2.8 The Mac Lane strictness theorem.

· Def. 2.8.1 : A monoidal cat. $\mathcal{C}$ is **strict** if for all $X, Y, Z \in obj(\mathcal{C})$

$$(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z), \quad a_{X,Y,Z} = id$$

$$X \otimes 1 = X = 1 \otimes X, \quad l_X = r_X = id.$$

· example :

(1) $End(\mathcal{C})$ is strict.

(2) Sets : not strict.

$n(A) = n \times A$.

$A \times B \cong card(A) \times B$

$\overline{Sets}$ be the cat. : $obj(\overline{Sets}) := \{ n \in \mathbb{Z}^{\geq 0} \}$.

$Hom_{\overline{Sets}}(m, n) := \{ f : \{0, \cdots, m-1\} \longrightarrow \{0, \cdots, n-1\} \mid f \text{ is a map} \}$.

Define the tensor product :

$m \otimes n = mn$.

$\{0, \cdots, m_1 m_2 - 1\}$.

$\forall\ f_1 : m_1 \longrightarrow n_1$ and $f_2 : m_2 \longrightarrow n_2$, $\{0, \cdots, n_1 n_2 - 1\}$.

$f_1 \otimes f_2 : m_1 m_2 \longrightarrow n_1 n_2$ where $0 \leq x \leq m_1 - 1$, $0 \leq y \leq m_2 - 1$

$m_2 x + y \longmapsto n_2 f_1(x) + f_2(y)$ $\quad 0 \leq f_1(x) \leq n_1 - 1 \quad 0 \leq f_2(y) \leq n_2 - 1$.

Then $\overline{Sets}$ is a strict monoidal cat.

And we have $\overline{Sets} \overset{F}{\longleftrightarrow} Sets$ (monoidal equivalence.)

$0, \quad m_2 - 1$

$m_2 \quad \cdots \quad 2m_2$

$\vdots$

$(m_1 - 1) m_2 \quad m_1 m_2$.

$n \longmapsto n$元集合 $\{0, \cdots, n-1\}$

① F is cat. equivalence.

② F is monoidal functor.

$J_{n,m} : F(n) \times F(m) \longrightarrow F(n \otimes m)$ is natural isom.

$$\begin{array}{ccc}
F(n) \times F(m) & \longrightarrow & F(n \otimes m) \\
(y, x) & \longmapsto & my + x \\
\downarrow f_1 \times f_2 & & \downarrow f_1 \otimes f_2 \\
F(n') \times F(m') & \longrightarrow & F(n' \otimes m') \\
(f_1(y), f_2(x)) & \longmapsto & m' f_1(y) + f_2(x)
\end{array}$$

$f_1 : n \to n'$, $f_2 : m \to m'$

symmetric. $c_{n,m}$:

$$\begin{array}{ccc}
nm & \longrightarrow & mn \\
\downarrow & & \downarrow \\
n'm' & \longrightarrow & m'n'
\end{array}$$

(3) k-Vec : non strict.

$\overline{k\text{-Vec}}$, $obj(\overline{k\text{-Vec}}) := \{ n \in \mathbb{Z}^{\geq 0} \}$.

$Hom_{\overline{k\text{-Vec}}}(m, n) := \{ N \in M_{m \times n}(k) \}$.

Define: the tensor cat.
$m \otimes n = mn$.
for $f_1 : m_1 \to n_1$, $f_2 : m_2 \to n_2$.
define $f_1 \otimes f_2 : m_1 m_2 \to n_1 n_2$ kronecker product of f_1, f_2.

Then $k - \overline{Vec}$ is strict monoidal cat.
$k - \overline{Vec} \xhookrightarrow{F} k - Vec$ $\quad n \longmapsto$ 标准基下的 n维 v.s.

$J_{n,m} : F(n) \otimes_k F(m) \longrightarrow F(n \otimes m)$ $\quad 0 \le x \le n-1,\ 0 \le y \le m-1$

$e_x \otimes e_y \longmapsto e_{mx+y}$.

