Def 2.1.1 (monoidal cate):

A monoidal cate is a quintuple $(\mathcal{C}, \otimes, a, 1, \iota)$ where $\mathcal{C}$ is a cate, $\otimes : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ is a bifunctor called the tensor product bifunctor (def of bifunctor, see rotman ex 2.35). $a : (-\otimes-)\otimes- \xrightarrow{\sim} -\otimes(-\otimes-)$ is a <u>natural</u> isom:

(2.1) $\quad a_{X,Y,Z} : (X\otimes Y)\otimes Z \xrightarrow{\sim} X\otimes(Y\otimes Z), \quad X, Y, Z \in \mathcal{C}.$

called the associativity constraint, $1 \in \mathcal{C}$, an obj of $\mathcal{C}$. and $\iota : 1\otimes 1 \xrightarrow{\sim} 1$ is an isom, and satisfy the following two axioms.

1. Pentagon axiom The diagram:

(2.2)
$$\begin{tikzcd}
 & ((W\otimes X)\otimes Y)\otimes Z \arrow[dl, "a_{W,X,Y}\otimes \mathrm{id}_Z"'] \arrow[dr, "a_{W\otimes X,Y,Z}"] & \\
(W\otimes(X\otimes Y))\otimes Z \arrow[d, "a_{W,X\otimes Y,Z}"] & & (W\otimes X)\otimes(Y\otimes Z) \arrow[d, "a_{W,X,Y\otimes Z}"] \\
W\otimes((X\otimes Y)\otimes Z) \arrow[rr, "\mathrm{id}_W\otimes a_{X,Y,Z}"] & & W\otimes(X\otimes(Y\otimes Z))
\end{tikzcd}$$

2. The unit axiom: The functors:

(2.3). $\quad L_1 : X \longmapsto 1\otimes X$ and $\quad$ are autoequivalence of $\mathcal{C}$.

(2.4) $\quad R_1 : X \longmapsto X\otimes 1.$

We say the pair $(1, \iota)$ the unit obj of $\mathcal{C}$.

Why call it monoidal cate?

Reason: The set of isom classes of objs in a monoidal cate indeed has a natural structure of monoid, with multiplication $\otimes$ and unit 1.

2.1.4: A monoidal subcate of a monoidal cate $(\mathcal{C}, \otimes, a, 1, \iota)$ is a quintuple $(\mathcal{D}, \otimes, a, 1, \iota)$, where $\mathcal{D} \subset \mathcal{C}$ is a subcate closed under the tensor product of objs and morphisms and containing $1, \iota$.

2.1.5: $\mathcal{C}^{op} := (\mathcal{C}^{op}, \otimes^{op}, 1, a^{op}, \iota)$. As a category, <u>$\mathcal{C}^{op} = \mathcal{C}$</u>. Its tensor product: $X \otimes^{op} Y := Y \otimes X$ and associativity constraint of $\mathcal{C}^{op}$ is $a^{op}_{X,Y,Z} := a^{-1}_{Z,Y,X} : Z\otimes(Y\otimes X) \xrightarrow{\sim} (Z\otimes Y)\otimes X$. $\mathcal{C}^{op}$ is a monoidal cate :

pentagon axiom :

$$\begin{tikzcd}[column sep=small]
 & Z\otimes(Y\otimes(X\otimes W)) \arrow[dl, "\mathrm{id}_Z\otimes a^{-1}_{Y,X,W}"'] \arrow[dr, "a^{-1}_{Z\otimes Y,X,W}"] & \\
Z\otimes((Y\otimes X)\otimes W) \arrow[d, "a^{-1}_{Z,Y\otimes X,W}"] & & (Z\otimes Y)\otimes(X\otimes W) \arrow[d, "a^{-1}_{Z,Y,X\otimes W}"] \\
(Z\otimes(Y\otimes X))\otimes W \arrow[rr, "a^{-1}_{Z,Y,X}\otimes \mathrm{id}_W"] & & ((Z\otimes Y)\otimes X)\otimes W
\end{tikzcd}$$

