1.11 Deligne's tensor product of locally finite abelian categories

Let $\mathcal{C}$, $\mathcal{D}$ be two locally finite abelian categories over a field $\Bbbk$. (essentially small) ($\Bbbk$-linear)

finite

All categories considered in this book will be locally small (except in the section on 2-categories) and most of them will be essentially small.

(Def) Deligne's tensor product $\mathcal{C}\boxtimes\mathcal{D}$ is an abelian $\Bbbk$-linear category

universal : functor $\boxtimes : \mathcal{C}\times\mathcal{D}\to\mathcal{C}\boxtimes\mathcal{D}$ which is right exact in both variables.

$$(X,Y)\mapsto X\boxtimes Y.$$

s.t. for $\forall$ $\Bbbk$-linear abelian category $\mathcal{A}$, and for any right exact in both variables bifunctor $F : \mathcal{C}\times\mathcal{D}\to\mathcal{A}$, $\exists!$ right exact functor $\bar F : \mathcal{C}\boxtimes\mathcal{D}\to\mathcal{A}$ satisfying $\bar F\circ\boxtimes = F$.

$$\begin{tikzcd} \mathcal{C}\times\mathcal{D} \arrow[r, "\boxtimes"] \arrow[d, "F"'] & \mathcal{C}\boxtimes\mathcal{D} \arrow[dl, dashed, "\exists!\,\bar F"] \\ \mathcal{A} & \end{tikzcd}$$

finite

Prop 1.11.2 (i) A Deligne's tensor product $\mathcal{C}\boxtimes\mathcal{D}$ exists and is a locally finite abelian category.

pf:

DEFINITION 1.8.5. A $\Bbbk$-linear abelian category $\mathcal{C}$ is said to be *finite* if it is equivalent to the category A–mod of finite dimensional modules over a finite dimensional $\Bbbk$-algebra A.

① $C\simeq R\text{-mod}_f$ $\quad D\simeq S\text{-mod}_f$. R, S f.d. alg

$R\text{-mod}_f\times S\text{-mod}_f \xrightarrow{\boxtimes} R\otimes_k S\text{-mod}_f$ $\quad (= R\text{-mod}_f\boxtimes S\text{-mod}_f)$

$M \times N \xmapsto{\otimes_k} M\otimes_k N$ $\quad (M, N$ f.d., M_k, ${}_kN)$

$R\subseteq \mathrm{End}(M)$, $S\subseteq\mathrm{End}(N)$

define $R\otimes S \cdot M\otimes_k N$

$$(r\otimes s)\cdot(m\otimes n) = (r\cdot m)\otimes(s\cdot n)$$

$$(r\otimes s)(\bar r\otimes\bar s)\cdot(m\otimes n) = (r\bar r\otimes s\bar s)\cdot(m\otimes n) = (r\bar r\cdot m)\otimes(s\bar s\cdot m)$$
$$= (r\otimes s)((\bar r\otimes\bar s)\cdot(m\otimes n))$$

$(1_R\otimes 1_S)\cdot(m\otimes n) = m\otimes n$

$\therefore M\otimes_k N$ is a $R\otimes S$-mod. $\quad M\otimes_k N$ f.d. $\quad R\otimes S$ f.d.

$R\otimes S\text{-mod}_f$: finite $\Bbbk$-linear abelian category.

② $\mathrm{Hom}_{R\text{-mod}}(X_1, Y_1) \times \mathrm{Hom}_{S\text{-mod}}(X_2, Y_2) \longrightarrow \mathrm{Hom}_{R\otimes S\text{-mod}}(X_1 \otimes_k X_2,\ Y_1 \otimes_k Y_2)$

$(f \quad , \quad g) \longmapsto f \otimes g$

Corollary 2.5.1 *Let $f : M_1 \to M_2$ be a right R-module homomorphism, and $g : N_1 \to N_2$ a left R-module homomorphism. Then there is a unique group homomorphism $f \otimes g$ from $M_1 \otimes_R N_1$ to $M_2 \otimes_R N_2$ such that $(f \otimes g)(m \otimes n) = f(m) \otimes g(n)$ for $m \in M_1$ and $n \in N_1$.*

$f\otimes g\,((r\otimes s)\cdot(m\otimes n)) = f(r\cdot m) \otimes g(s\cdot n) = rf(m) \otimes sg(n) = (r\otimes s)\cdot(f(m)\otimes g(n))$

$\mathrm{Hom}_{R\text{-mod}}(X_1, Y_1) \otimes \mathrm{Hom}_{S\text{-mod}}(X_2, Y_2) \overset{\text{i}}{\cong} \mathrm{Hom}_{R\otimes S\text{-mod}}(X_1 \otimes_k Y_1,\ X_2 \otimes_k Y_2)$

③ by THEOREM 1.3.8 (Mitchell; [**Fr**]). *Every abelian category is equivalent, as an additive category, to a full subcategory of the category of left modules over an associative unital ring A.*

$\forall$ $\mathbb{k}$-linear abelian category $\mathcal{A}$, $\mathrm{ob}\mathcal{A}$: $\mathbb{k}$-mod. (L)

by the universal prop of $\otimes_k$

$$\begin{tikzcd} M\times N \arrow[r, "\otimes_k"] \arrow[d, "f"', "\text{middle bilinear}"] & M\otimes_k N \arrow[dl, dashed, "\exists!\, \bar{f}"] \\ L & \end{tikzcd} \Longrightarrow \begin{tikzcd} R\text{-mod}\times S\text{-mod} \arrow[r, "\boxtimes"] \arrow[d, "F"'] & R\otimes_k S\text{-mod} \arrow[dl, dashed, "\exists!\, \bar{F}"] \\ \mathcal{A} & \end{tikzcd}$$

obviously $\boxtimes$ right exact in both variables

(M free $\mathbb{k}$-module $\Rightarrow$ flat $M\boxtimes\_$ exact, $\_\boxtimes M$ exact.)

next we show $\bar{F}$ right exact.

suppose $0 \to L_1 \to L_2 \to L_3 \to 0$ is exact in $R\otimes S$-mod

next we show $\bar{F}(L_1) \to \bar{F}(L_2) \to \bar{F}(L_3) \to 0$

pf: (locally finite)

(by Thm 1.9.15) Any essentially small locally finite abelian category $\mathcal{C}$ over a field $\Bbbk$ is equivalent to the category C-comod for a unique pointed coalgebra C.

$\therefore$ we can take coalg. in Thm 1.9.15, s.t. $\mathcal{C} \simeq C$-comod, $\mathcal{D} \simeq D$-comod (finite)

then one can define $\mathcal{C} \boxtimes \mathcal{D} = (C \otimes D)$-comod, next we show that it satisfies the required condition

$C \otimes D$: $\Delta_{C\otimes D}(c \otimes d) = \sum (c_{(1)} \otimes d_{(1)}) \otimes (c_{(2)} \otimes d_{(2)})$

$\varepsilon_{C\otimes D}(c \otimes d) = \varepsilon_C(c)\, \varepsilon_D(d)$

$$C\text{-comod} \times D\text{-comod} \longrightarrow C\otimes D\text{-comod}$$
$$M \times N \longmapsto M \otimes_{\Bbbk} N$$

$\because$ C pointed, D pointed, then $C \otimes D$ poin

(Radford Prop 4.1.7. (c))

