose $F \in \mathcal{F}_l(P)$)

$$\boxed{\delta(x) = \pi(x^1) \otimes \underbrace{x^2}_{\in F}}$$

Now we assume the situation of Thm 12.3.3

$(G, B, \langle\ ,\ \rangle)$ is a dual pair ...

$(\Omega, \omega) : {}^{B}_{B}\mathcal{YD}(\boxed{\mathcal{C}})_{\mathrm{rat}} \xrightarrow{\overset{G}{G}\mathcal{YD}} {}^{A}_{A}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$

K is a Hopf algebra in ${}^{B}_{B}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$

$$\underbrace{(S, \pi, \gamma)}_{K}, \quad \underbrace{(\tilde{S}, \tilde{\pi}, \tilde{\gamma})}_{\Omega(K)}$$

$$\begin{tikzcd} & B \arrow[dl, bend right, "\gamma"'] \arrow[d, phantom, "\parallel"] \\ S = K \# B \arrow[r, "\pi"'] & B \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{C} \arrow[d, phantom, "\parallel"] \\ \underbrace{\Omega(K)\#A}_{\tilde{S}} \arrow[r, "\tilde{\pi}"'] & A \arrow[l, bend right, "\tilde{\gamma}"'] \end{tikzcd} \quad \mathcal{C}$$

$(\tilde{S}^{\mathrm{cop}}, \underline{\tilde{\pi}}, \underline{\tilde{\gamma}})$ over A^{cop}

and $\tilde{\mathcal{C}}$ denotes the set of right coinvariant elements in $\tilde{\mathcal{C}}$

$$(\tilde{S}^{\mathrm{cop}} \xrightarrow{\tilde{\pi}} A^{\mathrm{cop}})$$

By Thm 12.3.3

$\boxed{T} : \underline{\tilde{\mathcal{C}}} \to \underline{K}$ Hopf algebra isom in $\mathcal{C}$

$$\tilde{\mathcal{C}} \to \underline{D(K^{\mathrm{cop}})} = \underline{K}$$

Thm 12.4.5 (12.4.6)

Under the assumptions of Thm 12.3.3,

the map $\Sigma_r(\underline{\tilde{S}}, \underline{\tilde{C}}) \to \underline{\Sigma_r^+(S)}$

$E \mapsto T(E)B$

is an inclusion-preserving bijection

with inverse $E \mapsto T^{-1}(E \cap K)$

For all $E \in \Sigma_r(\tilde{S}, \tilde{C})$,

the multiplication map

$T(E) \otimes B \to T(E)B$ is bijective

Prof:

By Thm 12.3.3

T is an A^{cop}-colinear iso of algebras and coalgebras in $\mathcal{C}$

$T : \tilde{C} \to D(K^{cop})$

is an iso of Hopf algebras

in $\underline{{}^{A^{cop}}_{A^{cop}}\mathcal{YD}(\bar{\mathcal{C}})}$

The A^{cop}-comodule structure of K

is $(S_A \otimes id)\underline{\delta_2}$

where the category iso $D_+^l : {}^A\mathcal{C} \to {}_B\mathcal{C}_{rat}$

$\cong$

$\underline{\langle\,,\rangle^+}$

maps $(K, \underline{\delta_2})$ onto $\underline{(K, \lambda)}$

$D_2 : {}^{A^{cop}}\bar{\mathcal{C}} \to {}_B\mathcal{C}_{rat}$

$D_2(V, \bar{\delta}) = (V, \lambda)$

$$B \otimes V \xrightarrow{id \otimes \bar{\delta}} B \otimes A \otimes V \xrightarrow{\langle\,,\rangle^{+cop} \otimes id} V$$

$(S_A \otimes id)\delta_2$

$$B \otimes K \xrightarrow{id \otimes \delta_2} \underline{B \otimes A \otimes K} \xrightarrow{\langle\,,\rangle^{+} \otimes id} K$$

$= \lambda$

$\underline{\langle\,,\rangle^{+cop}} = \langle\,,\rangle^{+}(id_B \otimes S_A^{-1})$

$$B \otimes V \xrightarrow{id \otimes (S_A \otimes id)\delta_2} B \otimes A \otimes V \xrightarrow{\langle\,,\rangle^{+cop} \otimes id} V$$