By (2.2)

$\text{LHS} = (\mathrm{id}_W \overset{op}{\otimes} a^{op}_{X,Y,Z})(a^{op}_{W,X\overset{op}{\otimes}Y,Z})(a^{op}_{W,X,Y} \overset{op}{\otimes} \mathrm{id}_Z)$

$\text{RHS} = (a^{op}_{W,X,Y\overset{op}{\otimes}Z}) \cdot (a^{op}_{W\overset{op}{\otimes}X,Y,Z})$

2.1.6 <u>Caution !</u> $C^\vee$ (dual cate of $\mathcal{C}$) "$\neq$" $\mathcal{C}^{op}$ (opposite cate of $\mathcal{C}$)

obtained from $\mathcal{C}$ by reversing arrows. $\mathcal{C}^\vee$ is a monoidal cate of $\mathcal{C}$ is.

$L_1(l_X)$:

$(\mathcal{C}, \otimes, a, 1, \iota)$: monoidal cate. Define <u>natural isom</u> :

(2.5) $\quad l_X : 1\otimes X \longrightarrow X \quad$ and $\quad r_X : X\otimes 1 \longrightarrow X$

in such a way s.t $L_1(l_X)$ and $R_1(r_X)$ are equal, resp. to the composition.

(2.6) : $L_1(l_X) (= \mathrm{id}_1 \otimes l_X) \quad = \quad 1\otimes(1\otimes X) \xrightarrow{a^{-1}_{1,1,X}} (1\otimes 1)\otimes X \xrightarrow{\iota\otimes \mathrm{id}_X} 1\otimes X$

(2.7) : $R_1(r_X) (= r_X \otimes \mathrm{id}_1) \quad = \quad (X\otimes 1)\otimes 1 \xrightarrow{a_{X,1,1}} X\otimes(1\otimes 1) \xrightarrow{\mathrm{id}_X\otimes \iota} X\otimes 1.$

i.e. $\mathrm{id}_1\otimes l_X = (\iota\otimes \mathrm{id}_X)\, a^{-1}_{1,1,X}$

$r_X\otimes \mathrm{id}_1 = (\mathrm{id}_X\otimes \iota)\, a_{X,1,1}.$

2.2.1 : Isom (2.5) are called left and right unit constraints.

2.2.2 : For any obj $X\in\mathcal{C}$, $\exists$ equalities :

(2.8) $\quad l_{1\otimes X} = \mathrm{id}_1\otimes l_X \quad$ and $\quad r_{X\otimes 1} = r_X\otimes \mathrm{id}_1.$

pf : It follows from the naturality of unit constraint :

$$\begin{tikzcd}
1\otimes(1\otimes X) \arrow[r, "\mathrm{id}_1\otimes l_X"] \arrow[d, "l_{1\otimes X}"'] & 1\otimes X \arrow[d, "l_X"] \\
1\otimes X \arrow[r, "l_X"] & X
\end{tikzcd}
\quad \overset{l_X \text{ is isom}}{\Longrightarrow} \quad \mathrm{id}_1\otimes l_X = l_{1\otimes X}$$

Similarly

$$\begin{tikzcd}
(X\otimes 1)\otimes 1 \arrow[r, "r_X\otimes \mathrm{id}_1"] \arrow[d, "r_{X\otimes 1}"'] & X\otimes 1 \arrow[d, "r_X"] \\
X\otimes 1 \arrow[r, "r_X"] & X
\end{tikzcd}$$

2.2.3 The "triangle" diag:

$$\begin{tikzcd}
(X\otimes 1)\otimes Y \arrow[rr, "a_{X,1,Y}"] \arrow[dr, "r_X\otimes id_Y"'] & & X\otimes(1\otimes Y) \arrow[dl, "id_X\otimes l_Y"] \\
& X\otimes Y &
\end{tikzcd}$$