$\rho_1 : M \to M \otimes C$ $\qquad$ $\rho_2 : N \to N \otimes D$

$\rho_1(m) = \sum m_0 \otimes m_1$ $\qquad$ $\rho_2(n) = \sum n_0 \otimes n_1$

$\rho_{C\otimes D}(m \otimes n) = \sum m_0 \otimes n_0 \otimes m_1 \otimes n_1$

$$\begin{tikzcd} M\otimes N \arrow[r, "\rho_{M\otimes N}"] \arrow[d, "\rho_{M\otimes N}"'] & M\otimes N\otimes C\otimes D \arrow[d, "id_{M\otimes N}\otimes \Delta_{C\otimes D}"] \\ M\otimes N\otimes C\otimes D \arrow[r, "\rho_{M\otimes N}\otimes id_{C\otimes D}"] & M\otimes N\otimes C\otimes D\otimes C\otimes D \end{tikzcd}$$

$$\begin{tikzcd} M\otimes N \arrow[r, "\rho_{M\otimes N}"] \arrow[d, "\rho_M\otimes\rho_N"'] & M\otimes N\otimes C\otimes D \\ M\otimes C\otimes N\otimes D \arrow[ur, "id_M\otimes\tau\otimes id_D"'] & \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\rho_M"] \arrow[dr, "\otimes 1"'] & M\otimes C \arrow[d, "id\otimes\varepsilon"] \\ & M\otimes\Bbbk \end{tikzcd}$$

$id_{M\otimes N} \otimes \Delta_{C\otimes D}(\sum m_0 \otimes n_0 \otimes m_1 \otimes n_1) = \sum m_0 \otimes n_0 \otimes m_{11} \otimes n_{11} \otimes m_{12} \otimes n_{12}$

$\rho_{M\otimes N} \otimes id_{C\otimes D}(\sum m_0 \otimes n_0 \otimes m_1 \otimes n_1) = \sum m_{00} \otimes n_{00} \otimes m_{01} \otimes n_{01} \otimes m_1 \otimes n_1$

$$\begin{tikzcd} M\otimes N \arrow[r, "\rho_{M\otimes N}"] \arrow[dr, "\otimes 1"'] & M\otimes N\otimes C\otimes D \arrow[d, "id\otimes\varepsilon_{C\otimes D}"] \\ & M\otimes N\otimes\Bbbk \end{tikzcd}$$

$$\begin{aligned} id \otimes \varepsilon_{C\otimes D}(\textstyle\sum m_0 \otimes n_0 \otimes m_1 \otimes n_1) &= \textstyle\sum m_0 \otimes n_0\ \varepsilon_C(m_1)\,\varepsilon_D(m_2) \\ &= \textstyle\sum m_0 \varepsilon_C(m_1) \otimes n_0 \varepsilon_D(m_2) \\ &= m \otimes n \end{aligned}$$

$\therefore$ $M \otimes N$ is a $(C\otimes D)$-comodule, where $C \otimes D$ pointed ($M \otimes N$ finite)

$C \otimes D$-comod locally finite abelian category.

Next we show F right exact. i.e. $\forall\ 0 \to L_1 \to L_2 \to L_3 \to 0$ in $C\otimes D$-comod

We show $\bar{F}(L_1)\to\bar{F}(L_2)\to\bar{F}(L_3)\to 0$

(ii) It is unique up to a unique equivalence.

pf:
$$\begin{tikzcd}
\mathcal{C}\times\mathcal{D} \arrow[r, "\boxtimes"] \arrow[d, "\boxtimes'"'] & \mathcal{C}\boxtimes\mathcal{D} \\
\mathcal{C}\boxtimes'\mathcal{D} \arrow[ur, dashed, "\exists!\,\overline{\boxtimes}"'] &
\end{tikzcd}$$
$\overline{\boxtimes}\circ\boxtimes'=\boxtimes$ ①

利用$\mathcal{C}\boxtimes'\mathcal{D}$泛性质

$$\begin{tikzcd}
\mathcal{C}\times\mathcal{D} \arrow[r, "\boxtimes"] \arrow[d, "\boxtimes'"'] & \mathcal{C}\boxtimes\mathcal{D} \arrow[dl, dashed, "\exists!\,\overline{\boxtimes}'"] \\
\mathcal{C}\boxtimes'\mathcal{D} &
\end{tikzcd}$$
$\overline{\boxtimes}'\circ\boxtimes=\boxtimes'$ ②

利用$\mathcal{C}\boxtimes\mathcal{D}$泛性质

② $\boxtimes'=\overline{\boxtimes}'\circ\boxtimes$ 代入①

$\overline{\boxtimes}\circ\overline{\boxtimes}'\circ\boxtimes=\boxtimes$

$\overline{\boxtimes}\circ\overline{\boxtimes}'=\mathrm{id}_{\mathcal{C}\boxtimes\mathcal{D}}$

①代入②

$\overline{\boxtimes}'\circ\overline{\boxtimes}\circ\boxtimes'=\boxtimes'$

$\overline{\boxtimes}'\circ\overline{\boxtimes}=\mathrm{id}_{\mathcal{C}\boxtimes'\mathcal{D}}$.

(iii) Let C, D be coalg. and let $\mathcal{C}=C$-comod and $\mathcal{D}=D$-comod. Then $\mathcal{C}\boxtimes\mathcal{D}=(C\otimes D)$-comod

pf: by (i) (ii)

(iv) The bifunctor $\boxtimes$ is exact in both variables and satisfies

$$\mathrm{Hom}_{\mathcal{C}}(X_1,Y_1)\otimes\mathrm{Hom}_{\mathcal{D}}(X_2,Y_2)\cong\mathrm{Hom}_{\mathcal{C}\boxtimes\mathcal{D}}(X_1\boxtimes X_2,\ Y_1\boxtimes Y_2)$$
$$(f\ ,\ g)\longmapsto f\otimes g$$

$$\begin{tikzcd}
X_1 \arrow[r, "f"] \arrow[d, "\rho_{X_1}"'] & Y_1 \arrow[d, "\rho_{Y_1}"] \\
X_1\otimes C \arrow[r, "f\otimes id"] & Y_1\otimes C
\end{tikzcd}
\qquad
\begin{tikzcd}
X_2 \arrow[r, "g"] \arrow[d, "\rho_{X_2}"'] & Y_2 \arrow[d, "\rho_{Y_2}"] \\
X_2\otimes D \arrow[r, "g\otimes id"] & Y_2\otimes D
\end{tikzcd}$$

$$\begin{tikzcd}
X_1\otimes C\otimes X_2\otimes D \arrow[dr, "\mathrm{id}\otimes\tau\otimes\mathrm{id}"'] & X_1\otimes_k X_2 \arrow[l, "\rho_{X_1}\otimes\rho_{X_2}"'] \arrow[r, "f\otimes g"] \arrow[d, "\rho_{X_1\otimes X_2}"'] & Y_1\otimes_k Y_2 \arrow[r, "\rho_{Y_1}\otimes\rho_{Y_2}"] \arrow[d, "\rho_{Y_1\otimes Y_2}"] & Y_1\otimes C\otimes Y_2\otimes D \arrow[dl, "\mathrm{id}\otimes\tau\otimes\mathrm{id}"] \\
& X_1\otimes_k X_2\otimes C\otimes D \arrow[r, "f\otimes g\otimes\mathrm{id}"] & Y_1\otimes_k Y_2\otimes C\otimes D &
\end{tikzcd}$$

$$\rho_{Y_1\otimes Y_2}\circ(f\otimes g)\,(x_1\otimes x_2) = \mathrm{id}\otimes\tau\otimes\mathrm{id}(\rho_{Y_1}\otimes\rho_{Y_2})\,(f\otimes g)\,(x_1\otimes x_2)$$

$$= \mathrm{id}\otimes\tau\otimes\mathrm{id}\big[((f\otimes\mathrm{id})\circ\rho_{X_1})\otimes((g\otimes\mathrm{id})\circ\rho_{X_2})\big]\,(x_1\otimes x_2)$$

$$= f\otimes g\otimes\mathrm{id}\,(\rho_{X_1\otimes X_2}(x_1\otimes x_2)) = \sum f(x_{10})\otimes g(x_{20})\otimes x_{11}\otimes x_{21}$$

$\therefore$ $f\otimes g$ is $C\otimes D$-comod-map.

as $\Bbbk$-module. $f\otimes g$ unique

$$\mathrm{Hom}_{\mathcal{C}}(X_1,Y_1)\otimes\mathrm{Hom}_{\mathcal{D}}(X_2,Y_2) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}\boxtimes\mathcal{D}}(X_1\boxtimes X_2,\, Y_1\boxtimes Y_2)$$

(v) Any bilinear bifunctor $F:\mathcal{C}\times\mathcal{D}\to\mathcal{A}$ exact in each variable defines an exact functor $\overline{F}:\mathcal{C}\boxtimes\mathcal{D}\to\mathcal{A}$.

pf: the same as before.

