CONSTRUCTION OF COEND AND THE RECONSTRUCTION THEOREM OF BIALGEBRAS

Abstract. Assume k is a field and let $\mathcal{F}:\mathcal{C}\to Vect_k$ be a small k-linear functor from a k-linear abelian category $\mathcal{C}$ to the category of vector spaces over the field k, the purpose of this note is to use a little knowledge of linear algebra and category to give the description of $\mathrm{end}(\mathcal{F})$ and $\mathrm{coend}(\mathcal{F})$, and then we give the reconstruction theorem of bialgebras by using this description. We use the constructive method to make $\mathrm{end}(\mathcal{F})$ and $\mathrm{coend}(\mathcal{F})$ easy to understand, and some of the proofs are new in this note.

1. Introduction

Let k be a field and let $\mathcal{F}:\mathcal{C}\to Vect_k$ be a k-linear functor from a small k-linear abelian category $\mathcal{C}$ to the category of vector spaces over the field k, and we will use the $\mathcal{F}$ and the $Vect_k$ without explanation in the following content. This paper is organized as follows. In Section 2, we will give the description of $\mathrm{end}(\mathcal{F})$ (resp. $\mathrm{coend}(\mathcal{F})$) as vector spaces, and we will give an example of $\mathrm{coend}(\mathcal{F})$ as our motivation for this note. Then we will give the coalgebra structure of $\mathrm{coend}(\mathcal{F})$ when $\dim(\mathcal{F}(X))<\infty$ for all $X\in\mathrm{Ob}(\mathcal{C})$, and then we will prove that $\mathrm{end}(\mathcal{F})=\mathrm{coend}(\mathcal{F})^*$ as algebra by using a different way from that in [3, Section 1.10]. If $(\mathcal{C},\otimes,I)$ is a strict tensor category and $(\mathcal{F},\mathrm{Id}_I,\mathcal{F}_2)$ is a tensor functor such that $\mathrm{Im}\mathcal{F}$ are finite dimensional, then we will use a constructive approach to give the bialgebra structure of $\mathrm{coend}(\mathcal{F})$ in Section 3. At last, we give the reconstruction theorem of bialgebras by using our description of the bialgebra $\mathrm{coend}(\mathcal{F})$. We must point out that this theorem has been obtained by many articles, what we have done is to describe the bialgebra structure of $\mathrm{coend}(\mathcal{F})$ concretely.

2. $\mathrm{end}(\mathcal{F})$ and $\mathrm{coend}(\mathcal{F})$ as vector spaces

Let A be an algebra over k and let $\mathcal{G}:A\text{-Mod}\longrightarrow Vect_k$ be the forgetful functor from the category of left A modules to $Vect_k$, then we can use the the algebra $\mathrm{End}(\mathcal{G})=\mathrm{Nat}(\mathcal{G},\mathcal{G})$ of natural transformations from $\mathcal{G}$ to $\mathcal{G}$ to reconstruct A through the algebra isomorphism $\varphi:A\to\mathrm{End}(\mathcal{G})$, where $\varphi(a)_{\mathrm{V}}:=a_{\mathrm{V}}$ for $\mathrm{V}\in\mathrm{Ob}(A\text{-Mod})$. Dually, if C is a coalgebra over k and let $\mathcal{E}:C\text{-Comod}\longrightarrow Vect_k$ be the forgetful functor from the category of right C comodules to $Vect_k$, then we can ask if we can reconstruct C by using the functor $\mathcal{E}$? To answer this question, we need the dual object of $\mathrm{End}(\mathcal{F})$ to reconstruct C. But the definition of $\mathrm{End}(\mathcal{F})$ depends on elements of it, and thus we

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recall the equivalent definition of $\mathrm{End}(\mathcal{F})$ by using morphisms to get the dual concept of it.

2.1. **Definition of** $\mathrm{end}(\mathcal{F})$ **and** $\mathrm{coend}(\mathcal{F})$ **as vector spaces.** For convenience, we introduce a k-linear bifunctor $\mathrm{Hom} : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow Vect_k$, which is defined by $\mathrm{Hom}(X, Y) := \mathrm{Hom}_k(\mathcal{F}(X), \mathcal{F}(Y))$ for $X, Y \in \mathrm{Ob}(\mathcal{C})$ and $\mathrm{Hom}(f^{op}, g)(T) := F(g) \circ T \circ F(f)$ for $f \in \mathrm{Hom}_{\mathcal{C}}(X', X)$, $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Y')$, $T \in \mathrm{Hom}_k(\mathcal{F}(X), \mathcal{F}(Y))$, here f^{op} is the morphism f in $\mathcal{C}^{op}$. In particular, $\mathrm{Hom}(f^{op}, \mathrm{Id}_Y)(T) = T \circ F(f)$ and $\mathrm{Hom}(\mathrm{Id}_X, g)(T) = F(g) \circ T$. Let $\mathcal{H} : \mathcal{C} \to \mathcal{D}$ be a k-linear functor, where $\mathcal{C}$, $\mathcal{D}$ are k-linear abelian categories, then the general definition of $\mathrm{end}(\mathcal{H})$ and $\mathrm{coend}(\mathcal{H})$ can be found in [1, 2.1.6. Definition]. But for our purposes, we consider only the case where $\{\mathcal{H}(X)\}_{X \in \mathrm{Ob}(\mathcal{C})}$ are vector spaces in this note.