Pf: Consider

$$\begin{tikzcd}
((X\otimes 1)\otimes 1)\otimes Y \arrow[rrrr, "a_{X,1,1}\otimes id_Y"] \arrow[dr, "r_{X\otimes 1}\otimes id_Y"] \arrow[dd, "a_{X\otimes 1,1,Y}"'] & & & & (X\otimes(1\otimes 1))\otimes Y \arrow[dlll, "id_X\otimes l\otimes id_Y"'] \arrow[dd, "a_{X,1\otimes 1,Y}"] \\
& (X\otimes 1)\otimes Y \arrow[dr, "a_{X,1,Y}"] & & & \\
(X\otimes 1)\otimes(1\otimes Y) \arrow[rr, "r_X\otimes id_{1\otimes Y}"] \arrow[drr, "a_{X,1,1\otimes Y}"'] & & X\otimes(1\otimes Y) & & X\otimes((1\otimes 1)\otimes Y) \arrow[ll, "id_X\otimes(l\otimes id_Y)"'] \arrow[dll, "id_X\otimes a_{1,1,Y}"] \\
& & X\otimes(1\otimes(1\otimes Y)) \arrow[u, "id_X\otimes l_{1\otimes Y}"] & &
\end{tikzcd}$$

It suffices to show the commutativity of "↪" (Since $\forall Y\in\mathcal{C}$, Y is iso to $1\otimes Y$).

1° The outside pentagon is commutative.

2° "↻" commutes by naturality of asso constraint.

3° "↪" commutes by def of r.

4° "↪" commutes by 2.2.2. Since $id_1\otimes l_Y \overset{(2.6)}{=} (l\otimes id_Y)\, a_{1,1,Y}^{-1}$
$\quad\|$ 2.2.2
$\quad l_{1\otimes Y}$.

2.2.4:

(2.12)
$$\begin{tikzcd}
(1\otimes X)\otimes Y \arrow[rr, "a_{1,X,Y}"] \arrow[dr, "l_X\otimes id_Y"'] & & 1\otimes(X\otimes Y) \arrow[dl, "l_{X\otimes Y}"] \\
& X\otimes Y &
\end{tikzcd}$$

(2.13)
$$\begin{tikzcd}
(X\otimes Y)\otimes 1 \arrow[rr, "a_{X,Y,1}"] \arrow[dr, "r_{X\otimes Y}"'] & & X\otimes(Y\otimes 1) \arrow[dl, "id_X\otimes r_Y"] \\
& X\otimes Y &
\end{tikzcd}$$

Pf: We only show (2.12). Consider:

$$\begin{tikzcd}
((X\otimes 1)\otimes Y)\otimes Z \arrow[rrrr, "a_{X,1,Y}\otimes id_Z"] \arrow[drr, "(r_X\otimes id_Y)\otimes id_Z"'] \arrow[dd] & & & & (X\otimes(1\otimes Y))\otimes Z \arrow[dll, "(id_X\otimes l_Y)\otimes id_Z"] \arrow[dd] \\
& & (X\otimes Y)\otimes Z \arrow[d, "a_{X,Y,Z}"] & & \\
(X\otimes 1)\otimes(Y\otimes Z) \arrow[rr, "r_X\otimes id_{Y\otimes Z}"] \arrow[drr, "a_{X,1,Y\otimes Z}"'] & & X\otimes(Y\otimes Z) & & X\otimes((1\otimes Y)\otimes Z) \arrow[ll, "id_X\otimes(l_Y\otimes id_Z)"'] \arrow[dll] \\
& & X\otimes(1\otimes(Y\otimes Z)) \arrow[u, "id_X\otimes l_{Y\otimes Z}"'] & &
\end{tikzcd}$$

We need to show the "$\circlearrowright$" commute.

