PBW basis of $\mathcal{B}(V)$ V diagonal type

Lyndon words:

Let $A = \{1, 2, \cdots, d\}$ $1 < 2 < 3 < \cdots < d$

We think A as an alphabet

Let $\mathbb{A}$ be set of nonempty words

For a word

$u = a_1 a_2 \cdots a_r$ $a_i \in A$ $1 \le i \le r$ (length)

Consider on $\mathbb{A}$ the lexicographic order:

$u < v \Leftrightarrow v = uw'$

or $u = w_i u'$, $v = w_j v'$, $i < j$

A word $u \in \mathbb{A}$ is called a lyndon word

if $u = u_1 u_2$ $u_1, u_2 \in \mathbb{A}$ $\Rightarrow u < u_2$

For example

12122 ✓

1212 ✗

Write $L = \{u \in A \mid u \text{ is a lyndon word}\}$

Prop: A word $u \in L \Leftrightarrow u \in A$ or $u = vw$, $v, w \in L$, $v < w$

More precisely,

if w is the proper right factor of maximal length of $u = vw$, $w \in L$,

then $v \in L$, $v < vw < w$

Shirshov decomposition

We take on L the lexicographic order

We say that the elements in L are called super-letters

Super-words

A monotonic super-word is a nonincreasing word on the set of superletters

i.e $v_1 \cdots v_n$ s.t $v_i \in L$

$v_1 \geq v_2 \geq \cdots \geq v_n$

M – monotonic super-word

$L_{\geq u} = \{v \in L \mid v \geq u\}$

$M_{\geq u} = \{v_1 \cdots v_r \in M \mid v_1 \geq v_2 \geq \cdots \geq v_r \geq u\}$

Thm:

A word in $\mathbb{A}$ can be written in a unique way as a monotonic super-word

Moreover, if $u = ~~v_1 \cdots v_n~~$ $v_1 \cdots v_n \in M$

then v_n is the smallest right factor of u

$123123312312 2123$

$= (1231233)(123)(122123)$

Let H_0 be a Hopf algebra with bijective S

Let $V = \bigoplus_{i=1}^{d} V_i$ be a direct sum of Y_D module over H_0

$c : T(V) \otimes T(V) \longrightarrow T(V) \otimes T(V)$

$c(x \otimes y) = x_{-1} \cdot y \otimes x_0$

c^{-1} - - - -

$c(v_i \otimes v_j) = v_j \otimes v_i$

Define $[x, y] = xy - m \circ c^{-1}(x \otimes y)$ (TV)

$[[x, y]] = xy - m \circ c(x \otimes y)$

Def:

$A = \{1, \cdots, d\}$

$V = \bigoplus_{i=1}^{d} V_i$

let $a_1, \cdots, a_m \in A$

$u = a_1 \cdots a_m \in \mathbb{A}$

Write $V^u = V_{a_1} V_{a_2} \cdots V_{a_m} \simeq V_{a_1} \otimes V_{a_2} \otimes \cdots \otimes V_{a_m}$

The elements in V^u is called u-vectors

We shall define inductively bracket operation:

$[\ ] : \bigoplus_{n \geq 0} V^{\otimes n} \to T(V)$

$[[\ ]]$

Let x_u be a u-vector

(1) If $|u| = 1$ $[x_u] = [[x_u]] = x_u$

(2) If $u \in L$ $u = vw$ is the shirshov

$[x_u] = [[x_v], [x_w]]$

(3) M if a word $u = v_1 \cdots v_r$ $x \in V^u$

$[x] = [x_{v_1}][x_{v_2}] \cdots [x_{v_r}]$ (TV)

Def: If u is a Lyndon word, x is a u-vector

then $[x]$ will be called a u-$[\,]$-letter

If u is monotonic super-word x is a u-vector

$[x]$ will be called a u-$[\,]$-word

Let $V^{[u]} = [V^u]$ denote the space of u-$[\,]$-words

A $[\,]$-letter ($[\,]$-word) is a u-$[\,]$-letter (u-$[\,]$-word) for some super-letter u (monotonic super-word)

Lemma 1:

Let $u \in L$

Any product of $[\,]$-letters corresponding to superletters in $L_{\geq u}$ is a linear combination of (monotonic) $[\,]$-words corresponding to super-word in $M_{\geq u}$

Def:

Let $x \in TV \setminus \{0\}$

$x = \sum_n x_n$ where $x_n \in V^{\otimes n}$

the greatest n s.t $x_n \neq 0$ is called the degree of x

Let v be the least u s.t $x_u \neq 0$ $|u| = \text{degree } x$

v – leading vector

Lemma 2:

Let $x \in TV$ be a nonzero u-vector for a monotonic super-word u

then x is the leading vector of $[x]$

Thm: $T(V) = \bigoplus_{u \in M} V^{[u]}$

Proof: Since the letters in A are also super-letters, $V^{[u]}$ generate $T(V)$.

The linear independence follows lemma 2.

Def of PBW-basis of an algebra:

Let A be an algebra, $P, S \subset A$

$h : S \to \mathbb{N} \cup \{\infty\}$

Let $<$ be a total order on S

denote by $B(P, S, <, h)$ the set

$\{p\, s_1^{e_1} s_2^{e_2} \cdots s_t^{e_t} : t \in \mathbb{N}_0,\ s_1 > s_2 > \cdots > s_t,\ p \in P,\ s_i \in S,\ 0 < e_i < h(s_i)\}$

$T(V)$ $B(\{1\}, \{[L]\}, <, h)$ ∞

(V, c) is of diagonal type

$x_1, \cdots, x_\theta$ $\mathbb{X}$ – word

bracket: $[x, y]_c = m \circ (id - c)(x \otimes y)$

Def: Let $u, v \in \mathbb{X}$

we say $u \succ v \Leftrightarrow l(u) < l(v)$

or $l(u) = l(v)$ but $u > v$ 字典

Let now I be a proper Hopf ideal of $T(V)$

set $H = T(V)/I$

Let $\pi : T(V) \to H$ be - - - -

Consider a subset of $\mathbb{X}$

$G_I := \{u \in \mathbb{X} : u \notin \mathbb{X}_{\succ u} + I\}$

Prop: $\pi(G_I)$ is a basis of H.

Let $S_I := G_I \cap L$

define $h_I : S_I \to \{2, 3, \cdots\} \cup \{\infty\}$

by $h_I(u) := \min\{t \in \mathbb{N} : u^t \in \Bbbk \mathbb{X}_{\succ u^t} + I\}$

Thm:

$B_I' := B(\{1 + I\}, [S_I]_c + I, <, h_I)$

is a PBW-basis of $H = T(V)/I$. ($\mathcal{B}(V)$)

Prop: If $v \in S_I$ s.t $h_I(v) < \infty$

then $q_{v,v}$ is a root of unit

order $= h_I(v)$

We can even assume the height of a PBW-generator $[u] < \infty$

$\Leftrightarrow 2 \leq \text{ord}(q_{u,u}) < \infty$ ($h = \text{ord}$)

for a PBW-generator x if $h_I(x) = \infty$

but $\text{ord}(q_{x,x}) = m$

we can restrict the height of x to m

and add x^m to the generator

$h(x^m) = \infty$

$\text{ord}(q_{x^m, x^m}) = (q_{x,x})^{m^2} = 1$

$\notin [2, +\infty)$