Deligne's tensor product can also be applied to functors. If $F:\mathcal{C}\to\mathcal{C}'$ and $G:\mathcal{D}\to\mathcal{D}'$ are right exact functors between locally finite abelian categories then one defines the functor $F\boxtimes G:\mathcal{C}\boxtimes\mathcal{D}\to\mathcal{C}'\boxtimes\mathcal{D}'$ using the defining universal property (see Definition 1.11.1) of $\mathcal{C}\boxtimes\mathcal{D}$. Namely, the bifunctor

$$F\times G:\mathcal{C}\times\mathcal{D}\to\mathcal{C}'\boxtimes\mathcal{D}':(V,\,W)\mapsto F(V)\boxtimes G(W)$$

canonically extends to a right exact functor $F\boxtimes G:\mathcal{C}\boxtimes\mathcal{D}\to\mathcal{C}'\boxtimes\mathcal{D}'$.

$$\begin{tikzcd}
\mathcal{C}\times\mathcal{D} \arrow[r, "\boxtimes"] \arrow[d, "F\times G"'] & \mathcal{C}\boxtimes\mathcal{D} \arrow[dl, dashed] \\
\mathcal{C}'\boxtimes\mathcal{D}' &
\end{tikzcd}$$

$\exists!\ \overline{F\times G} = F\boxtimes G$.

1.12 (A, m, u) alg.

(Def) The finite dual $A^*_{fin} = \{f \in A^* \mid f(I)=0$ for some ideal of A s.t. $\dim A/I < \infty\}$.

(A^*_{fin}, m^*, u^*) coalg.

Remark 1.12.3 Note that if A does not have finite dimensional modules $(\neq 0)$, then $A^*_{fin} = 0$.

pf: Suppose $A^*_{fin} \neq 0$, $\exists\, 0 \neq f \in A^*$. $f(I)=0$ for some ideal I of finite codimension.

$\because$ ${}_AA$, ${}_AI$ are A-mod $\therefore$ ${}_AA/I$ is A-mod A/I finite dimensional

but A does not have finite dimensional modules. $A/I = 0$ $A = I$

$f \in A^*_{fin}$ $f(A) = 0$ $f = 0$. contradiction.

1.13 Pointed coalg. and the coradical filtration

Let $\mathcal{C}$ be a locally finite abelian category.

Any object $X \in \mathcal{C}$ has a canonical filtration $0 = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n = X$

s.t. X_{i+1}/X_i is the socle (i.e. the maximal semisimple subobject) of X/X_i

(in other words, X_{i+1}/X_i is the sum of all simple subobjects of X/X_i).

pf:

$0 = X_0$，在X (X/X_0)中找出所有 simple subobject $\{R_i^{(0)}\}$. Let $X_1 = \sum R_i$

在 X/X_1 中找出所有 simple subobject $\{R_i^{(1)}\}$ Let $R'/X_1 = \sum R_i^{(1)}$

$\because$ Every abelian category is equivalent, as an additive category, to a full subcategory of the category of left modules over an associative unital ring A.

设N为M的子模，$\pi: M \to M/N$，则在π下，M的包含N的子模与M/N的子模是一一对应的.

$\therefore$ we can find X_2, continue

$\because$ X has finite length, any filtration of X can be extended to Jordan-Hölder series

$\therefore$ X has a canonical filtration.

(Def) The filtration of X by X_i is called the socle filtration or the coradical filtration.

It is easy to show by induction that the socle filtration is a filtration of X of the smallest possible length, s.t. the successive quotients are semisimple. The length of the socle filtration of X is called the Loewy length of X, and denoted $Lw(X)$.

Then we have a filtration of the category $\mathcal{C}$ by Loewy length of objects: $\mathcal{C}_0 \subset \mathcal{C}_1 \subset \cdots$, where $\mathcal{C}_i$ denotes the full subcategory of objects of $\mathcal{C}$ of Loewy length $\leq i+1$.

Clearly, the Loewy length of any subquotient of an object X does not exceed the Loewy length of X, so the categories $\mathcal{C}_i$ are closed under taking subquotient.

(对于包含N的子模H. $M/H \cong (M/N)/(H/N)$ $\quad x+H \mapsto \pi(x)+H/N$)

(Def) The filtration of the category $\mathcal{C}$ by $\mathcal{C}_i$ is called the socle filtration or the coradical filtration of $\mathcal{C}$.

If $\mathcal{C}$ is endowed with an exact faithful functor $F: \mathcal{C} \to \mathrm{Vec}$ then we can define the coalg. $C = \mathrm{Coend}(F)$ and its subcoalg. $C_i = \mathrm{Coend}(F|_{\mathcal{C}_i})$, and we have $C_i \subset C_{i+1}$ and $C = \bigcup_i C_i$

(alternative, we can say that C_i is spanned by matrix elements of C-comodules $F(X)$, $X \in \mathcal{C}_i$.

Let $\mathcal{C}$ be a $\Bbbk$-linear abelian category, and $F : \mathcal{C} \to \mathsf{Vec}$ an exact, faithful functor. In this case one can define the space $\mathsf{Coend}(F)$ as follows:

$$\mathsf{Coend}(F) := (\oplus_{X \in \mathcal{C}} F(X)^* \otimes F(X))/E \tag{1.9}$$

where E is spanned by elements of the form $y_* \otimes F(f)x - F(f)^* y_* \otimes x$, $x \in F(X)$, $y_* \in F(Y)^*$, $f \in \mathsf{Hom}(X, Y)$; in other words,

$$\mathsf{Coend}(F) = \varinjlim \mathsf{End}(F(X))^*.$$

[5] If M is a right C-comodule with coaction $\pi : M \to M \otimes C$ then a matrix element of M is an element $(f \otimes 1, \pi(m)) \in C$, where $f \in M^*$, $m \in M$.

① I. $\mathcal{C}_0$: $\Bbbk$-linear abelian cat.

$\because \mathcal{C}$ additive, $\therefore \mathcal{C}_0$ additive

next we show $\mathcal{C}_0$ has kernel.

$\because \mathcal{C}_0$ s.s. and $\mathcal{C}$ abelian cat. $\therefore \forall X \in \mathcal{C}_0$, $Y \hookrightarrow X$ in $\mathcal{C}$. we show Y s.s.

• X s.s. $\Longleftrightarrow$ X是它 Jordan-Hölder 列中单商因子直和.

pf: $\Leftarrow$ ✓ $\quad \Rightarrow$ $X = \bigoplus Y_i$ 是X的 s.s 分解，则 $0 \subset Y_1 \subset Y_1 \oplus Y_2 \subset \cdots \subset Y_1 \oplus \cdots \oplus Y_{n-1} \subset \cdots \quad \subset X$ 是X的合成列.