Definition 2.1. [1, 2.1.6. Definition] *The* $\mathrm{end}(\mathcal{F})$ *is a vector space such that following conditions*

(i) *there is a family of linear maps* $\{\pi_X : \mathrm{end}(\mathcal{F}) \longrightarrow \mathrm{Hom}(X, X)\}_{X \in \mathrm{Ob}(\mathcal{C})}$*,*

(ii) *the following diagrams commute for all* $X, Y \in \mathrm{Ob}(C)$ *and* $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$*,*

$$\begin{array}{ccc} \mathrm{Hom}(Y,Y) & \xrightarrow{\mathrm{Hom}(f^{op},\mathrm{Id}_Y)} & \mathrm{Hom}(X,Y) \\ \uparrow{\scriptstyle \pi_Y} & & \uparrow{\scriptstyle \mathrm{Hom}(\mathrm{Id}_X, f)} \\ \mathrm{end}(\mathcal{F}) & \xrightarrow{\pi_X} & \mathrm{Hom}(X,X) \end{array}$$

Here π_X *and* π_Y *are linear maps of* (i)*.*

(iii) $\mathrm{end}(\mathcal{F})$ *is a final object of* $Vect_k$ *such that* (i) *and* (ii)*, i.e if there is a vector space* U *and there are linear maps* $\{\pi^U_X : U \longrightarrow \mathrm{Hom}(X, X)\}_{X \in \mathrm{Ob}(\mathcal{C})}$ *making the following diagrams commute for all* $X, Y \in \mathrm{Ob}(\mathcal{C})$ *and* $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$*,*

$$\begin{array}{ccc} \mathrm{Hom}(Y,Y) & \xrightarrow{\mathrm{Hom}(f^{op},\mathrm{Id}_Y)} & \mathrm{Hom}(X,Y) \\ \uparrow{\scriptstyle \pi^U_Y} & & \uparrow{\scriptstyle \mathrm{Hom}(\mathrm{Id}_X, f)} \\ U & \xrightarrow{\pi^U_X} & \mathrm{Hom}(X,X) \end{array}$$

then we can find a unique linear map $\varphi : U \longrightarrow \mathrm{end}(\mathcal{F})$ *making the following diagrams commute for all* $X \in \mathrm{Ob}(\mathcal{C})$*,*

$$\begin{array}{ccccc} U & & \xrightarrow{\varphi} & & \mathrm{end}(\mathcal{F}) \\ & {\scriptstyle \pi^U_X}\searrow & & \swarrow{\scriptstyle \pi_X} & \\ & & \mathrm{Hom}(X,X) & & \end{array}$$

It can be seen that the above condition (ii) is equivalent to $\pi_Y(x) \circ \mathcal{F}(f) = \mathcal{F}(f) \circ \pi_X(x)$ for $X, Y \in \mathrm{Ob}(\mathcal{C})$, $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $x \in \mathrm{end}(\mathcal{F})$. If U is a vector space with a family of linear maps $\{\pi^U_X : U \longrightarrow \mathrm{Hom}(X, X)\}_{X \in \mathrm{Ob}(\mathcal{C})}$ and these linear maps satisfy (ii), (iii) of Definition 2.1, then we will call that U is a realization of $\mathrm{end}(\mathcal{F})$. Naturally we have following proposition

Proposition 2.2. *The* $\mathrm{end}(\mathcal{F})$ *above exists and it is unique up to isomorphism.*

Proof. If we define $\{\pi_X^{\mathcal{F}} : \mathrm{End}(\mathcal{F}) \longrightarrow \mathrm{Hom}(X,X)\}_{X\in \mathrm{Ob}(\mathcal{C})}$ by $\pi_X^{\mathcal{F}}(\eta) = \eta_X$ where $\eta \in \mathrm{End}(\mathcal{F})$ and $X \in \mathrm{Ob}(\mathcal{C})$, then it is easy to see that these linear maps satisfy the conditions (ii),(iii) of Definition 2.1. Therefore $\mathrm{End}(\mathcal{F})$ is a realization of $\mathrm{end}(\mathcal{F})$. Because $\mathrm{end}(\mathcal{F})$ is a final object, we know $\mathrm{end}(\mathcal{F})$ is unique up to isomorphism as vector space. □

Dually, we can now define the $\mathrm{coend}(\mathcal{F})$ by changing the direction of arrows of diagrams in Definition 2.1.

Definition 2.3. [1, 2.1.6. Definition] *The* $\mathrm{coend}(\mathcal{F})$ *is a vector space such that following conditions*

(i) *there is a family of linear maps* $\{i_X : \mathrm{Hom}(X,X) \longrightarrow \mathrm{coend}(\mathcal{F})\}_{X\in\mathrm{Ob}(\mathcal{C})}$,
(ii) *the following diagrams commute for all* $X, Y \in \mathrm{Ob}(C)$ *and* $f \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$,

$$\begin{array}{ccc} \mathrm{Hom}(Y,Y) & \xleftarrow{\mathrm{Hom}(f^{op},\mathrm{Id}_Y)} & \mathrm{Hom}(X,Y) \\ {\scriptstyle i_Y}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(\mathrm{Id}_X,f)} \\ \mathrm{coend}(\mathcal{F}) & \xleftarrow{\quad i_X \quad} & \mathrm{Hom}(X,X) \end{array}$$

Here i_X *and* i_Y *are linear maps of* (i).

(iii) $\mathrm{coend}(\mathcal{F})$ *is an initial object of* $Vect_k$ *such that* (i) *and* (ii), *i.e if there is a vector space* U *and there are linear maps* $\{i_X^U : \mathrm{Hom}(X,X) \longrightarrow U\}_{X\in\mathrm{Ob}(\mathcal{C})}$ *making the following diagrams commute for all* $X, Y \in \mathrm{Ob}(C)$ *and* $f \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$,

$$\begin{array}{ccc} \mathrm{Hom}(Y,Y) & \xleftarrow{\mathrm{Hom}(f^{op},\mathrm{Id}_Y)} & \mathrm{Hom}(X,Y) \\ {\scriptstyle i_Y^U}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(\mathrm{Id}_X,f)} \\ U & \xleftarrow{\quad i_X^U \quad} & \mathrm{Hom}(X,X) \end{array}$$

then we can find a unique linear map $\varphi : \mathrm{coend}(\mathcal{F}) \longrightarrow U$ *making the following diagrams commute for all* $X \in \mathrm{Ob}(\mathcal{C})$,

$$\begin{array}{ccc} \mathrm{coend}(\mathcal{F}) & \xrightarrow{\qquad \varphi \qquad} & U \\ & {\scriptstyle i_X}\nwarrow \quad \nearrow{\scriptstyle i_X^U} & \\ & \mathrm{Hom}(X,X) & \end{array}$$

