Ex 3.1.5

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Exercise 3.1.5.

(i) Show that in a $\mathbb{Z}_+$-ring, $i, j \in I_0,\ i \neq j$ implies that $b_i^2 = b_i,\ b_i b_j = 0$, and in a based ring $i^* = i$ for $i \in I_0$.

(ii) Show that for a given $\mathbb{Z}_+$-ring A,being a (unital) based ring is a property, not an additional structure.

Proof. (i) Given a $\mathbb{Z}_+$-ring A. Let $1 = \sum_{i \in I_0} a_i b_i$ be the decomposition of 1. Then for given $j \in I_0$

$$b_j = 1 \cdot b_j = \sum_{i \in I_0} a_i (b_i b_j)$$
$$= \sum_{k \in I} \sum_{i \in I_0} a_i c_{ij}^k b_k. \tag{1}$$

Since $\{b_i\}$ forms a basis of A, $\sum_{i \in I_0} a_i c_{ij}^k = 0$ unless k=j. Note $a_i > 0$ for any $i \in I_0$, $\sum_{i \in I_0} a_i c_{ij}^k = 0$ is equivalent to $c_{ij}^k = 0,\ \forall i \in I_0$. Let j run through I_0, we see that if $i,\ j \in I_0$, $c_{ij}^k = 0$ when $k \neq j$.

Similarly, writing $1 = \sum_{j \in I_0} a_j b_j$, and viewing b_i as $b_i \cdot 1$, $\forall i \in I_0$, we will see if $i,\ j \in I_0$, $c_{ij}^k = 0$ when $k \neq i$.

That is to say, when $i,\ j \in I_0$, $c_{ij}^k \neq 0$ if and only if $i = j = k$. It is easy to see from (1) that $c_{ii}^i = 1$. Thus the first statement is proved.

For the second statement, note that the number $\tau(b_i \cdot x)$ is exact the coefficient of b_i^* in the decomposition of x, *i.e.* $x = \sum_{i \in I} \tau(b_i \cdot x) b_i^*$. By the equations $\tau(b_i \cdot b_i) = \tau(b_i) = 1$ and $\tau(b_j \cdot b_i) = \tau(0) = 0,\ \forall j \neq i$, we obtain $b_i = b_i^*$, hence $i = i^*$.

(ii) Let A be a based ring, $*'$ be another involution of I satisfying the conditions in Definition 3.1.3, we will show $* = *'$.

Recall that for any $x \in A$ we have $x = \sum_{i \in I} \tau(b_i \cdot x) b_i^*$. For any $i \in I$, by the equations $\tau(b_i \cdot b_i^{*'}) = 1$ and $\tau(b_j \cdot b_i^{*'}) = \tau(0) = 0,\ \forall j \neq i$, we obtain $b_i^{*'} = b_i^*$. Hence $* = *'$. □

As application of the above proof, we get an alternative proof for Proposition 3.1.4 which shows that we **don't need the condition "based ring"** and we just need that it is a $\mathbb{Z}_+$-ring.

Proof. We have

$$1 = \sum_{i \in I_0} a_i b_i,\ \ a_i > 0. \tag{2}$$

Multiply (2) by b_j, $j \in I_0$. By the result of Exercise 3.1.5 (i), we have $b_j = a_j b_j$, thus $a_j = 1,\ \forall j \in I_0$. □