1° The pentagon commutes

2° "$\circlearrowright$" naturality of asso constraint

3° "$\circlearrowright$" follows from prop 2.2.3

4° Set $X = 1$ and applying L_1^{-1} to the lower right triangle

Cor 2.2.5: In any monoidal cate $l_1 = r_1 = \iota$.

Pf: Set $Y = X = 1$ in (2.12). Then, $l_1 \otimes id_1 = l_{1\otimes 1}\, a_{1,1,1} \overset{(2.8)}{=} (id_1 \otimes l_1)\circ a_{1,1,1}$.

Set $X = Y = 1$ in (2.10) Then $r_1 \otimes id_1 = (id_1 \otimes l_1)\, a_{1,1,1}$

By def of unit constraint $(id_1 \otimes l_1)\circ a_{1,1,1} = \iota \otimes id_1$ $\Rightarrow r_1 \otimes id_1 = l_1 \otimes id_1 = \iota \otimes id_1$.

Since R_1 is an autoequivalence. (apply R_1^{-1})

$\Rightarrow r_1 = l_1 = \iota$.

2.2.6: The unit obj in a monoidal cate is unique up to a unique isom.

Pf: Let $(1, \iota)$ and $(1', \iota')$ be two unit objs. Let (r, l) and (r', l') be the corresponding unit constraints. Then we have the isom: $\eta := l_{1'} \circ (r'_1)^{-1} : 1 \xrightarrow{\sim} 1'$

$$\begin{tikzcd}
1 \arrow[r, "\sim"] \arrow[d] & 1' \\
1\otimes 1' \arrow[ur] &
\end{tikzcd}$$

Claim: η maps ι to ι'

We want to show:

$$
\begin{tikzcd}
1\otimes 1 \arrow[d, "id_1\otimes\eta"'] \arrow[dr, "\iota = r_1"] & \\
1\otimes 1' \arrow[r, "r'_1"] \arrow[d, "\eta\otimes id_{1'}"'] & 1 \arrow[d, "\eta"] \\
1'\otimes 1' \arrow[r, "r'_1"] & 1'
\end{tikzcd}
$$

The whole diag com $\Rightarrow$ η maps ι to ι'

The rectangle com since r' is natural isom.

We want to show:

$$
\begin{tikzcd}
X\otimes(1\otimes 1') \arrow[rr, "id_X\otimes r'_1"] \arrow[dr] \arrow[ddr, "id_X\otimes l_{1'}"'] & & X\otimes 1 \arrow[dd, "r_X"] \\
 & (X\otimes 1)\otimes 1' \arrow[ur, "r'_{X\otimes 1}"] \arrow[d, "r_X\otimes id_{1'}"] \arrow[u, phantom, "2.2.4\ \circlearrowleft"] \arrow[l, phantom, "2.2.3\ \circlearrowleft"] \arrow[r, phantom, "\circlearrowleft"] & \\
 & X\otimes 1' \arrow[r, "r'_X"'] & X
\end{tikzcd}
$$

$\Rightarrow r'_1\circ(id_1\otimes l_{1'})\circ(id_1\otimes r'_1)^{-1}$
$= r_1$

" $\circlearrowleft$ " naturality of r'

" $\circlearrowleft$ " prop 2.2.3. + prop 2.2.4

It remains to show η is the only isom with the property.

It suffices to show if $b: 1 \xrightarrow{\sim} 1$ is the isom s.t the diag commutes.

(2.15)
$$
\begin{tikzcd}
1\otimes 1 \arrow[r, "b\otimes b"] \arrow[d, "\iota"'] & 1\otimes 1 \arrow[d, "\iota"] \\
1 \arrow[r, "b"] & 1
\end{tikzcd}
$$

then $b = id_1$. To see this, it suffices to note: for any morphism $c: 1 \to 1$ the diag commutes

(2.16)
$$
\begin{tikzcd}
1\otimes 1 \arrow[r, "c\otimes id_1"] \arrow[d, "r_1 = \iota"'] & 1\otimes 1 \arrow[d, "\iota = r_1"] \\
1 \arrow[r, "c"] & 1.
\end{tikzcd}
$$
(since $\iota = r_1$, r_1 is natural isom.)