(4) $k - Vec_G$: non strict. $\quad G$ is a group.

$k - \overline{Vec_G}$: $\mathrm{obj}(k - \overline{Vec_G}) := \{ x : G \longrightarrow \mathbb{Z}_+ \mid x$ is a map with finitely many nonzero values $\}$

$\mathrm{Hom}_{k-\overline{Vec_G}}(x, y) := \{ G$-graded linear map $f : \bigoplus_{g \in G} X_g \longrightarrow \bigoplus_{g \in G} Y_g \mid$ where $\dim(X_g) = x(g)$, $\dim(Y_g) = y(g)$. $\}$

Define the tensor product:

$$(x \otimes y)(g) := \sum_{a, b \in G, ab = g} x(a) y(b), \quad \text{unit}: x(1_G) = 1,\ x(g) = 0,\ g \neq 1_G.$$

for $f_1 : x_1 \longrightarrow y_1$, $f_2 : x_2 \longrightarrow y_2$.

$f_1 \otimes f_2$ the tensor product of G-graded linear map.

Then $\overline{Vec_G}$ is strict. $\quad \overline{Vec_G} \xhookrightarrow{F} Vec_G$.

$x \longmapsto V = \bigoplus_{g \in G} V_g$ where $\dim_k(V_g) = x(g)$, basis of V_g is standard basis.

- $\overline{Vec_G}$ is a strict.

$$((x \otimes y) \otimes z))(g) = \sum_{ab = g} \sum_{a_1 a_2 = a} x(a_1) y(a_2) z(b) = \sum_{abc = g} x(a) y(b) z(c)$$

$$(x \otimes (y \otimes z))(g) = \sum_{ab = g} \sum_{b_1 b_2 = b} x(a) y(b_1) z(b_2) = \sum_{abc = g} x(a) y(b) z(c)$$

- F is a monoidal functor.

$J_{x,y} : F(x) \otimes_k F(y) \longrightarrow F(x \otimes y)$

$V \otimes W \longrightarrow H = \bigoplus_{g \in G} H_g \quad \dim(H_g) = \dim((V \otimes W)_g)$

$1 \le i \le x(a)$, $1 \le j \le y(b)$. $\quad e_i^a \otimes e_j^b \longmapsto e^{ab}_{i \cdot y(b) + j}$

$e_i^a \in V_a$, $e_j^b \in V_b$.

· Vec_G^{ω}, Vec_G^{ω} is not strict.

Theorem 2.8.5 (Mac Lane strictness theorem)
Any monoidal category is monoidally equivalent a strict monoidal category.

proof: Aim: $\mathcal{C}$: monoidal category, $\mathcal{C} \simeq \underline{\mathrm{End}}(\mathcal{C})$ (right $\mathcal{C}$-module endofunctors)
↙ strict monoidal

· The right $\mathcal{C}$-module endofunctor $\underline{\mathrm{End}}(\mathcal{C})$ defined as follows:

① The objects: (F, c): where $F: \mathcal{C} \to \mathcal{C}$ is a functor and $c_{X,Y}: F(X) \otimes Y \xrightarrow{\sim} F(X \otimes Y)$ is a natural isom. such that for $\forall X, Y, Z \in \mathrm{obj}(\mathcal{C})$, the following is commuting:

$$\begin{array}{ccc} & (F(X)\otimes Y)\otimes Z & \\ {\scriptstyle c_{X,Y}\otimes id_Z}\swarrow & & \searrow{\scriptstyle a_{F(X),Y,Z}} \\ F(X\otimes Y)\otimes Z & & F(X)\otimes (Y\otimes Z) \\ {\scriptstyle c_{X\otimes Y,Z}}\downarrow & & \downarrow{\scriptstyle c_{X,Y\otimes Z}} \\ F((X\otimes Y)\otimes Z) & \xrightarrow{F(a_{X,Y,Z})} & F(X\otimes (Y\otimes Z)) \end{array} \qquad (2.38)$$

