张量范畴课程习题及答案

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§0.1 Practice: Week 01 - Categorical Concepts

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

1.1. Let $F, G : \mathcal{C} \times \mathcal{C} \to \mathcal{D}$ be (covariant)functors. Suppose that $\eta = \{\eta_{X,Y} : F(X,Y) \to G(X,Y) \mid X, Y \in \mathcal{C}\}$ is a *transformation* from F to G. Show that the followings are equivalent:

(i) η is a natural transformation from F to G;

(ii) $\eta = \{\eta_{X,Y}\}$ is natural in both variables X and Y. In other words, for any fixed $Y \in \mathcal{C}$, $\eta_{-,Y}$ is a natural transformation from $F(-,Y)$ to $G(-,Y)$; meanwhile for any fixed $X \in \mathcal{C}$, $\eta_{X,-}$ is a natural transformation from $F(X,-)$ to $G(X,-)$.

Try to generalize this proposition more or less.

Answer. *(i)⇒(ii) is direct.*

(ii)⇒(i): For any morphism $(f,g) : (X,Y) \to (X',Y')$ in $\mathcal{C} \times \mathcal{C}$, the naturalities of $\eta_{-,Y}$ and $\eta_{X',-}$ imply that both small rectangles in the following diagram commute:

$$
\begin{array}{ccc}
(X,Y) & \qquad & F(X,Y) & \xrightarrow{\eta_{X,Y}} & G(X,Y) \\
\downarrow{\scriptstyle (f,\mathrm{id}_Y)} & & \downarrow{\scriptstyle F(f,\mathrm{id}_Y)} & & \downarrow{\scriptstyle G(f,\mathrm{id}_Y)} \\
(X',Y) & & F(X',Y) & \xrightarrow{\eta_{X',Y}} & G(X',Y) \\
\downarrow{\scriptstyle (\mathrm{id}_{X'},g)} & & \downarrow{\scriptstyle F(\mathrm{id}_{X'},g)} & & \downarrow{\scriptstyle G(\mathrm{id}_{X'},g)} \\
(X',Y') & & F(X',Y') & \xrightarrow{\eta_{X',Y'}} & G(X',Y')
\end{array}
$$

As a conclusion, the large rectangle

$$
\begin{array}{ccc}
(X,Y) & \qquad & F(X,Y) & \xrightarrow{\eta_{X,Y}} & G(X,Y) \\
\downarrow{\scriptstyle (f,g)} & & \downarrow{\scriptstyle F(f,g)} & & \downarrow{\scriptstyle G(f,g)} \\
(X',Y') & & F(X',Y') & \xrightarrow{\eta_{X',Y'}} & G(X',Y')
\end{array}
$$

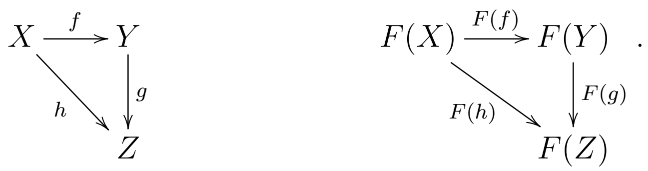
commutes as well, which shows that η is natural from F to G.

1.2. Suppose that $\mathcal{C}$ and $\mathcal{D}$ are equivalent. Try to complete and prove the following claims:

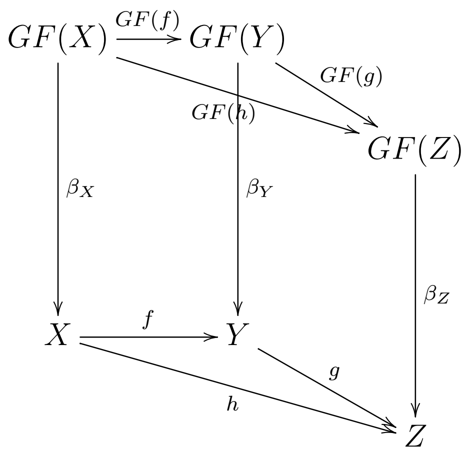
(1) A diagram commutes in $\mathcal{C}$, if and only if the corresponding diagram commutes in $\mathcal{D}$;

(2) A morphism in $\mathcal{C}$ has certain properties (such as monic, epic or isomorphism), if and only if the corresponding morphism in $\mathcal{D}$ does as well.