$Y_1 \oplus Y_2 / Y_1 \cong Y_2$，这是因为
$$\begin{tikzcd} Y_1 \arrow[r, hook, "i_1"] & Y_1 \oplus Y_2 \arrow[r, shift left, "p_2"] \arrow[dr, "\forall f"'] & Y_2 \arrow[l, shift left, dashed, "i_2"] \arrow[d, dashed] \\ & & Z \end{tikzcd}$$

$p_2 i_1 = 0$. $\forall f$, s.t. $f i_1 = 0$, $\exists !$ $f i_2 : Y_2 \to Z$

($(f i_2) p_2 = f(id_{Y_1 \oplus Y_2} - i_1 p_1) = f$

设 $g: Y_2 \to Z$, s.t. $g p_2 = f$

$f i_2 = g p_2 i_2 = g$ $\quad \therefore \exists ! f i_2$.)

$\mathrm{Coker}(Y_1 \hookrightarrow Y_1 \oplus Y_2) \cong Y_1 \oplus Y_2 / Y_1$ $\qquad Y_2 = \mathrm{Coker}(Y_1 \hookrightarrow Y_1 \oplus Y_2)$

II. $\mathcal{C}_1$ abelian cat. $\quad 0 \subset X_0 \subset X_1$, X_0 s.s $\quad X_1/X_0$ s.s. $\quad \forall Y \subset X_1$.

$\because \mathcal{C}_0$ abelian cat. + pull back
$$\begin{tikzcd} 0 \arrow[r, hook] & X_0 \arrow[r, hook] & X_1 \arrow[r] & X_1/X_0 \arrow[r] & 0 \\ 0 \arrow[r, hook] & Y \cap X_0 \arrow[r, hook] \arrow[u] & Y \arrow[r] \arrow[u] & Y/Y \cap X_0 \arrow[r] \arrow[u, dashed] & 0 \end{tikzcd}$$
$\to \mathrm{coker}(Y \to X_1/X_0)$

$\Rightarrow Y \in \mathcal{C}_1$

III. induction, $\mathcal{C}_i$ abelian cat.

② $C_i \triangleq \mathrm{Coend}(F|\mathcal{C}_i) = \bigoplus_{X\in\mathcal{C}_i} F(X)^*\otimes F(X)/E_i$ $\quad F: \mathcal{C}_i \xrightarrow{\text{equivalence}} C_i\text{-comod}$ (the cat. of f.d. right comodules over C_i

$\bigoplus_{X\in\mathcal{C}_i} F(X)^*\otimes F(X) \subseteq \bigoplus_{X\in\mathcal{C}_{i+1}} F(X)^*\otimes F(X)$

$E_i = E_{i+1} \cap \left(\bigoplus_{X\in\mathcal{C}_i} F(X)^*\otimes F(X)\right)$

$\bigoplus_{X\in\mathcal{C}_i} F(X)^*\otimes F(X)/E_{i+1}\cap\left(\bigoplus_{X\in\mathcal{C}_i} F(X)^*\otimes F(X)\right) \cong \bigoplus_{X\in\mathcal{C}_i} F(X)^*\otimes F(X) + E_{i+1}/E_{i+1} \hookrightarrow \bigoplus_{X\in\mathcal{C}_{i+1}} F(X)^*\otimes F(X)/E_{i+1}$

$\therefore C_i \subseteq C_{i+1}$

$C = \bigcup_i C_i$

③ C_i is spanned by matrix elements of C-comod $(F(X))$ $X\in C_i$

$f\in F(X)^*,\quad F(x)\in F(X),\quad X\in\mathcal{C}_i$

$\langle (f\otimes id, \rho(F(x))\rangle = C_i = \sum_{X\in\mathcal{C}_i} (F(X)^*\otimes id)\circ \rho_{F(X)}(F(X))$

Thus we have defined an increasing filtration by subcoalg. of any coalg. C. This filtration is called the coradical filtration, and the term C_0 is called the coradical of C

The "linear alg" define of the coradical filtration is as follow. One says that a coalg. is simple if it does not have nontrivial subcoalg., i.e. if it is finite dimensional, and its dual is a simple (i.e. matrix) alg.

$\Rightarrow$ Any simple subcoalg. of C is finite dimensional.
$V \subseteq C$ is a subcoalg. iff $V^{\perp} \subseteq C^*$ is a two-sided ideal of C^*.
A是F上有限维单代数 $\Leftrightarrow$ $A = M_n(D)$，其中，D是F上有限维可除代数

$\Leftarrow$ C f.d., then all subspace X of C^* closed $X^{\perp\perp} = X$, $X = (X^{\perp})^{\perp}$

Then C_0 is the sum of all simple subcoalg. of C. The coalg. C_{n+1} for $n \geq \cancel{1}\, 0$ are then defined inductively to be the spaces of those $x \in C$ for which $\Delta(x) \in C_n \otimes C + C \otimes C_0$.

Let C_0 be the coradical of C and set $C_{n+1} = C_n \wedge C_0$ for $n \geq 0$. $\left(C_n = \wedge^{n+1} C_0 = (\wedge^n C_0) \wedge C_0.\right)$

$C \xrightarrow{\Delta} C \otimes C \xrightarrow{\pi_1 \otimes \pi_2} C/C_n \otimes C/C_0 \qquad C_n \wedge C_0 = \ker((\pi_1 \otimes \pi_2)\Delta) = \Delta^{-1}(C \otimes C_0 + C_n \otimes C)$

then $\{C_n\}_{n=0}^{\infty}$ is a filtration of C. (coradical filtration of C)

Let C_0 be the coradical of C and set $C_n = C_{n-1} \wedge C_0$ for $n \geq 1$. We will show that $C_0^{(\infty)} = C$.

Suppose that D is a finite-dimensional subcoalgebra of C. Since all subspaces of D^* are closed, $C^{\perp}$ is the intersection of the maximal ideals of C^* by part (d) of Proposition 2.3.7. Thus $D_0^{\perp} = \mathrm{Rad}(D^*)$. Since D is finite-dimensional, $\mathrm{Rad}(D^*)$ is nilpotent. Therefore $D_n = ((\mathrm{Rad}(D^*))^{n+1})^{\perp} = (0)^{\perp} = D$ for some $n \geq 0$. Since C is the sum of its finite-dimensional subcoalgebras $C_0^{(\infty)} = C$. By part (a) of Proposition 4.1.4:

Exercise 1.13.3 (i) Suppose that C is a finite dimensional coalg. and I is the Jacobson radical of C^*. Show that $C_n^{\perp} = I^{n+1}$, and generalize this statement to the infinite dimensional case. This justifies the term "coradical filtration".

pf: (1) If C is finite-dimensional. C_0 is the sum of the simple subcoalg. D_α of C, then the Jacobson radical of an algebra A, $\mathrm{Rad}(C^*) = C_0^{\perp}$, $\therefore I = C_0^{\perp}$. $\left(\bigcap_\alpha V_\alpha^{\perp} = (\sum_\alpha V_\alpha)^{\perp}\right.$ in U^*, $\bigcap_\alpha X_\alpha^{\perp} = (\sum_\alpha X_\alpha)^{\perp}$ in U, $V_\alpha \subseteq U$, $\left.X_\alpha \subseteq U^*\right)$

$\because (\sum_\alpha D_\alpha)^{\perp} = \bigcap_\alpha D_\alpha^{\perp}$ C f.d., the subspace of C^* is closed, M is maximal ideal of C^*. $M = (M^{\perp})^{\perp}$

$\therefore D_\alpha^{\perp}$ is all the maximal ideal of C^* $\therefore \mathrm{Rad}(C^*) = C_0^{\perp}$

$n=0$ $C_0 = I^{\perp}$ assume $C_{n-1} = (I^n)^{\perp}$, $C_n = (C_{n-1} \wedge C_0) = (C_{n-1}^{\perp} C_0^{\perp})^{\perp} = (I^n \cdot I)^{\perp}$

$C_{n-1}^{\perp} = (I^n)^{\perp\perp} = I^n$ (C f.d subspace of C^* closed) $\qquad C_n^{\perp} = (I^{n+1})^{\perp\perp} = I^{n+1}$

(Prop) Let $I = C_0^\perp$ in C^*, then

(1) $I = Rad(C^*)$

(2) $C_n = (I^{n+1})^\perp$

(3) $\bigcap_{n\geq 0} I^n = (0)$

pf: $C_0 = \sum D_\alpha$. D_α simple subcoalg. of C, $M_\alpha = D_\alpha^\perp$ is maximal ideal of C^* of f.d. codim

Then $I = (\sum D_\alpha)^\perp = \bigcap_\alpha D_\alpha^\perp = \bigcap_\alpha M_\alpha$ $\quad I \supseteq Rad(C^*)$

For the other containment, we first show (2). by induction on n.

