模论与表示论初步 第十六讲

Young 子群. λ ~ Young 表. λ' - 转置 Young 表. P_λ. Q_λ

1.3. Young 对称化子 (Symmetrizer).

给定 λ ~ Young 表. 我们得到 Young 子群 P_λ. Q_λ

令 $a_\lambda := \sum_{g\in P_\lambda} g \in \mathbb{C}S_n$; $b_\lambda := \sum_{g\in Q_\lambda} \mathrm{sgn}(g)\, g \in \mathbb{C}S_n$.

定义: 称 $c_\lambda := a_\lambda b_\lambda \in \mathbb{C}S_n$ 为 λ 对应的 Young 对称化子.

例: (1). $\lambda = \boxed{1}\boxed{2}\boxed{3}\boxed{\cdots\cdots}\boxed{n}$. $P_\lambda = S_n$. $Q_\lambda = 1$.

$$a_\lambda = \sum_{g\in S_n} g, \quad b_\lambda = 1. \quad c_\lambda = a_\lambda$$

$$c_\lambda \cdot c_\lambda = \sum_{g\in S_n} g \cdot \sum_{h\in S_n} h = |G|\, c_\lambda$$

$$\underset{\text{left module}\to}{\mathbb{C}S_n \cdot c_\lambda} = \mathbb{C}c_\lambda \leftarrow \text{left module}.$$

c_λ provides a 1-dimensional representation of S_n: $\mathbb{C}S_n c_\lambda = \mathbb{C}c_\lambda$ which is irreducible.

(2) $\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline \vdots\\\hline n\\\hline\end{array}$ Young-表

$P_\lambda = 1$, $Q_\lambda = S_n$. $a_\lambda = 1$. $b_\lambda = \sum_{g\in S_n} \mathrm{sgn}(g)\, g$

$c_\lambda = a_\lambda b_\lambda = b_\lambda$. $\mathbb{C}S_n \cdot c_\lambda$

$$h\cdot \sum_{g\in S_n} \mathrm{sgn}(g)\, g = \sum_{g\in S_n} \mathrm{sgn}(g)\, hg = \mathrm{sgn}(h)\sum_{g\in S_n} \mathrm{sgn}(h)\,\mathrm{sgn}(g)\, hg = \mathrm{sgn}(h)\sum_{g\in S_n} \mathrm{sgn}(hg)\, hg$$

$= \mathrm{sgn}(h)\, c_\lambda$. $\mathbb{C}S_n c_\lambda = \mathbb{C}c_\lambda$.

$\therefore$ c_λ 也提供了一个一维不可约的表示 —— 符号表示.

(3) S_3. $\begin{array}{|c|c|}\hline 1 & 2\\\hline 3 & \\\cline{1-1}\end{array}$ ~ λ

$P_\lambda = \{1, (12)\}$. $\begin{array}{|c|c|}\hline 1 & 3\\\hline 2 & \\\cline{1-1}\end{array}$ $Q_\lambda = \{1, (13)\}$.

$a_\lambda = 1 + (12)$, $b_\lambda = 1 - (13)$.

$$c_\lambda = (1+(12))(1-(13)) = 1 + (12) - (13) - (132)$$

$\mathbb{C}S_3 \cdot c_\lambda$: $(12)c_\lambda = (12) + (1) - (132) - (13)$; $(13)c_\lambda = (13) + (123) - 1 - (23)$; $(23)c_\lambda = (23) + (132) - (123) - (12)$

$(123)c_\lambda = (123) + (13) - (23) - 1$; $(132)c_\lambda = (132) + (23) - (12) - (123)$.

$\mathbb{C}S_3 \cdot c_\lambda$ 是二维表示 (这是个不可约的表示) ~ Standard Representation.