Answer. *(1) Denote an equivalence (functor) by $F : \mathcal{C} \to \mathcal{D}$. The left-side diagram commutes in $\mathcal{C}$ if and only if the right side diagram commutes in $\mathcal{D}$:*



We only need to show the sufficiency, which could be deduced from the following diagram:



where $G : \mathcal{D} \to \mathcal{C}$ is a quasi-inverse of F, and $\beta : GF \cong \mathrm{Id}_{\mathcal{C}}$ is a natural isomorphism.

Just note that the three rectangles are commutative since β is a natural isomorphism. This follows that one of the triangles commutes if and only if *the other does as well. Here the situation is:*

$$GF(h) = GF(g)GF(f) \implies h = gf.$$

(2) This is direct since F is fully faithful (the proposition on equivalent functors).

1.3. (<u>Universal property v.s. Adjoint functors: Part I</u>)

Definition 0.1. *(Free vector space on a set) Let $\Bbbk$ be a field and S be a set. Define*

$$\begin{aligned} \Bbbk S &:= \{\sum_{s\in S} \alpha_s s \mid \alpha_s \in \Bbbk, \textit{ only finitely many } \alpha_s \textit{ are nonzero}\} \\ &= \{\alpha : S \to \Bbbk \textit{ (a map)} \mid \textit{Finitely many } \alpha_s \textit{ are nonzero}\}. \end{aligned}$$

A characteristic property of the free vector space $\Bbbk S$ is that: For any $\Bbbk$-vector space V and any map $f : S \to V$, then f could be extended uniquely to a $\Bbbk$-linear map from $\Bbbk S$ to V. Try to define a functor from **Set** to $\Bbbk$-**Vec** according to this so-called *universal property*, and find some relation with the forgetful functor

$$U : \Bbbk\text{-}\mathbf{Vec} \to \mathbf{Set}.$$

Answer. *Please check the concepts in [Jacobson - Basic Algebra II, Section 1.7]. For the definition of this functor F : **Set**$\to$ $\Bbbk$-**Vec**, specific steps are as follows: Let $\iota_X : X \hookrightarrow \Bbbk X$ denotes the inclusion map for each $X \in$**Set**.*

i) The correspondence of objects is $F(X) := \Bbbk X$;

*ii) The map on morphisms: For any $X, Y \in$**Set** and $f \in \mathrm{Hom}_{\mathbf{Set}}(X, Y)$, since $\iota_Y f$ is a map from X to $\Bbbk Y$, there <u>exists</u> a <u>unique</u> $\Bbbk$-linear map $F(f) : \Bbbk X \to \Bbbk Y$ such that*

$$\begin{array}{ccc} X & \xrightarrow{\iota_X} & \Bbbk X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \exists! F(f)} \\ Y & \xrightarrow[\iota_Y]{} & \Bbbk Y \end{array}$$

*commutes in **Set**.*

iii) $F(gf) = F(g)F(h)$ *and* $F(id_X) = id_{F(X)}$ *are also followed by the existence and uniqueness of* F*. Please fill the details (by diagrams).*

§0.2 Practice: Week 02 - Abelian Categories

Let $\mathcal{C}$ be an additive category with the direct sum $\oplus$ and zero object 0.

2.1. (1) Show that any zero morphism 0 (the zero element in the "Hom sets" as abelian groups) has the following property:

$$0 \circ f = 0, \ g \circ 0 = 0$$

as long as the compositions make sense;

(2) Show that any morphism from 0 (object) or to 0 must be a zero morphism.

Answer. *(1) Let* $X, Y, Z \in \mathcal{C}$ *and* $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$*. Denote the zero in* $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ *by* 0*, and* $0 = 0 + 0$ *holds evidently. Then by the biadditivity of composition* $\circ$*, we obtain*

$$g \circ 0 = g \circ (0 + 0) = g \circ 0 + g \circ 0.$$

Thus $g \circ 0 = 0 \in \mathrm{Hom}_{\mathcal{C}}(X, Z)$*.*

(2) Let $X \in \mathcal{C}$ *and* $f \in \mathrm{Hom}_{\mathcal{C}}(X, 0)$*. Since* $\mathrm{Hom}_{\mathcal{C}}(0, 0) = 0$*, the identity morphism* $\mathrm{id}_0 \in \mathrm{Hom}_{\mathcal{C}}(0, 0)$ *must be the zero morphism* 0 *itself. Thus*

$$f = \mathrm{id}_0 \circ f = 0 \circ f = 0$$

by the conclusion in (1).