Now $C_0 = C_0^{\perp\perp} = I^\perp$ $\quad n=0$ ✓

Assume true for $n-1$, $c \in C$.

$\langle I^{n+1}, c\rangle = 0 \Leftrightarrow \langle I \cdot I^n, c\rangle = 0 \Leftrightarrow \langle I \otimes I^n, \Delta c\rangle = 0$

$\Leftrightarrow \Delta c \in (I \otimes I^n)^\perp$

$\Leftrightarrow \Delta c \in C \otimes (I^n)^\perp + I^\perp \otimes C$

$\Leftrightarrow \Delta c \in C \otimes C_{n-1} + C_0 \otimes C$

$\Leftrightarrow c \in C_{n-1} \wedge C_0 = C_n$ $\quad (X \wedge Y = \Delta^{-1}(C \otimes Y + X \otimes C))$

(2) ✓

return to (1) Assume $f \in I$, by (2) $C_n = (I^{n+1})^\perp$ $\quad \langle f^{n+1}, C_n\rangle = 0 \quad \forall n \geq 0$

$g = \sum_{n=0}^\infty f^n$ is defined on all of C, where $f^0 = \varepsilon$. But $g = (\varepsilon - f)^{-1}$ in C^*; that is

$\varepsilon - f$ is invertible for all $f \in I$. It follows that $I \subseteq Rad(C^*)$

(3) $I^n \subseteq (I^n)^{\perp\perp}$ $\quad C = \bigcup_{n\geq 0} C_n$

(ii) Show that the coproduct respects the coradical filtration, i.e. $\Delta(C_n) \subset \sum_{i=0}^n C_i \otimes C_{n-i}$

pf: $C_n = C_{n-1} \wedge C_0$ $\quad ((X \wedge Y) \wedge Z = X \wedge (Y \wedge Z))$

$C_n = (\wedge^i C_0) \wedge (\wedge^{n+1-i} C_0)$

$\therefore \forall\ 1 \leq i \leq n$ $\quad \Delta C_n \subseteq C \otimes \wedge^{n+1-i} C_0 + \wedge^i C_0 \otimes C$

$= C \otimes C_{n-i} + C_{i-1} \otimes C$ $\quad (*)$

$i = 0, n+1$, $*$ ✓ by $\Delta C_n \subseteq C_n \otimes C_n$ $\quad (\wedge^0 X = \{0\},\ \wedge^1 X = X,\ \wedge^n X = (\wedge^{n-1} X) \wedge X$.

by lemma, if V is a vector space with subspaces $\{0\} = V_0 \subset V_1 \subset V_2 \subset \cdots$ then

$\bigcap_{i=0}^n (V \otimes V_{n-i} + V_i \otimes V) = \sum_{i=1}^n V_i \otimes V_{n+1-i}$ $\quad$ let $V_i = C_{i-1}$.

(iii) Show that C_0 is the direct sum of simple subcoalg. of C. In particular, grouplike elements of any coalg. C are linearly independent.

pf: 略.

Def 1.3.4 A coalg. C is said to be cosemisimple if C is a direct sum of simple subcoalg. ($C = \operatorname{Corad}(C)$)

Clearly, a coalg. C is cosemisimple iff C-comod is a semisimple category.

Definition 3.4.9. Let C be a coalgebra over the field k. A *completely reducible C-comodule* is a C-comodule M which is the sum of its simple

subcomodules.

Cosemisimple coalgebras over k are characterized much in the same way as are semisimple artinean algebras over k.

Theorem 3.4.10. *Suppose that C is a coalgebra over the field k. Then the following are equivalent:*

(a) *All right C-comodules are completely reducible.*
(b) $C = C_0$.
(c) *All left C-comodules are completely reducible.*

Proof. We need only show the equivalence of parts (a) and (b). For the equivalence of parts (c) and (b) is the equivalence of parts (a) and (b) for C^{cop}.

Suppose that all right C-comodules are completely reducible. Then C itself is the sum of simple right coideals of C. Therefore $C = C_0$ by part (a) of Theorem 3.4.2.

On the other hand, suppose that $C = C_0$ and let $\{D_i\}_{i\in I}$ be the set of simple subcoalgebras of C. Then any right C-comodule (M, ρ) can be written $M = \bigoplus_{i\in I} M_i$, where $\rho(M_i) \subseteq M_i \otimes D_i$ for all $i \in I$, by Exercise 3.2.11. To complete the proof we may assume that C is simple. In this case C^* is a finite-dimensional simple algebra over k by Corollary 2.3.8 and thus all C^*-modules are completely reducible. Therefore all right C-comodules are completely reducible and the theorem is proved. □

Suppose that (M, ρ) is a simple right C-comodule. Then $\rho(M) \subseteq M \otimes D$ for some simple subcoalgebra D of C by part (d) of Theorem 3.2.11. Thus the simple C-comodules can be understood in terms of the sum of the simple subcoalgebras of C.

Theorem 3.4.2. *Let C be a coalgebra over the field k. Then C_0 is*

(a) *the sum of the simple right coideals of C and is also*
(b) *the sum of the simple left coideals of C.*

Proof. Since the coradicals of C and C^{cop} are the same, we need only establish part (a). Let N be a simple right coideal of C. As noted

above $\Delta(N) \subseteq N \otimes D$, where D is a simple subcoalgebra of C. Since $\mathrm{I}_C = (\epsilon \otimes \mathrm{I}_C) \circ \Delta$ it follows that $N \subseteq \epsilon(N) D \subseteq C_0$.

Let D be a simple subcoalgebra of C. To complete the proof of the theorem we need only show that D is the sum of simple right coideals of C. Since D is a non-zero finite-dimensional right coideal of C, it follows that D contains a minimal right coideal N of C. Let $c^* \in C^*$. By (2.19) the linear endomorphism $\mathrm{R}(c^*)$ of C defined by $\mathrm{R}(c^*)(c) = c \leftharpoonup c^*$ for all $c \in C$ is a map of right C-comodules. Thus $N \leftharpoonup c^*$ is a homomorphic image of N. Consequently $N \leftharpoonup c^* = (0)$ or $N \leftharpoonup c^* \simeq N$ since N is simple. Let $E = N \leftharpoonup C^*$. Then $E \subseteq D$ and is the sum of simple right coideals of C. Since

$$C^* \rightharpoonup E \leftharpoonup C^* = C^* \rightharpoonup N \leftharpoonup C^* = N \leftharpoonup C^* = E$$ $\Rightarrow E$ is C^*-sub-bimod

if follows by part (b) of Proposition 2.3.5 that E is a subcoalgebra of C. Since D is a simple subcoalgebra of C we conclude $D = E$ and thus is the sum of simple right coideals of C. □

V is right coideal of C iff V is a left C^*-submodule

V is a subcoalg. of C iff V both left and right coideal

Exercise 3.2.11. Let (M, ρ) be a right C-comodule. Suppose that $C = \bigoplus_{i\in I} D_i$ is the direct sum of subcoalgebras. Show that:

(a) $M = \bigoplus_{i\in I} M_i$, where $\rho(M_i) \subseteq M_i \otimes D_i$.
(b) For such a decomposition of M necessarily $M_i = \rho^{-1}(M \otimes D_i)$.