$\Rightarrow K$ 的 A^{cop} 余模 $\underline{(S_A \otimes id)\delta_2}$

Note that a subobject $\underline{F \subseteq K}$ in $\mathcal{C}$

is a B-submodule

$\Leftrightarrow$ F is an A^{cop}-subcomodule

since D_+^l and $S_A : A \to A^{cop}$ is a coalgebra

$\mathcal{F}$ isomorphism

($D_T^L : B \otimes K^{\mathcal{F}} \xrightarrow{\mathrm{id} \otimes \delta_L} B \otimes A \otimes K \xrightarrow{\mathcal{F}\ \kappa . S \otimes \mathrm{id}} K^{\mathcal{F}}$

${}_A\mathcal{C} \to {}_B\mathcal{C}_{\mathrm{rat}}$

A-comod $\Leftrightarrow$ A^{cop}-comodule

$S_A \otimes \mathrm{id}$

$S_A^{-1} \otimes \mathrm{id}$

Hence T induces a bijection between

$\mathcal{F}_L(\tilde{S}^{\mathrm{cop}}) = \{F \mid F \subseteq \tilde{L}$ subalgebra in ${}^H_H\mathcal{YD}$,

$\Delta_{\tilde{L}}(F) \subseteq \tilde{L} \otimes F$,

$F \subseteq \tilde{L}$, A^{cop}- subcomodule$\}$

and

$\mathcal{F} = \{F \mid F \subseteq K$ subalgebra in ${}^H_H\mathcal{YD}$

$\cancel{c^{{}^B_B\mathcal{YD}(\mathcal{C})}_{K,K}}$, $\Delta_K(F) \subseteq \overset{F \otimes K}{K \otimes F}$

$\overset{T(\ )}{\Rightarrow}$: $T : \tilde{L} \to K$ $F \subseteq K$, B - submodule$\}$

$D(K^{\mathrm{cop}})$ ${}^B_B\mathcal{YD}$

Let $F \subseteq \tilde{L}$ subalgebra in ${}^H_H\mathcal{YD}$

$T(F) \subseteq K$ subalgebra in ${}^H_H\mathcal{YD}$

$\Delta_{\tilde{L}}(F) \subseteq \tilde{L} \otimes F$

$\overset{T(\ )}{\Rightarrow}$ $c^{{}^B_B\mathcal{YD}(\mathcal{C})}_{K,K}$ $\Delta_K(T(F)) \subseteq K \otimes T(F)$

F is an A^{cop}- subcomodule

$\mathcal{F}$ $\Rightarrow$ $T(F)$ is an A^{cop}-subcomodule

$\Leftrightarrow$ $T(F)$ is a B-submodule

If $F \subseteq K$ is a subobject in $\mathcal{C}$ and a left B-submodule

then $c^{{}^B_B\mathcal{YD}(\mathcal{C})}_{K,K}$ $\Delta_K(F) \subseteq K \otimes F$

$\Leftrightarrow$ $\Delta_K(F) \subseteq F \otimes K$

Since by Prop 3.4.5

the braiding of $K \otimes K$ in ${}^B_B\mathcal{YD}(\mathcal{C})$

defines an iso $K \otimes F \cong F \otimes K$

for B-stable subobj $F \subseteq K$

Here $F = \mathcal{F}_r(S)$

Lemma 2.4.4 for the projection $\tilde{S}^{\mathrm{cop}} \xrightarrow{} \tilde{\mathcal{F}_r}$

$/+^{op}$

gives the equality

$$\underline{\mathcal{E}_r}(\tilde{S},\tilde{L}) = \underline{\mathcal{E}_l}(\tilde{S}^{\boxed{op}},\tilde{L}) \underset{12.4.4}{\boxed{=}} F_l(S^{op})$$

$\mathcal{Y}_0$

By lem 12.4.3 (2)

$$F_r(S) \to \mathcal{E}_r^+(S) \text{ bijection}$$

$$F \mapsto FQ$$

Then

$$\boxed{\mathcal{E}_r(\tilde{S},\tilde{L})} = \mathcal{E}_l(\tilde{S}^{op},\tilde{L}) \underset{12.4.4}{=} F_l(\boxed{\tilde{S}^{op}}) \overset{T()}{\boxed{=}} F = F_r(S) \xrightarrow{12.4.3(2)} \boxed{\mathcal{E}_r^+(S)}$$

$$F \mapsto FQ$$

$$\boxed{\mathcal{E}_r(\tilde{S},{}^{\alpha}\tilde{L}) \to \mathcal{E}_r^+(S)}$$

$$\underline{E} \mapsto \underline{T(E)}\underline{B}$$

[illegible] $T(E) \otimes B \to T(E)B$ bijection

12.4.3 (1)