Let $\Bbbk$ be a field of char 0, G - abelian

Let $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ be a finite dimensional YD-module with completely reducible $\Bbbk G$-action.

$\delta : V \to \Bbbk G \otimes V$

$\cdot : \Bbbk G \otimes V \to V$

Set $n = \dim V$

By assumption, $\exists\ q_{ij} \in \Bbbk$

a basis $\{x_i \mid 1 \le i \le n\}$

$\forall i\ \exists\ g_i \in G$.

s.t. $g_i \cdot x_j = q_{ij} x_j$

$\delta(x_j) = g_j \otimes x_j$

$\sigma : \mathrm{End}_{\Bbbk}(V \otimes V)$

$\sigma(v \otimes w) = v_{-1} \cdot w \otimes v_0$

$\sigma^{-1}(v \otimes w) = w_0 \otimes S^{-1}(w_{-1}) \cdot v$

$\mathcal{B}(V)$ - Nichols algebra

$\mathcal{B}(V) = \Bbbk \oplus V \oplus \bigoplus_{m \ge 2}^{\infty} V^{\otimes m} / \ker S_m$

Ω_n

$\mathcal{B}(V)$ is $\mathbb{Z}^n$-graded, where the degrees of the generator x_i are $\deg x_i = e_i$

$\{e_i \mid 1 \le i \le n\}$ is a basis of $\mathbb{Z}$-mod $\mathbb{Z}^n$.

$T(V)$ / $J(V)$ — $\mathbb{Z}^n$-graded

Note that V is additionally a Y-D module over $\Bbbk \mathbb{Z}^n$ where

$e_i \triangleright x_j = q_{ji} x_j$

$\gamma(x_j) = e_j \otimes x_j$

In order to avoid misunderstandings, denote by e_i^{-1} the inverse of e_i in $\mathbb{Z}^n$

Claim:

σ commutes both with the action of g_i and e_i

$\sigma(g_i \cdot (x_j \otimes x_m)) = g_i \cdot \sigma(x_j \otimes x_m)$

$\sigma(e_i \triangleright (x_j \otimes x_m)) = e_i \triangleright \sigma(x_j \otimes x_m)$

LHS $= \sigma(q_{ij} q_{im} (x_j \otimes x_m))$

$=$ RHS (域的乘法是交换的)

Let $\{y_i \mid 1 \le i \le n\}$ denote the basis of V^* dual to $\{x_i\}$

Left and right skew-differential operator y_i^L, y_i^R $(1 \le i \le n)$

$\mathcal{B}(V) \cong \Bbbk \oplus V \oplus \bigoplus_{m \ge 2}^{\infty} V^{\otimes m} / \ker S_m$

$$\boxed{S_m = \prod_{j=1}^{m-1} (id^{\otimes m-j-1} \otimes S_{1,j})}$$

$$= \prod_{j=1}^{m-1} (S_{j,1} \otimes id^{\otimes m-j-1})$$

$S_{1,j} = id + \sigma_{12}^{-1} + \sigma_{12}^{-1}\sigma_{23}^{-1} + \cdots + \sigma_{12}^{-1}\sigma_{23}^{-1} \cdots \sigma_{j,j+1}^{-1}$

$S_{j,1} = id + \sigma_{j,j+1}^{-1} + \sigma_{j,j+1}^{-1}\sigma_{j-1,j}^{-1} + \sigma_{j,j+1}^{-1}\sigma_{j-1,j}^{-1} \cdots \sigma_{23}^{-1}\sigma_{12}^{-1}$

取 $m=2$ $\quad S_m = S_{1,1}$

$S_{1,1} = id + \sigma_{12}^{-1}$

$S_2 = id + \sigma_{12}^{-1}$

$\Omega_2 = id + \sigma$

for $m \in \mathbb{N}$, $\rho \in \mathcal{B}(V)_m$, $i \in \{1, 2, \ldots, n\}$

Set $y_i^L(1) = y_i^R(1) = 0$

$y_i^L(\rho) = \rho_i'$ $\quad y_i^R(\rho) = \rho_i''$

where $S_{1,m-1}(\rho) = \sum_{l=1}^{n} x_l \otimes \rho_l'$

$S_{m-1,1}(\rho) = \sum_{l=1}^{n} \rho_l'' \otimes x_l$

$y_3^L(x_1 x_2 x_3 x_4 x_5)$ $\quad m = 5$

$= q_{32} q_{31} x_1 x_2 x_4 x_5$

One can consider

$\mathrm{Lin}_{\Bbbk}\{y_i^L \mid 1 \le i \le n\} \overset{\text{as V.S}}{=} V^*$

as a Y-D module over $\Bbbk G$ dual to V

$g_i \cdot y_j^L = q_{ij}^{-1} y_j^L$

$\delta(y_j^L) = g_j^{-1} \otimes y_j^L$

Similarly, $\mathrm{Lin}_{\Bbbk}\{y_i^R \mid 1 \le i \le n\}$ becomes a YD-module over $\Bbbk \mathbb{Z}^n$ dual to V

$e_i \triangleright y_j^R = q_{ji}^{-1} y_j^R$

$\gamma(y_j^R) = e_j^{-1} \otimes y_j^R$

Moreover, y_i^L, y_i^R and σ satisfies

$(y_i^L \otimes id)(\sigma^{-1}(x_j \otimes x_m)) = (g_i^{-1} \cdot x_j)\, y_i^L(x_m)$

$(id \otimes y_i^R)(\sigma^{-1}(x_j \otimes x_m)) = y_i^R(x_j)\, e_i^{-1} \triangleright x_m$

$(id \otimes y_i^R)(x_m \otimes q_{mj}^{-1} x_j)$

LHS $= (y_i^L \otimes id)(x_m \otimes S^{-1}(g_m) \cdot x_j)$

RHS $= (g_i^{-1} \cdot x_j)\, y_i^L(x_m)$ $\quad i = m$

therefore equation (4) give that

$y_i^L(\rho_1 \rho_2) = y_i^L(\rho_1)\rho_2 + g_i^{-1} \cdot \rho_1\, y_i^L(\rho_2)$

$y_i^R(\rho_1 \rho_2) = \rho_1\, y_i^R(\rho_2) + y_i^R(\rho_1)\, e_i^{-1} \triangleright \rho_2$

$\rho_1, \rho_2 \in \mathcal{B}(V)$

y_i^L $\quad \overbrace{a_{i_1} \cdots a_{i_m}}^{\rho_1}\ \overbrace{b_{j_1} \cdots b_{j_n}}^{\rho_2}$

x_i

$\exists x_i$ $\quad \exists x_i$

Note that by def

$x_j x_m$