If U is a vector space with a family of linear maps $\{i_X^U : \mathrm{Hom}(X,X) \longrightarrow U\}_{X\in\mathrm{Ob}(\mathcal{C})}$ which satisfy the conditions (ii), (iii) of Definition 2.3, then we will call U is a realization of $\mathrm{coend}(\mathcal{F})$. Note that the above condition (ii) is equivalent to $i_X(\mathcal{F}(f) \circ T) = i_Y(T \circ \mathcal{F}(f))$ for $X, Y \in \mathrm{Ob}(\mathcal{C})$, $f \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$ and $T \in \mathrm{Hom}(X,Y)$, which we will use frequently. Similar to Proposition 2.2, we have

Proposition 2.4. *The* $\mathrm{coend}(\mathcal{F})$ *above exists and it is unique up to isomorphism.*

Proof. Let $V := \bigoplus_{X \in \mathrm{Ob}(\mathcal{C})} \mathrm{Hom}(X, X)$ and we denote J be the subspace of V which is linear spanned by $\{F(f) \circ T - T \circ F(f) |\ X, Y \in \mathrm{Ob}(\mathcal{C}), f \in \mathrm{Hom}_{\mathcal{C}}(X, Y), T \in \mathrm{Hom}(Y, X)\}$. If we define $\{j_X : \mathrm{Hom}(X, X) \longrightarrow V/J\}_{X \in \mathrm{Ob}(\mathcal{C})}$ by $j_X(S) = S + J$ for $X \in \mathrm{Ob}(\mathcal{C})$, then it is easy to see that these linear maps satisfy the conditions (ii),(iii) of Definition 2.1. Therefore the quotient space V/J is a realization of coend$(\mathcal{F})$. Since coend$(\mathcal{F})$ is an intial object, we know coend$(\mathcal{F})$ is unique up to isomorphism as vector space. □

Returning to the original problem at the beginning of this section, that is can we reconstruct the coalgebra C by using the forgetful functor $\mathcal{E} : C\text{-Comod} \longrightarrow Vect_k$? For technical reasons, we use another forgetful functor $\mathcal{T} : C\text{-Comod}_f \longrightarrow Vect_k^f$ instead of using the $\mathcal{E}$, where $C\text{-Comod}_f$ is the category of finite dimensional right C comodules and $Vect_k^f$ is the category of finite dimensional vector spaces over k. The following example shows that the coalgebra C can be reconstructed from coend$(\mathcal{T})$ as vector space. Furthermore, we will prove that the coalgebra C can be reconstructed from coend$(\mathcal{T})$ as coalgebra in Example 2.9 and this example is the main motivation of this note. Recall that if U, V are finite dimensional vector spaces and if we denote the linear dual space of U by U^*, then $\mathrm{Hom}_k(U, V) = \{\beta \otimes x |\ \beta \in U^*, x \in V, (\beta \otimes x)(y) := \beta(y)x\}$. In order to describe the following results more convenient, we will use this fact often without further comment. Moreover we agree that $\beta \otimes x, \gamma \otimes y ...$ means that $(\beta \otimes x)(u) := \beta(u)x, (\gamma \otimes y)(u) := \gamma(u)y ...$, where English letters represent elements of a given vector space and Greek alphabet represent elements of dual space of this vector space.

Example 2.5. Let $\mathcal{T} : C\text{-Comod}_f \longrightarrow Vect_k^f$ be the forgetful functor above, and if we define a family of linear maps $\{i_X : \mathrm{Hom}_k(X, X) \longrightarrow C\}_{X \in \mathrm{Ob}(\mathcal{C})}$ by $i_X(\beta \otimes x) := \beta(x_{(0)})x_{(1)}$ then we will show that C is a realization of coend $\mathcal{T}$. Therefore the coalgebra C can be reconstructed from coend$(\mathcal{T})$ as vector space.
Let $g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and taking $(\gamma \otimes y) \in \mathrm{Hom}_k(Y, X)$, then we have $g(y_{(0)}) \otimes y_{(1)} = g(y)_{(0)} \otimes g(y)_{(1)}$ by definition of the g. So $i_Y[(\gamma \otimes y) \circ g] = i_X[g \circ (\gamma \otimes y)]$ and this implies that $\{i_X\}_{X \in \mathrm{Ob}(\mathcal{C})}$ such that (ii) of Definition 2.3.
Assume U is a vector space with linear maps $\{i_X^U : \mathrm{Hom}_k(X, X) \longrightarrow U\}_{X \in \mathrm{Ob}(\mathcal{C})}$ and these maps satisfy (ii) of Definition 2.3. Define $\varphi : C \longrightarrow U$ by $\varphi(c) := i_{N_c}^U(\epsilon \otimes c)$ for $c \in C$, where N_c is the smallest subcoalgebra of C which contains c and its comodule structure is given by Δ. Then we will show φ satisfy the diagram of (iii) in Definition 2.3 and it is unique.
Taking $(\beta \otimes x) \in \mathrm{Hom}_k(X, X)$, directly we have $\varphi \circ i_X(\beta \otimes x) = i_{N_c}^U(\epsilon \otimes c)$ where $c = \beta(x_{(0)})x_{(1)}$. Since X is finite dimensional, we can choose a finite dimensional subcoalgebra $C_1 \subseteq C$ such that $\rho_X(X) \subseteq X \otimes C_1$. Then $\rho_X \in \mathrm{Hom}_{\mathcal{C}}(X, X \otimes C_1)$ where $\rho_{X \otimes C_1} := (\mathrm{Id} \otimes \Delta)$. Let $(\beta \otimes \epsilon) \otimes x \in \mathrm{Hom}_k(X \otimes C_1, X)$ defined by $((\beta \otimes \epsilon) \otimes x)(y \otimes d) := \beta(y)\epsilon(d)x$, then $((\beta \otimes \epsilon) \otimes x) \circ \rho_X = \beta \otimes x$ and $\rho_X \circ ((\beta \otimes \epsilon) \otimes x) = (\beta \otimes \epsilon) \otimes (x_{(0)} \otimes x_{(1)})$. Since (ii) of Definition 2.3, we get