② A morphism: $\theta: (F^1, c^1) \longrightarrow (F^2, c^2)$ in $\underline{\mathrm{End}}(\mathcal{C})$ is a natural transformation $\theta: F^1 \longrightarrow F^2$ s.t. the following commutes.

$$\begin{array}{ccc} F^1(X)\otimes Y & \xrightarrow{c^1_{X,Y}} & F^1(X\otimes Y) \\ \downarrow{\scriptstyle \theta_X\otimes id_Y} & & \downarrow{\scriptstyle \theta_{X\otimes Y}} \\ F^2(X)\otimes Y & \xrightarrow{c^2_{X,Y}} & F^2(X\otimes Y) \end{array} \qquad (2.39).$$

③ $(\theta^2 \circ \theta^1)_X = \theta^2_X \circ \theta^1_X$
(2.39) is satisfied

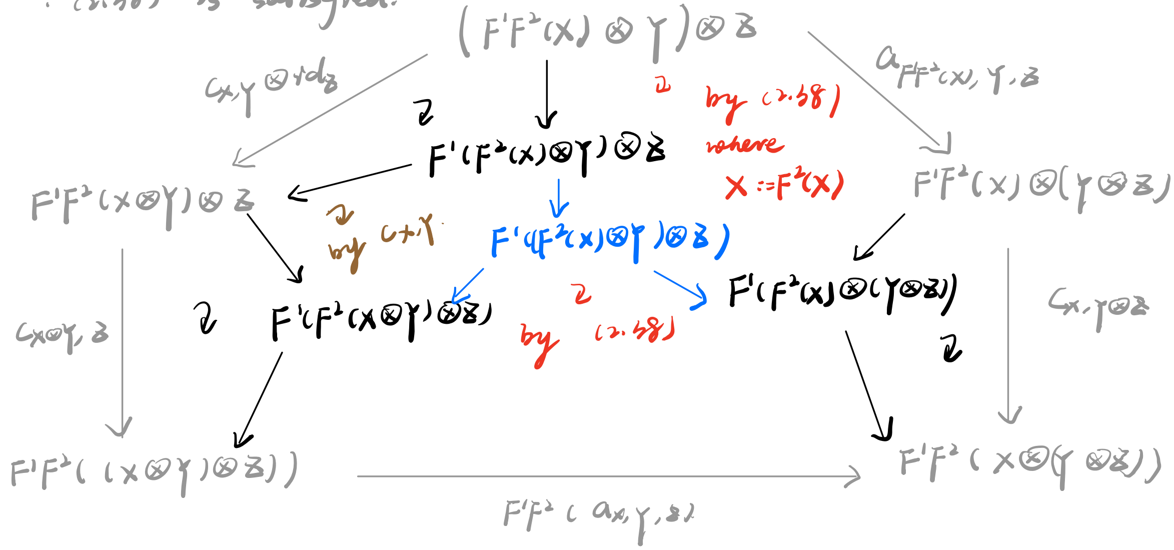
④ The tensor product of objects:

$(F^1, c^1) \otimes (F^2, c^2) = (F^1F^2, c)$ where c is given by.

$$F^1F^2(X)\otimes Y \xrightarrow{c^1_{F^2(X),Y}} F^1(F^2(X)\otimes Y) \xrightarrow{F^1(c^2_{X,Y})} F^1F^2(X\otimes Y). \qquad (2.40)$$

· c is natural isom

· (2.38) is satisfied.