$b\circ\iota = \iota\circ(b\otimes b) = \iota\circ(b\otimes id_1) \Rightarrow b\otimes b = b\otimes id_1 \Rightarrow b = id_1.$

2.2.8: (a traditional def) A monoidal cate is a sextuple $(\mathcal{C}, \otimes, a, l, r)$ satisfying the pentagon axiom (2.2) and triangle axiom (2.10)

2.2.9: ① prop 2.2.6 implies for a triple $(\mathcal{C}, \otimes, a)$ satisfying a pentagon axiom, being a monoidal cate is a property and not a structure

② Furthermore, one can show the commutativity of the triangle implies implies that in a monoidal cate, one can identify $1 \otimes X$ and $X \otimes 1$ with X using the unit constraints, and assume that the unit morphism are identity

Take $Y = 1$ in (2.10).

$$\begin{tikzcd}
(X \otimes 1) \otimes 1 \arrow[rr] \arrow[dr] & & X \otimes (1 \otimes 1) \arrow[dl] \\
& X \otimes 1 &
\end{tikzcd}$$

$\Rightarrow r_{X \otimes 1} = R_1(r_X)$

R_1 is autoequivalence $\Rightarrow$ we can identify $X \otimes 1$ with X.

Prop 2.2.10: Let $\mathcal{C}$ be a monoidal cate. Then $End_{\mathcal{C}}(1)$ is a com monoid under composition. Furthermore.

$f \otimes g = \iota^{-1} \circ (f \circ g) \circ \iota$ for all $f, g \in End_{\mathcal{C}}(1)$.

Pf: By naturality of unit constraint:

$f \otimes id_1 = r_1^{-1} \circ f \circ r_1$. $\quad id_1 \otimes g = l_1^{-1} \circ g \circ l_1$.

since $r_1 = l_1 = \iota$. $\Rightarrow$

$f \otimes g = (f \otimes id_1) \circ (id_1 \otimes g) = \iota^{-1} \circ (f \circ g) \circ \iota$

$g \otimes f = (id_1 \otimes f) \circ (g \otimes id_1) = \iota^{-1} \circ (f \circ g) \circ \iota$

$f \otimes g = \iota^{-1} \circ (g \circ f) \circ \iota$ $\quad \Rightarrow f \circ g = g \circ f$

$\Rightarrow f \otimes g = g \otimes f$ unit: ok.

$\Rightarrow End_{\mathcal{C}}(1)$ is com monoid

2.3: Examples:

2.3.1: The cate **Sets** form a monoidal cate.

① tensor product: Cartesian product;

② unit obj: one element set;

Caution: The Cartesian product isn't associative, i.e. **Sets** is not strict.

③ $a: (A \times B) \times C \longrightarrow A \times (B \times C)$

$((a, b), c) \longmapsto (a, (b, c))$

④ $l: \{1\} \times A \longrightarrow A$. $\quad r: \cdots$

$(1, a) \longmapsto a$.

pentagon axiom
triangle axiom 同各.

2.3.2: Any additive cate. is monoidal,

① tensor product: direct sum $\oplus$

② unit obj: zero obj

③ $a: (A \oplus B) \oplus C \cong A \oplus (B \oplus C)$

$((a, b), c) \longmapsto (a, (b, c))$

④ $l: 1 \oplus A \longmapsto A$ $\quad (1, a) \mapsto a$.

2.3.3: The cate $\mathbb{k}-Vec$ of all $\mathbb{k}$-v.s. where

① $\otimes = \otimes_{\mathbb{k}}$,

② $\mathbb{1} = \mathbb{k}$.