[Hint: Since $\sum_{i\in I} M \otimes D_i$ is direct and ρ is one-one, $\sum_{i\in I} M_i$ is direct, where $M_i = \rho^{-1}(M \otimes D_i)$. To show that $\sum_{i\in I} M_i = M$ we may assume that I is finite and without loss of generality let $I = \{1, \ldots, r\}$. For each $i \in I$ define $e_i \in C^*$

by $e_i|D_j = \delta_{i,j}\epsilon|D_i$. For $m \in M$ show that $m = \epsilon \rightharpoonup m = e_1 \rightharpoonup m + \cdots + e_r \rightharpoonup m \in M_1 + \cdots + M_r$.]

THEOREM 4.4. *Any module for a semi-simple artinian ring R is completely reducible and there is a 1-1 correspondence between the isomorphism classes of irreducible modules for R and the simple components of the ring. More precisely, if $R = R_1 \oplus \cdots \oplus R_s$ where the R_j are the simple components and I_j is a minimal left ideal in R_j, then $\{I_1, \ldots, I_s\}$ is a set of representatives of the isomorphism classes of irreducible R-modules.*

Let C be a coalgebra and $M \in \mathcal{M}^C$. We recall that the socle of M, denoted by $s(M)$, is the sum of all simple subcomodules of M. Then $s(M)$ is a semisimple subcomodule of M. Since any non-zero comodule contains a simple subcomodule, we see that $s(M)$ is essential in M. We can define recurrently an ascending chain $M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n \subseteq \ldots$ of subcomodules of M as follows. Let $M_0 = s(M)$, and for any $n \geq 0$ we define M_{n+1} such that $s(M/M_n) = M_{n+1}/M_n$. This ascending chain of subcomodules is called the Loewy series of M. Since M is the union of all subcomodules of finite dimension, we have that $M = \cup_{n\geq 0} M_n$.
If I is a two-sided ideal of C^*, we denote by $ann_M(I) = \{x \in M | Ix = 0\}$, which is clearly a left C^*-submodule of M.

Corollary 3.1.10 *Let C be a coalgebra and $C_0, C_1, \ldots$ the Loewy series of the right (or left) C-comodule C. Then C_0 is the coradical of C, $C_n = \wedge^{n+1} C_0$ and C_n is a subcoalgebra of C for any $n \geq 0$.*

Proof: We have seen in Proposition 3.1.4 that the coradical of C is just the socle of the right C-comodule C. Lemma 3.1.9 shows that $C_n = ann_C(J(C^*)^{n+1})^\perp$. By Proposition 2.5.3(i) we have $C_n = (J(C^*)^{n+1})^\perp$, and by Lemmma 2.5.7 we see that $C_n = \wedge^{n+1} C_0$. By Lemma 1.5.23 C_n is a subcoalgebra. ∎

Proposition 3.1.4 *Let C be a coalgebra. Then $C_0 = s({}_C C) = s(C_C)$, where $s(C_C)$ is the socle of C as an object of $\mathcal{M}^C$, and $s({}_C C)$ is the socle of C as an object of ${}^C\mathcal{M}$.*

Proof: We will show that $C_0 = s(C_C)$. The proof of the fact that $C_0 = s({}_C C)$ is similar (or can bee seen directly by looking at the co-opposite coalgebra and applying the result about the right socle). A simple subcoalgebra A of C is a right C-subcomodule of C. Since A is a finite direct sum of simple right coideals of A, we see that A is semisimple of finite length when regarded as a right C-comodule. Thus $A \subseteq s(C_C)$, and then $C_0 \subseteq s(C_C)$.

$S \to S \otimes C(1)$

Conversely, let $S \subseteq s(C_C)$ be a simple right C-comodule, and let A be the coalgebra associated to S. By Exercise 3.1.2 A is a simple coalgebra, so $A \subseteq C_0$. But $S \subseteq A$, since for $c \in S$ we have $c = \sum \varepsilon(c_1)c_2 \in A$. Thus $S \subseteq A \subseteq C_0$, so $s(C_C) \subseteq C_0$. ∎

$S \subseteq C$

Lemma 2.5.7 *For any subspaces X and Y of the coalgebra C we have that $X \wedge Y = (X^\perp Y^\perp)^\perp$.*
In particular, if A is a subcoalgebra of C, then for any positive integer n we have that $\wedge^n A = (J^n)^\perp$, where $J = A^\perp$.

Lemma 3.1.9 *Let $I = J(C^*) = C_0^\perp$ and $M \in \mathcal{M}^C$. Then for any $n \geq 0$ we have $M_n = ann_M(I^{n+1})$.*

$IM_0 = M_0,\ M_0 = 0$

Proof: We use induction on n. For $n = 0$, we have $ann_M(I) = M_0 = s(M)$. Indeed, $IM_0 = J(C^*)M_0 = 0$, since the Jacobson radical of C^* annihilates all simple left C^*-modules. Thus $M_0 \subseteq ann_M(I)$. On the other hand $C_0^\perp ann_M(I) = I ann_M(I) = 0$, so by Proposition 2.5.3, $ann_M(I)$ is a right C_0-comodule. Since C_0 is a cosemisimple coalgebra, $ann_M(I)$ is a semisimple object of the category $\mathcal{M}^{C_0}$, and then also of the category $\mathcal{M}^C$. We obtain that $ann_M(I) \subseteq s(M) = M_0$.
Assume now that $M_{n-1} = ann_M(I^n)$ for some $n \geq 1$. Since $M_n/M_{n-1} = s(M/M_{n-1})$ is semisimple, we have that $I(M_n/M_{n-1}) = 0$, therefore $IM_n \subseteq M_{n-1}$. Then $I^{n+1}M_n = I^n(IM_n) \subseteq I^n M_{n-1} = 0$, so $M_n \subseteq ann_M(I^{n+1})$. If we denote $X = ann_M(I^{n+1})$, we have $I^{n+1}X = 0$, so $IX \subseteq ann_M(I^n) = M_{n-1}$. Then $I(X/M_{n-1}) = 0$ and by the same argument as above X/M_{n-1} is a right C_0-comodule, so X/M_{n-1} is a semisimple comodule.
We have that $s(M/M_{n-1}) = M_n/M_{n-1}$, so we obtain that $X \subseteq M_n$. Thus $M_n = ann_M(I^{n+1})$, which ends the proof. ∎

$X/M_{n-1} \subset M_n/M_{n-1}$

Proposition 2.5.3 *Let C be a coalgebra. Then the following assertions hold.*
(i) If I is a left ideal of C^*, then $I^\perp = ann_C(I) = \{c \in C | I \rightharpoonup c = 0\}$.
(ii) If X is a left coideal of C, then $X^\perp = ann_{C^*}(X)$, where

$$ann_{C^*}(X) = \{f \in C^* | f \rightharpoonup x = 0 \text{ for any } x \in X\}.$$

(iii) If $\rho : M \to M \otimes C$ is the comodule structure map of the right C-comodule M, and J is a two-sided ideal of C^* such that $JM = 0$, then $\rho(M) \subseteq M \otimes J^\perp$, i.e. M is a right comodule over the subcoalgebra $J^\perp$ of C.
(iv) If M is a right C-comodule and $A = (ann_{C^*}(M))^\perp$, then A is the smallest subcoalgebra of C such that $\rho(M) \subseteq M \otimes A$. The subcoalgebra A is called the coalgebra associated to the comodule M.