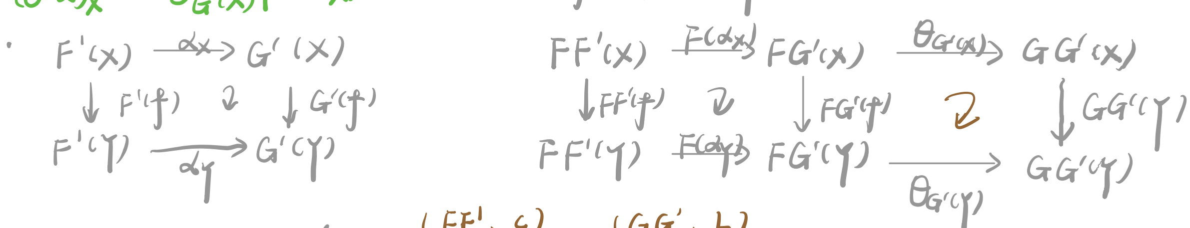


⑥ The tensor product of morphism (F, c^1), (F', c^2), (G, b^1), (G', b^2).

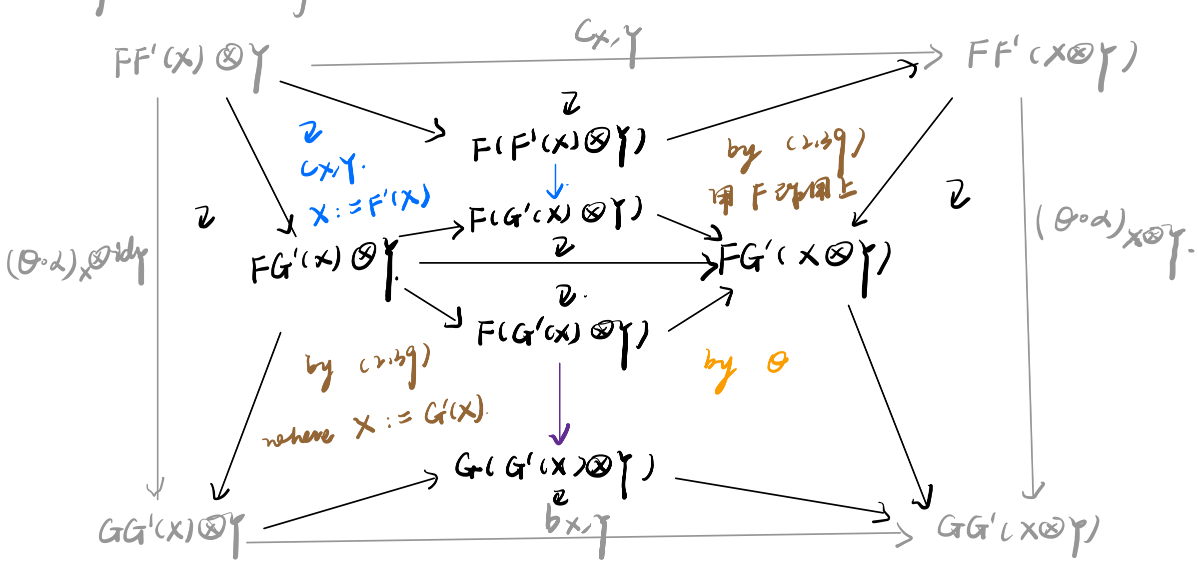
$\theta : F \longrightarrow G$, $\alpha : F' \longrightarrow G'$.

$\theta \otimes \alpha := \theta \circ \alpha$ (horizontal composition) : $FF' \longrightarrow GG'$

$(\theta\circ\alpha)_X := \theta_{G'(X)} F(\alpha_X)$. $\forall f : X \to Y$.



· (2.39) is satisfied. (FF', c) (GG', b)



⑥ $\mathrm{End}(\mathcal{C})$ is a strict monoidal cat.

(i) $\otimes$ is a bifunctor.

(ii) a is id.

(iii) $id_{\mathcal{C}}$

(iv) $l_F = r_F = id_F$

(v) "pentagon" and "triangle"

$\Big\} \Rightarrow$ strict.