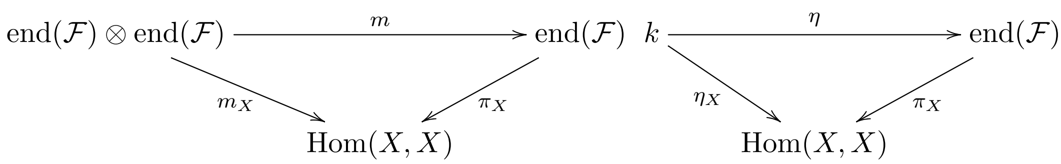
$$i_X^U(\beta \otimes x) = i_{X \otimes C_1}^U[(\beta \otimes \epsilon) \otimes (x_{(0)} \otimes x_{(1)})]. \tag{2.1}$$

Let $f := (\beta \otimes \mathrm{Id}) \in \mathrm{Hom}_{\mathcal{C}}(X \otimes C_1, C_1)$ and $T := [\epsilon \otimes (x_{(0)} \otimes x_{(1)})] \in \mathrm{Hom}_k(C_1, X \otimes C_1)$, where $(\beta \otimes \mathrm{Id})(x \otimes c_1) := \beta(x)c_1$ and $[\epsilon \otimes (x_{(0)} \otimes x_{(1)})](c_1) := \epsilon(c_1)x_{(0)} \otimes x_{(1)}$ for $x \in X, c_1 \in C$, then we know $i^U_{X \otimes C_1}(T \circ f) = i^U_{C_1}(f \circ T)$ since (ii) of Definition 2.3. Because $T \circ f = (\beta \otimes \epsilon) \otimes (x_{(0)} \otimes x_{(1)})$ and $f \circ T = \epsilon \otimes c$, we get

$$i^U_{X \otimes C_1}[(\beta \otimes \epsilon) \otimes (x_{(0)} \otimes x_{(1)})] = i^U_{C_1}(\epsilon \otimes c). \tag{2.2}$$

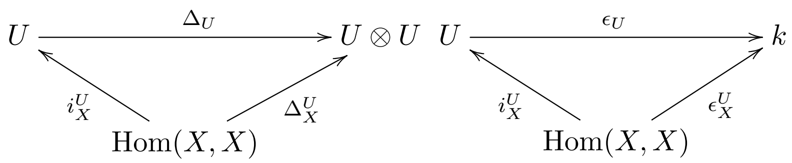
Owing to (2.1), (2.2), we know $i^U_X(\beta \otimes x) = i^U_{C_1}(\epsilon \otimes c)$. Because there is a natural inclusion $N_c \hookrightarrow C_1$, we get $i^U_{C_1}(\epsilon \otimes c) = i^U_{N_c}(\epsilon \otimes c)$. Since $\varphi \circ i_X(\beta \otimes x) = i^U_{N_c}(\epsilon \otimes c)$ and $i^U_X(\beta \otimes x) = i^U_{C_1}(\epsilon \otimes c)$, we have $\varphi \circ i_X(\beta \otimes x) = i^U_X(\beta \otimes x)$ and hence $\varphi \circ i_X = i^U_X$. Note that $\varphi \circ i_{N_c}(\epsilon \otimes c) = i^U_X(\epsilon \otimes c)$ which gives the uniqueness of φ.

2.2. **Coalgebra structure on** $\mathrm{coend}(\mathcal{F})$**.** We have known that the $\mathrm{End}(\mathcal{F})$ is a realization of $\mathrm{end}(\mathcal{F})$ and because the $\mathrm{End}(\mathcal{F})$ is an algebra, so $\mathrm{end}(\mathcal{F})$ has a natural algebra structure. At the same time, the algebra structure of $\mathrm{end}(\mathcal{F})$ can also be determined by the following way. Assume $\{\pi^U_X : U \longrightarrow \mathrm{Hom}(X, X)\}_{X \in \mathrm{Ob}(\mathcal{C})}$ is a realization of $\mathrm{end}(\mathcal{F})$, if we consider linear maps $\{m_X : \mathrm{end}(\mathcal{F}) \otimes \mathrm{end}(\mathcal{F}) \longrightarrow \mathrm{Hom}(X, X)\}_{X \in \mathrm{Ob}(\mathcal{C})}$, $\{\eta_X : k \longrightarrow \mathrm{Hom}(X, X)\}_{X \in \mathrm{Ob}(\mathcal{C})}$, where m_X, η_X are defined by $m_X(a \otimes b) := \pi_X(a)\pi_X(b)$ and $\eta_X(1) := \mathrm{Id}_X$ for $X \in \mathrm{Ob}(\mathcal{C})$, then we will get the multiplication and the unit of $\mathrm{end}(\mathcal{F})$ through these linear maps by using (iii) of Definition 2.1, i.e the the multiplication m and the unit η of $\mathrm{end}(\mathcal{F})$ are the unique linear maps making the following diagrams commute for all $X \in \mathrm{Ob}(\mathcal{C})$,



Dually, we can define coalgebra structure of $\mathrm{coend}(\mathcal{F})$. For technical reasons, we require $\mathcal{F}(X)$ are finite dimensional for all $X \in \mathrm{Ob}(\mathcal{C})$ in the following content. Let $\{x_i\}_{i=1}^n$ be a basis for $\mathcal{F}(X)$ and let $\{x^i\}_{i=1}^n$ be the dual basis for $\mathcal{F}(X)^*$, and we recall that $\mathrm{Hom}(X, X)$ has a natural coalgebra structure which is defined by $\Delta_X(\beta \otimes x) := \sum_{i=1}^n (\beta \otimes x_i) \otimes (x^i \otimes x)$ and $\epsilon_X(\beta \otimes x) := \beta(x)$, where $\beta \in X^*, x \in X$ and $(\beta \otimes x)(y) := \beta(y)x, (x^i \otimes x)(y) := x^i(y)x$. Now we can use this family of coalgebra algebras $\{\mathrm{Hom}(X, X)\}_{X \in \mathrm{Ob}(\mathcal{C})}$ to give coalgebra structure of $\mathrm{coend}(\mathcal{F})$.