Proof: (i) Let $c \in ann_C(I)$. Then $f \rightharpoonup c = 0$ for any $f \in I$. Then

$$\begin{aligned} f(c) &= f(\sum \varepsilon(c_1)c_2) \\ &= \sum \varepsilon(f(c_2)c_1) \\ &= \varepsilon(f \rightharpoonup c) \\ &= 0 \end{aligned}$$

so $c \in I^\perp$.
Conversely, if $c \in I^\perp$, then $f(c) = 0$ for any $f \in I$. Let $\Delta(c) = \sum_{1 \leq i \leq n} x_i \otimes y_i$ with $(x_i)_{1 \leq i \leq n}$ linearly independent. If $1 \leq t \leq n$, there exists $g \in C^*$ such that $g(x_t) = 1$ and $g(x_i) = 0$ for any $i \neq t$. Then $gf \in I$ and

$$\begin{aligned} 0 &= (gf)(c) \\ &= \sum_{1 \leq i \leq n} g(x_i)f(y_i) \\ &= f(y_t) \end{aligned}$$

so $f(y_t) = 0$. Then $f \rightharpoonup c = \sum_{1 \leq i \leq n} f(y_i)x_i = 0$, which shows that $c \in ann_C(I)$. Thus $I^\perp \subseteq ann_C(I)$.
(ii) If $f \in X^\perp$ then $f(X) = 0$. Let $x \in X$. Then $f \rightharpoonup x = \sum f(x_2)x_1 = 0$, thus $x \in ann_{C^*}(X)$.
Conversely, assume that $f \in ann_{C^*}(X)$. Then for any $x \in X$ we have that

$$\begin{aligned} f(x) &= f(\sum \varepsilon(x_1)x_2) \\ &= \varepsilon(\sum f(x_2)x_1) \\ &= \varepsilon(f \rightharpoonup x) \\ &= 0 \end{aligned}$$

so $f \in X^\perp$.
(iii) For $m \in M$ let $\rho(m) = \sum m_0 \otimes m_1$, and assume that the m_0's are linearly independent. If $f \in J$ we have that $0 = fm = \sum f(m_1)m_0$, so $f(m_1) = 0$ for any m_1, thus $m_1 \in J^\perp$. We obtain that $\rho(M) \subseteq M \otimes J^\perp$.
(iv) Denote $J = ann_{C^*}(M)$. Then J is a two-sided ideal of C^* and by (iii) we have $\rho(M) \subseteq M \otimes A$, and $A = J^\perp$ is a subcoalgebra of C.
Assume that B is a subcoalgebra of C such that $\rho(M) \subseteq M \otimes B$. If $f \in B^\perp$ and $m \in M$, then $fm = 0$, so $B^\perp \subseteq ann_{C^*}(M) = J$. Thus $J^\perp \subseteq (B^\perp)^\perp = B$, and we find that $A \subseteq B$. ∎

(iv) We have defined C_i in three ways: as $\mathrm{Coend}(F|_{\mathcal{C}_i})$, as the span of matrix elements of $F(X)$, $X \in \mathcal{C}_i$; and by the "linear alg" definition above. Show that these three definitions agree.

① C_0 cosemisimple

C_0 cosemisimple $\Leftrightarrow$ C_0-comod s.s.

$C_0\text{-comod} \cong \mathcal{C}_0$ $\quad$ $\mathcal{C}_0$: cosemisimple, C_0-comod cosemisimple

$\therefore$ C_0 cosemisimple

②
$$\begin{tikzcd} \mathcal{C}_0 \arrow[r, phantom, "\subset"] \arrow[d, "F|_{\mathcal{C}_0}"'] & \mathcal{C} \arrow[d, "\cong F"] \\ C_0\text{-comod} \arrow[r, dashed] & C\text{-comod} \end{tikzcd}$$

$\mathrm{corad}(C) = \bigoplus_{I \subseteq C,\ \text{simple coideal}} I$

all simple C-comodule M_i, $C_0 = \bigoplus M_i$

$C_0 = s(C^C)$

③
$$\begin{tikzcd} \mathcal{C}_i \arrow[r, phantom, "\subset"] \arrow[d] & \mathcal{C}_{i+1} \arrow[d] \\ C_i\text{-comod} \arrow[r, dashed] & C_{i+1}\text{-comod} \end{tikzcd}$$

?

$0 = X_0 \subset \cdots \subset X_i \subset X_{i+1} = X$

$0 = F(X_0) \subset \cdots \subset F(X_i) \subset F(X_{i+1}) = F(X)$ in C-comod

$F(X_{i+1}/X_i) \cong F(X_{i+1})/F(X_i)$ in C-comod

$0 \to X_i \to X_{i+1} \to X_{i+1}/X_i \to 0$

$0 \to F(X_i) \to F(X_{i+1}) \to F(X_{i+1}/X_i) \to 0$

$\mathrm{Coker}(F(X_i) \to F(X_{i+1})) = F(X_{i+1})/F(X_i)$

Let $gr(C) := \bigoplus_{n=0}^{\infty} \cancel{C_{i+1}/C_i}\ C_i/C_{i-1}$ be the associated graded coalg. of a coalg. C with respect to the coradical filtration. Then $gr(C)$ is a $\mathbb{Z}_+$-graded coalg.

Let Γ be a set. Comodules over $\Bbbk\Gamma$ are given by Γ-graded vector spaces. A **Γ-grading** of a vector space V is a family $\mathcal{V} = (V(g))_{g\in\Gamma}$ of subspaces of V such that

$$V = \bigoplus_{g\in\Gamma} V(g).$$

A **Γ-graded vector space** is a pair $(V, \mathcal{V})$, where V is a vector space with a grading (or a **gradation**) $\mathcal{V}$. For a graded vector space $V = (V, \mathcal{V})$ we denote by $\pi_g^V : V \to V(g)$, $g \in \Gamma$, the canonical projection. An element $v \in V$ is called **homogeneous of degree** $g \in \Gamma$ if $v \in V(g)$. We write $\deg(v) = g$, if $v \in V(g)$.

We also use the notation $V_g = V(g)$, in particular, when G is a monoid or a group.

Let $\Gamma\text{-Gr}\mathcal{M}_\Bbbk$ be the category of Γ-graded vector spaces, where a morphism $f : (V, \mathcal{V}) \to (W, \mathcal{W})$ is a **graded map** or a **homogeneous map** (of degree 0), that is a $\Bbbk$-linear map with $f(V(g)) \subseteq W(g)$ for all $g \in \Gamma$.

DEFINITION 1.2.26. (1) An $\mathbb{N}_0$-**graded coalgebra** is a pair $(C, \mathcal{C})$, where C is a coalgebra, $(C, \mathcal{C})$ is an $\mathbb{N}_0$-graded vector space, and

(1.2.3) $$\Delta(C(n)) \subseteq \bigoplus_{r+s=n} C(r) \otimes C(s) \text{ for all } n \geq 0,$$

(1.2.4) $$\varepsilon(C(n)) = 0 \text{ for all } n > 0.$$

We write

$$\Delta_{m,n} : C(m+n) \subseteq C \xrightarrow{\Delta} C \otimes C \xrightarrow{\pi_m^C \otimes \pi_n^C} C(m) \otimes C(n),\ m, n \in \mathbb{N}_0,$$

for the **components of the comultiplication** Δ.

Now suppose we start with a filtered coalgebra $C = \cup C_n$. We will define the <u>associated graded coalgebra</u>, denoted gr C, as follows:

$$\operatorname{gr} C(n) = C_n/C_{n-1} \quad \text{for } n \geq 1$$
$$\operatorname{gr} C(0) = C_0$$

①

$V_i = C_i/C_{i-1} \quad i \geq 1$

Let $v_n = c_n + C_{n-1} \in V_n$

$\Delta(c_n + C_{n-1}) = \sum (c_{n(1)} + C_{n-1}) \otimes (c_{n(2)} + C_{n-1})$

$\Delta(C_n) \subset \sum_{i=0}^{n} C_i \otimes C_{n-i} \quad \sum c_{n(1)} + c_{n(2)} \in \sum_{i=0}^{n} C_i \otimes C_{n-i}$

$\therefore \Delta(V_n) \subseteq \bigoplus_{r+s=n} V(r) \otimes V(s)$

EXERCISE. Show that if C is a coalgebra where $C = \bigoplus_{i=0}^{\infty} C(i)$ and $\Delta(C(n)) \subset \sum_{i=0}^{n} C(i) \otimes C(n-i)$ then $\epsilon|C(n) = 0$ for $n \geq 1$.