1.4 S_n 的所有不可约的表示

定理：设 $A=\mathbb{C}S_n$，则

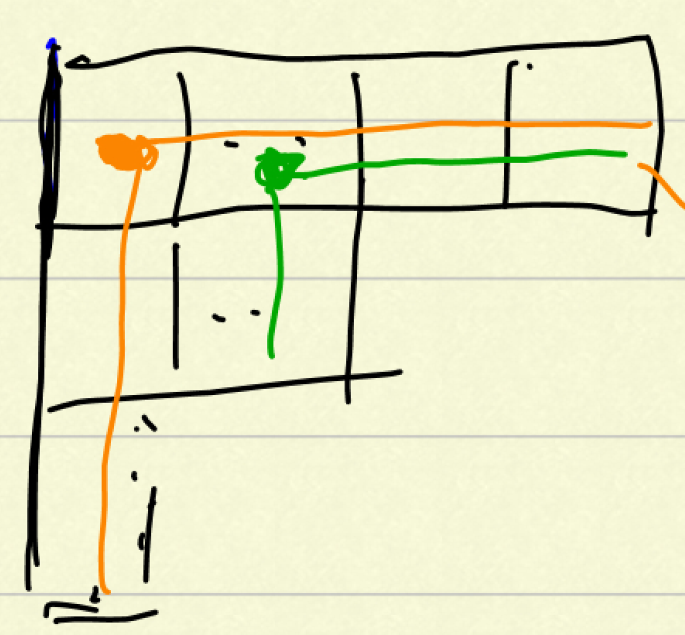
(1) Ac_λ 为 A 的一个不可约的表示.

(2) 如果 $\lambda \neq \mu$ (λ, μ 是两个 Young 图)，则 作为表示 $Ac_\lambda \not\cong Ac_\mu$.

说明：上述定理给出了 S_n 的所有的互不同构的不可约的表示.

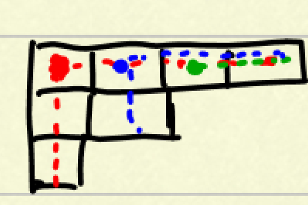
1.5 Character formula. (特征标公式).

① Hook length formula. — 确定不可约的表示的维数.



→ hook length. 这个 hook 里所有 box 的数目之和

Example:

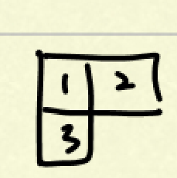


• → hook length = 6
• → hook length = 2
• → hook length = 4

定理：令 λ 为 Young 图，对应的不可约的表示 V_λ，则

$$\boxed{\dim V_\lambda = \frac{|S_n|}{\prod \text{hook length}}}$$

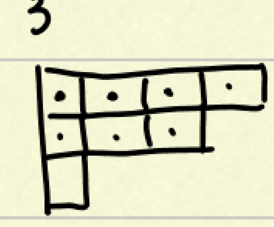
例 ① S_3.



$\dim Ac_\lambda = 2$.

$$\dim V_\lambda = \frac{|S_3|}{\prod \text{hook length}} = \frac{6}{3} = 2.$$

② S_8. $\lambda = (4,3,1)$.



6, 4, 3, 1
4, 2, 1
1

$$\dim V_\lambda = \frac{|S_8|}{6\times 4\times 3\times 4\times 2} = 70.$$

② 一般的特征标公式.

• Conjugacy class → λ (partition) S_n

For any partion λ, define $\vec{i}_\lambda := (i_1, i_2, \dots, i_n)$ 其中 $i_j := \#\{\text{cycles of length } j \text{ in } \lambda\}$.

Example: S_3. $(1) \dashrightarrow i_{(1)} = (3,0,0)$

$(12) \dashrightarrow i_{(12)} = (1,1,0)$. $(123) \dashrightarrow i_{(123)} = (0,0,1)$.

- $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ partition.

令 $x_1, x_2, \cdots x_k$ 为 k 个未定元.

定义: $P_j^\lambda(x) = x_1^j + x_2^j + \cdots + x_k^j \quad (1 \le j \le n)$

$$\Delta_\lambda(x) = \prod_{i<j} (x_i - x_j).$$

- 如果 $f(x) \in \mathbb{C}[x_1, \ldots, x_k]$.

$$[f(x)]_{l_1, \ldots, l_k} := x_1^{l_1} \cdots x_k^{l_k} \text{ 前面的系数}.$$

- 现在给定: 对应 partition λ, 令

$$l_1^\lambda = \lambda_1 + k - 1, \quad l_2^\lambda = \lambda_2 + k - 2, \cdots, l_k^\lambda = \lambda_k.$$