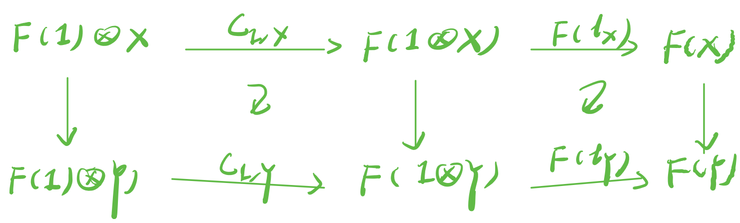
- Consider a functor of left mult. $L : \mathcal{C} \longrightarrow \mathrm{End}(\mathcal{C})$ given by $L(X) = (X\otimes -, a_{X,-,-})$, $L(f) = f \otimes -$

- L is a monoidal equivalence.

① dense : $\forall (F,c) \in \mathrm{End}(\mathcal{C})$, $L(F(1)) = (F(1)\otimes -, a_{F(1),-,-}) \cong (F,c)$.

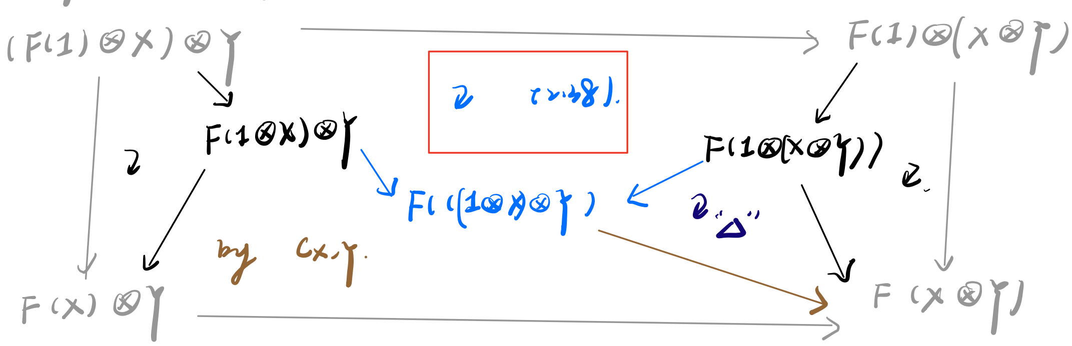
$F(1)\otimes - \xrightarrow{F(l_-)\circ c_{1,-}} F$ is natural isom.

$\forall f : X \to Y$.



引入 $c_{X,Y}$ 的用处!

(2.4.9) is satisfied.



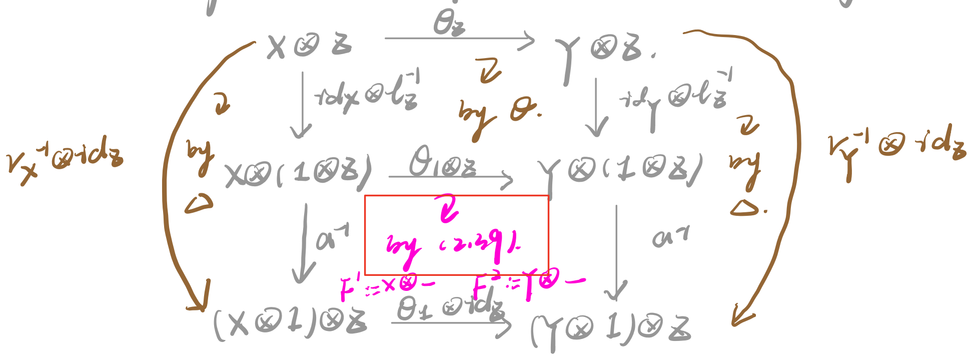
② L is full

let $L(X) \xrightarrow{\theta} L(Y)$ be a morphism in $\mathrm{End}(\mathcal{C})$

define: $f : X \to Y$.