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[dr, "\otimes 1"'] & C \otimes C \arrow[d, "id \otimes \varepsilon"] \\ & C \otimes \Bbbk \end{tikzcd}$$

$\Delta(C_1) \subseteq C(0) \otimes C(1) + C(1) \otimes C(0)$

$\forall c \in C_1 \quad \Delta(c) = \sum c_{(1)} \otimes c_{(2)}$
$= \sum c^0 \otimes c^1 + \sum c^2 \otimes c^3$
$c^0, c^3 \in C(1),\ c^1, c^2 \in C(0)$

$c = \sum c^0 \varepsilon(c^1) + \sum c^2 \varepsilon(c^3)$

$c \in C_1, \quad c \notin C_0 \quad \varepsilon(c^3) = 0$

$\therefore$ 同理 $\varepsilon(c^0) = 0$

$c = \sum c^0 \varepsilon(c^1) \quad \varepsilon(c) = \underbrace{\sum \varepsilon(c^0)\varepsilon(c^1)}_{=0}$

It is easy to see from Exercise 1.13.3(i) that the coradical filtration of $gr(C)$ is induced by its grading. ($C_0 \subset C_1/C_0 + C_0 \subset \cdots$)

DEFINITION 5.3.11. An $\mathbb{N}_0$-graded coalgebra $C = \bigoplus_{n\geq 0} C(n)$ is called **coradically graded** if the coradical filtration $(C_n)_{n\geq 0}$ of C is given by

$$C_n = \bigoplus_{i=0}^{n} C(i)$$

for all $n \geq 0$.

A graded coalg. $\bar{C}$ with this property (i.e., one isomorphic to $gr(C)$ for some coalg. C) is said to be coradically graded, and a coalg. C s.t. $gr(C) = \bar{C}$ is called a lifting of $\bar{C}$

Prop. 5.3.15 (Heckenberger) Let C be a coalg. then $\mathrm{gr}(C)$ is coradically graded.

Proposition 5.3.13. *Let $C = \bigoplus_{n\geq 0} C(n)$ be an $\mathbb{N}_0$-graded coalgebra. Assume that $C(0)$ is cosemisimple. Then the following are equivalent.*

(1) *C is coradically graded.*
(2) *For all $n \geq 2$, $\Delta_{1,n-1} : C(n) \to C(1) \otimes C(n-1)$ is injective.*

Proof. We denote the coradical filtration of C by $(C_n)_{n\geq 0}$.

(1) $\Rightarrow$ (2): Let $0 \neq x \in C(n)$, $n \geq 2$. Then $x \notin C_{n-1} = \bigoplus_{i=0}^{n-1} C(i)$, since C is coradically graded. Hence $\Delta_{1,n-1}(x) \neq 0$ by (5.3.1), since

$$\Delta(x) \in \bigoplus_{i=0}^{n} C(i) \otimes C(n-i) \subseteq C_0 \otimes C + C(1) \otimes C(n-1) + C \otimes C_{n-2}.$$

(2) $\Rightarrow$ (1): The natural filtration

$$C(0) \subseteq C(0) \oplus C(1) \subseteq C(0) \oplus C(1) \oplus C(2) \subseteq \cdots$$

is a coalgebra filtration. Hence $C_0 \subseteq C(0)$ by Proposition 5.2.4. Since $C(0)$ is cosemisimple, it follows that $C_0 = C(0)$.

Let $n \geq 1$. The inclusion $C(n) \subseteq C_n$ follows easily by induction, since

$$\Delta(C(n)) \subseteq \bigoplus_{i=0}^{n} C(i) \otimes C(n-i) \subseteq C(0) \otimes C + C \otimes \Big(\bigoplus_{i=0}^{n-1} C(i)\Big).$$

Hence $\bigoplus_{i=0}^{n} C(i) \subseteq C_n$. We prove equality by induction on $n \geq 0$. Suppose there are integers $n \geq 1$, $m > n$ and elements $x_i \in C(i)$, $0 \leq i \leq m$, with $x = \sum_{i=0}^{m} x_i \in C_n$. Then $\Delta(x) \in C_0 \otimes C + C \otimes C_{n-1}$ by (5.3.1). By induction, $C_{n-1} = \bigoplus_{i=0}^{n-1} C(i)$. Hence $\Delta_{1,m-1}(x) = 0$. ~~Then $\Delta_{1,m-1}(x_m) = 0$, and $x_m = 0$ by (2).~~ □

$C_n = \Delta^{-1}(C_0 \otimes C + C \otimes C_{n-1})$

$m-1 \geq n$, $x = 0 \Rightarrow x_m = 0$ ($\oplus$), $\therefore m \leq n$

Definition 1.2.26. (1) An $\mathbb{N}_0$-**graded coalgebra** is a pair $(C, \mathcal{C})$, where C is a coalgebra, $(C, \mathcal{C})$ is an $\mathbb{N}_0$-graded vector space, and

(1.2.3) $$\Delta(C(n)) \subseteq \bigoplus_{r+s=n} C(r) \otimes C(s) \text{ for all } n \geq 0,$$

(1.2.4) $$\varepsilon(C(n)) = 0 \text{ for all } n > 0.$$

We write

$$\Delta_{m,n} : C(m+n) \subseteq C \xrightarrow{\Delta} C \otimes C \xrightarrow{\pi_m^C \otimes \pi_n^C} C(m) \otimes C(n), \quad m, n \in \mathbb{N}_0,$$

for the **components of the comultiplication** Δ.

Proposition 5.3.15. *Let C be a coalgebra. Then $\mathrm{gr}\, C$ is coradically graded.*

Proof. By definition, C_0 is cosemisimple. By Proposition 5.3.13 it is enough to prove that $\Delta_{1,n-1}$ for $\mathrm{gr}\, C$ is injective for all $n \geq 2$. We choose subspaces $X_n \subseteq C$, $n \geq 1$, with $C_n = C_{n-1} \oplus X_n$ for all $n \geq 1$. Then

$$C_1 \otimes C_{n-1} = C_0 \otimes C_{n-1} + X_1 \otimes X_{n-1} + X_1 \otimes C_{n-2}$$

$C_1 = C_0 \oplus X_1$

$C_{n-1} = C_{n-2} \oplus X_{n-1}$

for all $n \geq 2$. Hence, by (1.3.3),

$$\Delta(C_n) \subseteq \sum_{i=0}^{n} C_i \otimes C_{n-i} \subseteq C_0 \otimes C_n + C_1 \otimes C_{n-1} + C \otimes C_{n-2}$$
$$\subseteq C_0 \otimes C + X_1 \otimes X_{n-1} + C \otimes C_{n-2}.$$

Since $\Delta^{-1}(C_0 \otimes C + C \otimes C_{n-2}) = C_{n-1}$, the map

$$\Delta' : C_n/C_{n-1} \to (X_1 \otimes X_{n-1} + C_0 \otimes C + C \otimes C_{n-2})/(C_0 \otimes C + C \otimes C_{n-2})$$

$\xrightarrow{\pi_1 \otimes \pi_{n-1}} C_1/C_0 \otimes C_{n-1}/C_{n-2}$

induced by Δ is injective. Thus $\Delta_{1,n-1}$ is injective. □

$\Delta_{1,n-1}(x) = 0 \quad x = 0$