Solution to Exercise 3.3.2 in Tensor category

October 31, 2021

Exercise. Show that any unital based ring is transitive.

Proof. For any $b_i := X, b_j := Z \in I$, $X^*Z = b_{i*}b_j = \sum_{k\in I} c^k_{i*,j} b_k$. Then $b_i(b_i^* b_j) = (b_i b_i^*)b_j$, Note, $b_i b_i^* = \sum_r c^r_{i,i^*} b_r$. Since A is unital, there exists $k_0 \in I$, s.t. $b_{k_0} = 1$ and $c^{k_0}_{i,i^*} = 1$, then

$$(b_i b_i^*)b_j = \cdots + b_j + \cdots$$

This means there existing some $b_n \in I$ s.t. $b_i b_n$ contains b_j with a positive coefficent, otherwise $b_i(b_{i*}b_j)$ doesn't contains b_j. So we take $Y_1 = b_n$, then XY_1 contains Z. Similary for $Y_2 X$ case. □