Definition 2.6. *Assume that* $\{i^U_X : \mathrm{Hom}(X, X) \longrightarrow U\}_{X \in \mathrm{Ob}(\mathcal{C})}$ *is a realization of* $\mathrm{coend}(\mathcal{F})$ *and let* $\{\Delta^U_X : \mathrm{Hom}(X, X) \longrightarrow U \otimes U\}_{X \in \mathrm{Ob}(\mathcal{C})}$, $\{\epsilon^U_X : \mathrm{Hom}(X, X) \longrightarrow k\}_{X \in \mathrm{Ob}(\mathcal{C})}$ *be linear maps defined by* $\Delta^U_X := (i^U_X \otimes i^U_X) \circ \Delta_X$ *and* $\epsilon^U_X := \epsilon_X$*. Then the coalgebra structure* $(U, \Delta_U, \epsilon_U)$ *is defined by the unique linear maps* Δ_U, ϵ_U *which making the following diagrams commute for all* $X \in \mathrm{Ob}(\mathcal{C})$*,*



The following two propositions show that existence of Δ_U, ϵ_U of Definition 2.6 and $(U, \Delta_U, \epsilon_U) \cong (V, \Delta_V, \epsilon_V)$, where $\{i_X^U : \mathrm{Hom}(X,X) \longrightarrow U\}_{X \in \mathrm{Ob}(\mathcal{C})}$ is another realization of coend$(\mathcal{F})$ and $(V, \Delta_V, \epsilon_V)$ is given by the Definition 2.6 above. For these reasons, the coalgebra coend$(\mathcal{F})$ given by the Definition 2.6 is unique up to isomorphism as coalgebra.

Proposition 2.7. *The Δ_U, ϵ_U of Definition 2.6 exist and $(U, \Delta_U, \epsilon_U)$ is a coalgebra.*

Proof. To show the Δ_U, ϵ_U exist, it is enough to check that $\{\Delta_X^U : \mathrm{Hom}(X,X) \longrightarrow U \otimes U\}_{X \in \mathrm{Ob}(\mathcal{C})}$, $\{\epsilon_X^U : \mathrm{Hom}(X,X) \longrightarrow k\}_{X \in \mathrm{Ob}(\mathcal{C})}$ satisfy the (ii) of Definition 2.3. Let $\{x_i\}_{i=1}^n$ (resp. $\{y_i\}_{i=1}^m$) be a basis for $\mathcal{F}(X)$ (resp. $\mathcal{F}(Y)$) and let $\{x^i\}_{i=1}^n$ (resp. $\{y^i\}_{i=1}^m$) be the dual basis for $\mathcal{F}(X)^*$ (resp. $\mathcal{F}(Y)^*$), and taking $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, $(\beta \otimes x) \in \mathrm{Hom}(Y,X)$, then we have

$$\begin{aligned}
\Delta_X^U((\beta \otimes x) \circ \mathcal{F}(f)) &= \sum_{i=1}^n i_X^U(\beta \circ \mathcal{F}(f) \otimes x_i) \otimes i_X^U(x^i \otimes x) \\
&= \sum_{i=1}^n i_Y^U(\beta \otimes \mathcal{F}(f)(x_i)) \otimes i_X^U(x^i \otimes x) \\
&= \sum_{i}^n \sum_{j=1}^m i_Y^U(\beta \otimes y_j < y^j, \mathcal{F}(f)(x_i) >) \otimes i_X^U(x^i \otimes x) \\
&= \sum_{i}^n \sum_{j=1}^m i_Y^U(\beta \otimes y_j) \otimes i_X^U(< y^j, \mathcal{F}(f)(x_i) > x^i \otimes x) \\
&= \sum_{j=1}^m i_Y^U(\beta \otimes y_j) \otimes i_X^U(y^j \circ \mathcal{F}(f) \otimes x) \\
&= \sum_{j=1}^m i_Y^U(\beta \otimes y_j) \otimes i_X^U(y^j \otimes \mathcal{F}(f)(x)) \\
&= \Delta_Y^U(\mathcal{F}(f) \circ (\beta \otimes x))
\end{aligned}$$

and $\epsilon_X^U((\beta \otimes x) \circ \mathcal{F}(f)) = \beta \circ \mathcal{F}(f)(x) = \epsilon_Y^U(\mathcal{F}(f) \circ (\beta \otimes x))$, hence we know the Δ_U, ϵ_U exist. Let $T \in \mathrm{Hom}(X,X)$ and we denote $i_X^U(T)$ of Definition 2.6 by $\overline{T}$, then $\Delta_U(\overline{T}) = (i_X^U \otimes i_X^U) \circ \Delta_X(T)$ and $\epsilon_U(\overline{T}) = \epsilon_X(T)$ by definition and so $(\Delta_U \otimes \mathrm{Id}) \circ \Delta_U(\overline{T}) = (\mathrm{Id} \otimes \Delta_U) \circ \Delta_U(\overline{T})$, $(\epsilon_U \otimes \mathrm{Id}) \circ \Delta_U(\overline{T}) = (\mathrm{Id} \otimes \epsilon_U) \circ \Delta_U(\overline{T}) = (\overline{T})$. Since Proposition 2.4, we know U is linear spanned by $\{\overline{T} |\ T \in \mathrm{Hom}(X,X)\}_{X \in \mathrm{Ob}(\mathcal{C})}$ and hence $(U, \Delta_U, \epsilon_U)$ is a coalgebra. □

Proposition 2.8. $(U, \Delta_U, \epsilon_U) \cong (V, \Delta_V, \epsilon_V)$ *as coalgebra.*

Proof. Because coend$(\mathcal{F})$ is an initial object of $Vect_k$, there exists invertible linear maps $\varphi : U \to V$ such that $\varphi(i_X^U(T)) = i_X^V(T)$. Then we will check that φ is a coalgebra map. Directly we have $\Delta_V(\varphi(i_X^U(T))) = (i_X^V \otimes i_X^V) \circ \Delta_X(T) = (\varphi \otimes \varphi) \circ \Delta_U(i_X^U(T))$ and $\epsilon_U(i_X^U(T)) = \epsilon_X(T) = \epsilon_V(\varphi(i_X^U(T)))$. Note that U is linear spanned by $\{i_X^U(T) |\ T \in \mathrm{Hom}(X,X)\}_{X \in \mathrm{Ob}(\mathcal{C})}$ and hence φ is a coalgebra isomorphism. □

We already saw in Example 2.5 that coend($\mathcal{T}$) can reconstruct C as vector space, furthermore, the following example shows that coend($\mathcal{T}$) also reconstruct C as coalgebra.