③ a, l, r are obvious.

more generally, if R is a com unital ring. Then R-Mod is a monoidal cate. (not strict)

2.3.4: Let G be a gp. The cate of $Rep_{\mathbb{k}}(G)$ of all rep of G over $\mathbb{k}$ is a monoidal cate., with $\otimes$ being the tensor product of representations,

1° For a rep V, we denote $\rho_V : G \to GL(V)$.

Then $V \otimes W$ corresponds to $\rho_{V\otimes W}(g) := \rho_V(g) \otimes \rho_W(g)$,

2° The unit obj: Trivial rep $\mathbb{1} = \mathbb{k}$.

2.3.5: Let G be an affine (pro) alg group over $\mathbb{k}$. (an affine algebraic variety with a group structure, s.t the mult map and inverse map are regular (also called a morphism) is a function between the varieties that's given locally by polynomials. The cate of $Rep(G)$. of algebraic representation of G over $\mathbb{k}$ are monoidal categories

the mult and counit are given similiarly

The $Rep(\mathfrak{g})$ of a Lie alg $\mathfrak{g}$. The tensor product: $\rho_{V\otimes W}(a) = \rho_V(a) \otimes id_W + id_V \otimes \rho_W(a)$

$(\rho_V : \mathfrak{g} \to \mathfrak{gl}(V))$

The unit: 1-dim rep of $\mathfrak{g}$, with zero action of $\mathfrak{g}$ i.e $\rho_1(a) = 0$

2.3.6: Let G be a monoid. A be an abelian group. Let $\mathcal{C}_G = \mathcal{C}_G(A)$ be the cate whose obj's δ_g are labelled by elements of G (so there's one element in each isom class). Define $Hom_{\mathcal{C}_G}(\delta_{g_1}, \delta_{g_2}) = \phi$ if $g_1 \neq g_2$.

and $Hom_{\mathcal{C}_G}(\delta_g, \delta_g) = A$.

① tensor product of obj $\delta_g \otimes \delta_h = \delta_{gh}$ tensor product of morphism: $a \otimes b = ab$

② unit obj δ_1.

$\Rightarrow \mathcal{C}_G$ being a monoidal cate with the asso isomorphism being the identity

Note: in $\mathcal{C}_G(A)$. $X \otimes Y \not\cong Y \otimes X$ (for example δ_{gh} may not be equal to δ_{hg})

A "linear" version: $\mathbb{k}-Vec_G$: G-graded v.s. over $\mathbb{k}$.

Vector spaces $V \in \mathrm{Vec}_G$ has a decomposition $V = \bigoplus_{g \in G} V_g$. Morphism in Vec_G preserves the grading

1° tensor product: $V \otimes W$ as vector space tensor product,

and $(V \otimes W)_g = \bigoplus_{\substack{x,y \in G \\ xy = g}} V_x \otimes W_y$

2° unit obj $\mathbb{1}$ by $\mathbb{1}_1 = \Bbbk$ and $\mathbb{1}_g = 0$ for $g \neq 1$.

3° a: $(U \otimes V) \otimes W \xrightarrow{a_{U,V,W}} U \otimes (V \otimes W)$ $\Rightarrow$ Vec_G is a monoidal cate

ι: $\mathbb{1} \otimes \mathbb{1} \longrightarrow \mathbb{1}$.

In Vec_G. We have pairwise obj δ_g defined by $(\delta_g)_x = \Bbbk$ if $x = g$ and $(\delta_g)_x = 0$ otherwise. By def.

$\delta_g \otimes \delta_h \cong \delta_{gh}$. Thus: $\mathcal{C}_G(\Bbbk^\times)$ is a $\underbrace{\text{non-full}}_{\text{(zero morphism miss)}}$ monoidal subcate of Vec_G

This subcate can be viewed as a "linear basis" of Vec_G. As any obj of $\Bbbk$-Vec_G is iso to a direct sum of

objs δ_g with non-negative integer multiplicities