定理 (Frobenius). Let μ be a partion of n. $\vec{i}_\mu = (i_1, i_2, \cdots, i_n)$. $g \in C_\mu$ (μ 对应的共轭类)

$$\chi_\lambda(g) = \left[\Delta_\lambda(x) \prod_{j=1}^{n} P_j^\lambda(x)^{i_j} \right]_{(l_1^\lambda, l_2^\lambda, \ldots, l_k^\lambda)}$$

例. S_3. $\lambda = (2, 1)$. ⊞ $l_1^\lambda = 3$, $l_2^\lambda = 1$

$$\Delta_\lambda(x) = x_1 - x_2. \qquad P_1(x) = x_1 + x_2. \cdots P_3(x) = x_1^3 + x_2^3.$$

- (1), now $\vec{i}_{(1)} = (3, 0, 0)$

$$\chi_\lambda((1)) = [(x_1 - x_2) P_1^3(x)]_{(3,1)} = 2.$$

- (12). $\vec{i}_{(12)} = (1, 1, 0)$.

$$\chi_\lambda((12)) = [(x_1 - x_2) P_1(x) P_2(x)]_{(3,1)} = 0$$

- (123) $\vec{i}_{(123)} = (0, 0, 1)$

$$\chi_\lambda((123)) = [(x_1 - x_2) P_3(x)]_{(3,1)} = -1.$$

2. Algebraic Group. Lie Group. Rep.

(1) 代数群.

Structure: $1 \to G^\circ \to G \to G/G^\circ \to 1$

G°: Connected component; G/G°: finite gp

$1 \to H \to G^\circ \to G^\circ/H \to 1$

H: Affine; G°/H: Abelian variety.

$1 \to R(H^\circ) \to H^\circ \to H^\circ/R(H^\circ) \to 1$

$R(H^\circ)$: 可解 (相对简单); $H^\circ/R(H^\circ)$: Semisimple

one can use Lie alg to study it.

3. Schur-Weyl duality.

Frobenius — Schur. 让 Schur 去考虑 $GL_n(\mathbb{C})$ 的表示

- 1901. Schur 定义了一类代数 $S_{\mathbb{C}}(n,r)$. — Schur 代数.
- 1927. Schur gave a new way to study the rep of $GL_n(\mathbb{C})$.

  $GL_n(\mathbb{C}) \curvearrowright V = \mathbb{C}^n$ Consider symmetric gp S_r

  $GL_n(\mathbb{C}) \curvearrowright V^{\otimes r} \curvearrowleft S_r$
- 1939. popularize schur's obeseration.

定理. (双中心化子定理).

设 E 是一个有限维的线性空间. $A, B \subseteq End(E)$ 的两个子代数. 假设 A 是半单的. $B = End_A(E)$. 则

(1) $A = End_B(E)$ (2) B 半单.

Schur-Weyl Duality

定理: $GL(V)$ 对角地作用在 $V^{\otimes r}$ 上 $(g\cdot(v_1\otimes\cdots\otimes v_r) = gv_1\otimes\cdots\otimes g\cdot v_r)$. S_r 从右侧以置换的形式作用在 $V^{\otimes r}$ 上 $((v_1\otimes\cdots\otimes v_r)\cdot\sigma = v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(r)})$. 故

$\rho_1: GL(V) \longrightarrow End(V^{\otimes r})$, $\rho_2: S_r \longrightarrow End(V^{\otimes r})$. 则

$A = End_B(V^{\otimes r})$, $B = End_A(V^{\otimes r})$. 其中 $A = \langle Im\,\rho_1 \rangle$, $B = \langle Im\,\rho_2 \rangle$ subalg.

考前辅导:

(1) Maschke 定理及其证明. (两种证明)

(2) 会计算一些简单群的特征标. (S_3, S_4, Q_8, D_8).

(3). 会给出诱导表示的不可约分解. ($V_{S_3}^{S_4}$, $V_{\cdot}^{Q_8}$)

Frobenius 互反律.

(4) 会确定简单紧群的不可约表示 (S^1, SU_2)

(5) 会画 Young 图 会写对应的 Young 对称子.

(6) 会计算张量积分解公式. (7) 探讨一下 $V \cong V^*$ 作用表示的充分必要条件.