Example 2.9. Let $\mathcal{T} : \mathcal{C} \longrightarrow Vect_k^f$ be the forgetful functor of Example 2.5, and we have known that C is a realization of coend $\mathcal{T}$ as vector space. Therefore we have a coalgebra structure on C by Definition 2.6 and we denote it by (Δ_C, ϵ_C). Now let's prove that $(\Delta_C, \epsilon_C) = (\Delta, \epsilon)$, where (Δ, ϵ) is the coalgebra structure of C itself. We still use the notation of Example 2.5 in the following context. Let $c \in C$ and we choose a finite dimensional subcoalgebra C_1 of C which contains the c, then $i_{C_1}(\epsilon \otimes c) = c$ by definition. For convenience, we denote $i_X(T)$ by $\overline{T}$ and hence $c = \overline{\epsilon \otimes c}$. Let $\{x_i\}_{i=1}^n$ be a basis for C_1 and let $\{x^i\}_{i=1}^n$ be the dual basis for C_1^*, then we have

$$\begin{aligned}
\Delta_C(\overline{\epsilon \otimes c}) &= \overline{(\epsilon \otimes c)_{(1)}} \otimes \overline{(\epsilon \otimes c)_{(2)}} \\
&= \sum_{i=1}^n \overline{(\epsilon \otimes x_i)} \otimes \overline{(x^i \otimes c)} \\
&= \sum_{i=1}^n x_i \otimes < x^i, c_{(1)} > c_{(2)} \\
&= c_{(1)} \otimes c_{(2)} = \Delta(c).
\end{aligned}$$

Due to $\epsilon_C(\overline{\epsilon \otimes c}) = \epsilon_{C_1}(\epsilon \otimes c) = \epsilon(c)$, we get $\epsilon_C = \epsilon$ and hence $(\Delta_C, \epsilon_C) = (\Delta, \epsilon)$. That is to say the coalgebra coend $\mathcal{T}$ reconstruct the coalgebra C.

Since coend($\mathcal{F}$) and end($\mathcal{F}$) are dual concept, we have the following proposition and we give its proof by using a different way from that in [3, Section 1.10].

Proposition 2.10. [3, Section 1.10] end($\mathcal{F}$) $\cong$ coend($\mathcal{F}$)* *as algebra.*

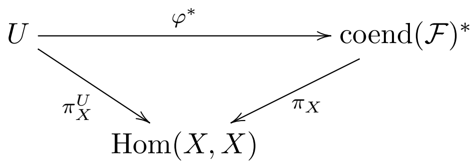
Proof. Assume $\{i_X : \mathrm{Hom}(X, X) \longrightarrow \mathrm{coend}(\mathcal{F})\}_{X \in \mathrm{Ob}(\mathcal{C})}$ is a realization of coend($\mathcal{F}$), and let $\{\pi_X : \mathrm{coend}(\mathcal{F})^* \longrightarrow \mathrm{Hom}(X, X)\}_{X \in \mathrm{Ob}(\mathcal{C})}$ be a family of linear maps which are defined by $\pi_X(\alpha) := \sum_{i,j=1}^n \alpha \circ i_X(x^i \otimes x_j) x^j \otimes x_i$ for $X \in \mathrm{Ob}(\mathcal{C})$, where $\{x^i\}_{i=1}^n$ is the dual basis of $\mathcal{F}(X)^*$ corresponding with a given basis $\{x_i\}_{i=1}^n$ of $\mathcal{F}(X)$ and $(x^i \otimes x_j)(x) := x^i(x) x_j$, then we can see that $\{\pi_X\}_{X \in \mathrm{Ob}(\mathcal{C})}$ are algebra maps due to $\{i_X\}_{X \in \mathrm{Ob}(\mathcal{C})}$ are coalgebra maps. To complete the proof, we only need to show that $\{\pi_X : \mathrm{coend}(\mathcal{F})^* \longrightarrow \mathrm{Hom}(X, X)\}_{X \in \mathrm{Ob}(\mathcal{C})}$ is a realization of end($\mathcal{F}$).

Firstly, we show that $\{\pi_X\}_{X \in \mathrm{Ob}(\mathcal{C})}$ satisfy (ii) of Definition 2.1. Let $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $\alpha \in \mathrm{coend}(\mathcal{F})^*$, and we assume $\{y^i\}_{i=1}^m$ is the dual basis of $\mathcal{F}(Y)^*$ corresponding with a given basis $\{y_i\}_{i=1}^m$ of $\mathcal{F}(Y)$. Since

$$\begin{aligned}
\mathcal{F}(f) \circ \pi_X(\alpha) &= \sum_{i,j=1}^n \alpha \circ i_X(x^i \otimes x_j) x^j \otimes \mathcal{F}(f)(x_i) \\
&= \sum_{i,j=1}^n \sum_{k=1}^m \alpha \circ i_X(x^i \otimes x_j) x^j \otimes y_k < y^k, \mathcal{F}(f)(x_i) >
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\sum_{k=1}^{m} \alpha \circ i_X(y^k \circ \mathcal{F}(f) \otimes x_j)x^j \otimes y_k \\
&= \sum_{i=1}^{n}\sum_{k=1}^{m} \alpha \circ i_X(y^k \otimes \mathcal{F}(f)(x_j))x^j \otimes y_k \\
&= \sum_{i=1}^{n}\sum_{j,k=1}^{m} \alpha \circ i_X(y^k \otimes < y^i, \mathcal{F}(f)(x_j) > y_i)x^j \otimes y_k \\
&= \sum_{i,k=1}^{m} \alpha \circ i_X(y^k \otimes y_i)y^i \circ \mathcal{F}(f) \otimes y_k \\
&= \pi_Y \circ \mathcal{F}(f)(\alpha),
\end{aligned}
$$