$X \xrightarrow{r_X^{-1}} X\otimes 1 \xrightarrow{\theta_1} Y\otimes 1 \xrightarrow{r_Y} Y$

· claim. $\theta = L(f)$. $\theta_Z = f\otimes id_Z$. $\forall\, Z \in obj(\mathcal{C})$.

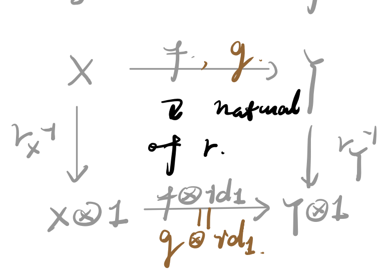


$\Rightarrow \theta_Z = (r_Y\otimes id_Z)\circ(\theta_1\otimes id_Z)\circ(r_X^{-1}\otimes id_Z) = f\otimes id_Z.$

③ L is faithful.

If $L(f) = L(g)$ then $f\otimes id_1 = g\otimes id_1$.

$f, g : X\to Y$.



$\Rightarrow f = g$.

Then L is an equivalence.

④ Define a monoidal functor structure on L.

Let $\phi : 1_{End(\mathcal{C})} \xrightarrow{\sim} L(1_{\mathcal{C}})$

$\|$

$l^{-1} : (id_{\mathcal{C}}, id) \longrightarrow (1\otimes -,\ a_{1,-,-})$

(2.39).

and $J_{X,Y} : L(X)\circ L(Y) \xrightarrow{\sim} L(X\otimes Y)$

$\|$

$a^{-1}_{X,Y,-} : ((X\otimes(Y\otimes -)),\ id_X\otimes a_{Y,-,-}\circ a_{X,Y\otimes -,-})$ 用到了 c.

$\longrightarrow ((X\otimes Y)\otimes -,\ a_{X\otimes Y,-,-})$.

(2.39). is satisfied.

· (2.24) the monoidal structure axiom "

$$\begin{array}{ccc} (L(X)\circ L(Y))\circ L(Z)(T) & = & L(X)\circ(L(Y)\circ L(Z))(T) \\ \downarrow & & \downarrow \\ L(X\otimes Y)\circ L(Z)(T) & & L(X)\circ L(Y\otimes Z)(T) \\ \downarrow & & \downarrow \\ L((X\otimes Y)\otimes Z)(T) & \longrightarrow & L(X\otimes(Y\otimes Z))(T) \end{array}$$

· (2.25) & (2.26) is satisfied.

(2.25)

$$\begin{array}{ccc} id_{\mathcal{C}}\circ L(X) & = & L(X) \\ \downarrow{\scriptstyle \varepsilon^{-1}\circ id_{L(X)}} & \wr & \downarrow{\scriptstyle L(l_X^{-1})} \\ L(1_{\mathcal{C}})\circ L(X) & \xrightarrow[(J_{1_{\mathcal{C}},X})_Y]{} & L(1_{\mathcal{C}}\otimes X) \end{array}$$

$\Downarrow$

$$\begin{array}{ccccc} & & X\otimes Y & & \\ & \swarrow{\scriptstyle l_{X\otimes Y}^{-1}} & \wr & \searrow{\scriptstyle l_X^{-1}\otimes id_Y} & \\ 1\otimes(X\otimes Y) & & \xrightarrow{a_{1,X,Y}^{-1}} & & (1\otimes X)\otimes Y \end{array}$$

$\Rightarrow$ L is monoidal functor.

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Rem: every monoid M, there is a monoid F s.t. $F \cong M$

where $F = \{ f : M \to M \mid f(mn) = f(m)n,\ \forall m, n \in M \}$.

Rem: $\mathcal{C}_G^{\omega}(A)$ (δ_n) ω is cohomologically nontrivial.

$\mathcal{C}$ is not a strict one ?

If $\mathcal{C} \underset{\text{isom.}}{\cong} \mathcal{D}$, $\mathcal{C}$ is strict. $\mathcal{D}$ is not strict.