we know $\{\pi_X\}_{X\in \mathrm{Ob}(\mathcal{C})}$ satisfy (ii) of Definition 2.1.
Secondly, we prove that $\{\pi_X\}_{X\in \mathrm{Ob}(\mathcal{C})}$ such that (iii) of Definition 2.1. If there is a vector space U with linear maps $\{\pi_X^U : U \longrightarrow \mathrm{Hom}(X, X)\}_{X\in \mathrm{Ob}(\mathcal{C})}$ and we assume that these maps satisfy (ii) of Definition 2.1, then $\{i_X^U : \mathrm{Hom}(X, X) \longrightarrow U^*\}_{X\in \mathrm{Ob}(\mathcal{C})}$ which are defined by $i_X^U(\beta \otimes x)(u) := \beta \circ \pi_X^U(u)(x)$ are linear maps satisfy (ii) of Definition 2.3. Therefore there is a unique linear map $\varphi : \mathrm{coend}(\mathcal{F}) \longrightarrow U^*$ such that $\varphi \circ i_X = i_X^U$. Define $\varphi^* : U \longrightarrow \mathrm{coend}(\mathcal{F})^*$ by $\varphi^*(u)(c) = \varphi(c)(u)$ for $u \in U$ and $c \in \mathrm{coend}(\mathcal{F})$, then we know $\pi_X \circ \varphi^* = \pi_X^U$ due to $\varphi \circ i_X = i_X^U$ and hence the following diagrams commute for all $X \in \mathrm{Ob}(\mathcal{C})$.



Let's prove that a linear map satisfying the above commutative diagrams is unique. Assume that there is another linear map $\phi : U \longrightarrow \mathrm{coend}(\mathcal{F})^*$ such that $\pi_X \circ \phi = \pi_X^U$, then $\pi_X(\phi(u)) = \pi_X(\varphi^*(u))$ since we have shown $\pi_X \circ \varphi^* = \pi_X^U$. But $\pi_X(\alpha) = \sum_{i,j=1}^{n} \alpha \circ i_X(x^i \otimes x_j)x^j \otimes x_i$ for $\alpha \in \mathrm{coend}(\mathcal{F})^*$ by definition, so $\phi(u) \circ i_X = \varphi^*(u) \circ i_X$ for $X \in \mathrm{Ob}(\mathcal{C})$. This implies $\phi(u) = \varphi^*(u)$ and hence φ^* is unique if it satisfies the above diagrams. □

3. Reconstruction theorem of bialgebras

We assume that $(\mathcal{C}, \otimes, I)$ is a strict tensor category and $(\mathcal{F}, \mathrm{Id}_I, \mathcal{F}_2)$ is a tensor functor from $\mathcal{C}$ to $Vect_k^f$ in this section. Recall that if $\mathcal{C}$ is only a k-linear abelian category and $\mathcal{F}$ is a k-linear functor then we can only get coalgebra structure of $\mathrm{coend}(\mathcal{F})$, but now we've added a tensor structure to $\mathcal{C}$, naturally we can think of adding some information to $\mathrm{coend}(\mathcal{F})$. In order to express this idea precisely, we need the following lemmas. Define $\mathcal{F} \otimes \mathcal{F} : \mathcal{C} \times \mathcal{C} \longrightarrow Vect_k^f$ by $(\mathcal{F} \otimes \mathcal{F})(X, Y) = \mathcal{F}(X) \otimes \mathcal{F}(Y)$, $(\mathcal{F} \otimes \mathcal{F})(f, g) = \mathcal{F}(f) \otimes \mathcal{F}(g)$ for $X, Y \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, X')$, $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Y')$, then we have the following result. The proof of it is different from that in [2, Section 8].

Lemma 3.1. [2, Section 8] $\mathrm{coend}(\mathcal{F}\otimes\mathcal{F}) \cong \mathrm{coend}(\mathcal{F})\otimes\mathrm{coend}(\mathcal{F})$ *as coalgebra, where* $\mathrm{coend}(\mathcal{F})\otimes\mathrm{coend}(\mathcal{F})$ *has the tensor product coalgebra structure.*

Proof. Assume $\{i_X : \mathrm{Hom}(X,X) \longrightarrow \mathrm{coend}(\mathcal{F})\}_{X\in\mathrm{Ob}(\mathcal{C})}$ is a realization of $\mathrm{coend}(\mathcal{F})$, and we define $i_{X,Y} : \mathrm{Hom}_k(\mathcal{F}(X)\otimes\mathcal{F}(Y), \mathcal{F}(X)\otimes\mathcal{F}(Y)) \longrightarrow \mathrm{coend}(\mathcal{F})\otimes\mathrm{coend}(\mathcal{F})$ by $i_{X,Y}(S\otimes T) = i_X(S)\otimes i_Y(T)$ for $S\in\mathrm{Hom}(X,X)$, $T\in\mathrm{Hom}(\mathcal{F}(Y),\mathcal{F}(Y))$. It can be seen that $\mathrm{coend}(\mathcal{F})\otimes\mathrm{coend}(\mathcal{F})$ with these maps is a realization of $\mathrm{coend}(\mathcal{F}\otimes\mathcal{F})$ and $\{i_{X,Y}\}_{X,Y\in\mathrm{Ob}(\mathcal{C})}$ are coalgebra maps, so $\mathrm{coend}(\mathcal{F}\otimes\mathcal{F}) \cong \mathrm{coend}(\mathcal{F})\otimes\mathrm{coend}(\mathcal{F})$ as coalgebra. □

Let $\{v_i\}_{i=1}^n$ be a basis for V and let $\{v^i\}_{i=1}^n$ be the dual basis for V^*, and we recall that the comatrix coalgebra $\mathrm{Hom}_k(V,V)$ is defined by $\Delta(\beta\otimes v) := \sum_{i=1}^n(\beta\otimes v_i)\otimes(v^i\otimes v)$ and $\epsilon(\beta\otimes v) := \beta(v)$, where $(\beta\otimes v)(w) := \beta(w)v$ for $\beta\in V^*$, $v,w\in V$, then we have

Lemma 3.2. *Assume* $P\in\mathrm{Hom}_k(V,V)$ *and it is invertible, then* $\varphi_P : \mathrm{Hom}_k(V,V)\longrightarrow \mathrm{Hom}_k(V,V)$ *is a coalgebra automorphism, where* $\varphi_P(T) := PTP^{-1}$.

Proof. Let $\{v_i\}_{i=1}^n$ be a basis for V, and we consider the non-degenerate dual pair $<,>: \mathrm{Hom}_k(V,V)\times\mathrm{Hom}_k(V,V)\longrightarrow k$ which is defined by $<\beta\otimes v, \gamma\otimes w> = \beta(w)\gamma(v)$, where $(\beta\otimes v)(y) := \beta(y)v$ and $(\gamma\otimes w)(y) := \gamma(y)w$. Then we can define $\phi_P : \mathrm{Hom}_k(V,V)\times\mathrm{Hom}_k(V,V)\longrightarrow k$ through the equation $<\phi_P(T), S> = <T, \varphi_P(S)>$, and we can check that $\phi_P(T) = P^tT(P^t)^{-1}$ where P^t is the transpose matrix of P for the given basis $\{v_i\}_{i=1}^n$ of V. Therefore ϕ_P is an algebra map, which implies φ_P is a coalgebra map. □

Given a realization of $\mathrm{coend}(\mathcal{F})$ by $\{i_X : \mathrm{Hom}(X,X)\longrightarrow\mathrm{coend}(\mathcal{F})\}_{X\in\mathrm{Ob}(\mathcal{C})}$, and we denote $i_X(T)$ by $\overline{T}$ for $T\in\mathrm{Hom}(X,X)$. Due to Lemma 3.1, we can use the following commute diagrams for all $X,Y\in\mathrm{Ob}(\mathcal{C})$ to define multiplication of $\mathrm{coend}(\mathcal{F})$

$$\begin{array}{ccc} \mathrm{Hom}_k(\mathcal{F}(X)\otimes\mathcal{F}(Y), \mathcal{F}(X)\otimes\mathcal{F}(Y)) & \xrightarrow{m_{X,Y}} & \mathrm{coend}(\mathcal{F}) \\ \downarrow{\scriptstyle \overline{\mathcal{F}_2}} & \nearrow{\scriptstyle i_{X\otimes Y}} & \\ \mathrm{Hom}_k(\mathcal{F}(X\otimes Y), \mathcal{F}(X\otimes Y)) & & \end{array},$$

where $\overline{\mathcal{F}_2}(S\otimes T) := \mathcal{F}_2(S\otimes T)\mathcal{F}_2^{-1}$ for $S\in\mathrm{Hom}(X,X)$, $T\in\mathrm{Hom}(Y,Y)$. Then the above commute diagrams determine the product of $\mathrm{coend}(\mathcal{F})$ by $m(\overline{S}\otimes\overline{T}) = \overline{\mathcal{F}_2(S\otimes T)\mathcal{F}_2^{-1}}$. Moveover if we define unit η of $\mathrm{coend}(\mathcal{F})$ by $\eta := i_I$, then we have

Proposition 3.3. $(\mathrm{coend}(\mathcal{F}), \Delta, \epsilon, m, \eta)$ *is a bialgebra.*

Proof. Firstly, we show $(\mathrm{coend}(\mathcal{F}), m, \eta)$ is an algebra. For simple, we denote the identity map from $\mathrm{coend}(\mathcal{F})$ to $\mathrm{coend}(\mathcal{F})$ by id. By definition, $m\circ(m\otimes id)(\overline{W}) = i_{(X\otimes Y)\otimes Z}[\mathcal{F}_2\circ(\mathcal{F}_2\otimes id)\circ W\circ(\mathcal{F}_2\otimes id)^{-1}\circ\mathcal{F}_2^{-1}]$ and $m\circ(id_X\otimes m)(\overline{W}) = i_{X\otimes(Y\otimes Z)}[\mathcal{F}_2\circ(id\otimes\mathcal{F}_2)\circ W\circ(id\otimes\mathcal{F}_2)^{-1}\circ\mathcal{F}_2^{-1}]$ where $W\in\mathrm{Hom}_k(\mathcal{F}(X)^{\otimes 3}, \mathcal{F}(X)^{\otimes 3})$. Since $\mathcal{C}$ is strict and $[\mathcal{F}_2\circ(\mathcal{F}_2\otimes id) = \mathcal{F}_2\circ(id\otimes\mathcal{F}_2)$ due to $(\mathcal{F}, \mathrm{Id}_I, \mathcal{F}_2)$ is a tensor functor, we know m satisfy associativity. Due to $m\circ(id\otimes\eta)(\overline{S}) = i_X[(\mathcal{F}_2)_{X,I}(S\otimes 1)(\mathcal{F}_2^{-1})_{X,I}] = \overline{S}$, we have $m \circ (id \otimes \eta) = id$. Similarly, we get $(id \otimes \eta) \circ m = id$ and hence $(\mathrm{coend}(\mathcal{F}), m, \eta)$ is an algebra.

Secondly, we show m, η are coalgebra maps. It can be seen that η is a coalgebra map by definition of coalgebra structure of $\mathrm{coend}(\mathcal{F})$. To prove m is a coalgebra map, we need only to check that $\{m_{X,Y}\}_{X,Y \in \mathrm{Ob}(\mathcal{C})}$ are coalgebra maps. We have known $i_{X \otimes Y}$ is a coalgebra map since the definition of coalgebra structure of $\mathrm{coend}(\mathcal{F})$. Because $m_{X,Y} = i_{X \otimes Y} \circ \overline{\mathcal{F}_2}$ and $\overline{\mathcal{F}_2}$ is a coalgebra map by Lemma 3.2, we get that $m_{X,Y}$ is a coalgebra map and hence we have completed the proof. □

Since $\mathrm{coend}(\mathcal{F})$ is a bialgebra, we know that finite dimensional right comodules of $\mathrm{coend}(\mathcal{F})$ is a strict tensor category and we denote it by $\mathcal{C}_{\mathcal{F}}$. Then the tensor functor $\mathcal{F}$ induces a unique tensor functor $\mathcal{E} : \mathcal{C} \longrightarrow \mathcal{C}_{\mathcal{F}}$ satisfies $\mathcal{F} = \mathcal{F}' \circ \mathcal{E}$ and $\mathcal{F}_2 = \mathcal{F}'_2 \circ \mathcal{E}_2$, where $\mathcal{F}' : \mathcal{C}_{\mathcal{F}} \longrightarrow Vect_k^f$ is the forgetful functor. A natural question is when $\mathcal{E}$ is an equivalent tensor functor? The following reconstruction theorem of bialgebras is the answer to this question.

Theorem 3.4. [2, Theorem 3] *If $\mathcal{F}$ is exact and faithful, then $\mathcal{E}$ is an equivalence of tensor functor.*

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