2.2. Some functorial properties on $\oplus$.

(1) Define the bifunctor $\oplus : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, especially on morphisms;

(2) Prove Proposition 1.2.4.

Answer. *(1) Let* $(X_1, X_2), (X_1', X_2') \in \mathcal{C}\times\mathcal{C}$ *and* $(f_1, f_2) \in \mathrm{Hom}_{\mathcal{C}}((X_1, X_2), (X_1', X_2'))$*. The existence of direct sums* $X_1 \oplus X_2$ *and* $X_1' \oplus X_2'$ *supply following morphisms:*

$$X_1 \underset{p_1}{\overset{i_1}{\rightleftarrows}} X_1 \oplus X_2 \underset{i_2}{\overset{p_2}{\rightleftarrows}} X_2 \text{ with relations } \begin{cases} p_1 i_1 = \mathrm{id}_{X_1}, \ p_2 i_2 = \mathrm{id}_{X_2} \\ i_1 p_1 + i_2 p_2 = \mathrm{id}_{X_1 \oplus X_2} \end{cases},$$

$$X_1' \underset{p_1'}{\overset{i_1'}{\rightleftarrows}} X_1' \oplus X_2' \underset{i_2'}{\overset{p_2'}{\rightleftarrows}} X_2' \text{ with relations } \begin{cases} p_1' i_1' = \mathrm{id}_{X_1'}, \ p_2' i_2' = \mathrm{id}_{X_2'} \\ i_1' p_1' + i_2' p_2' = \mathrm{id}_{X_1' \oplus X_2'} \end{cases}.$$

It is sufficient to define

$$f_1 \oplus f_2 := i_1' f_1 p_1 + i_2' f_2 p_2 \in \mathrm{Hom}_{\mathcal{C}}(X_1 \oplus X_2, X_1' \oplus X_2').$$

(2) The natural isomorphism $F(X) \oplus F(Y) \xrightarrow{\sim} F(X \oplus Y)$ *would be*

$$\eta = \{\eta_{X,Y} := F(i_X)p_{F(X)} + F(i_Y)p_{F(Y)} \mid X, Y \in \mathcal{C}\}.$$

2.3. Claim and prove that:

(1) An "additive equivalence" (i.e. an additive functor which is an equivalence) keeps kernels and cokernels;

(2) An "additive equivalence" between abelian categories are exact.

Answer. *(1) (This could be regard as an application of Practice 1.2 (1), and equivalences are fully faithful.)*

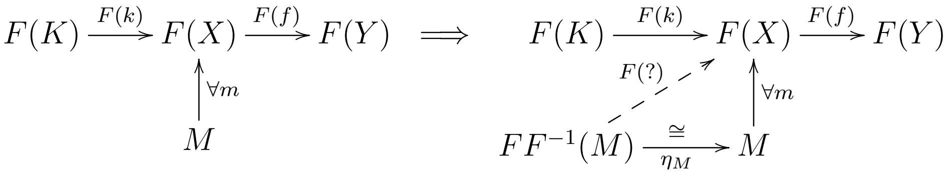
a) ***For kernels:***

Let $F : \mathcal{C} \to \mathcal{D}$ *be an additive equivalence. Suppose that* (K, k) *be the kernel of* $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$*. We aim to show that* $(F(K), F(k))$ *is the kernel of* $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ *by the definition.*

Firstly, since F *determines the group homomorphism from* $\mathrm{Hom}_{\mathcal{C}}(K, Y)$ *to* $\mathrm{Hom}_{\mathcal{D}}(F(K), F(Y))$*, it is evident that*

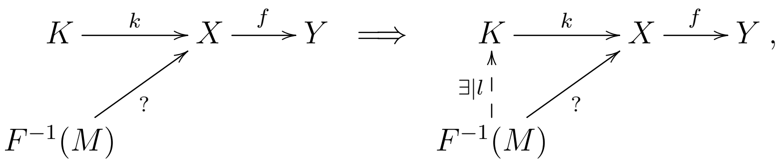
$$F(f)F(k) = F(fk) = F(0) = 0.$$

Now we assume that there is a morphism $m : M \to F(X)$ in $\mathcal{D}$ such that $F(f)m = 0$, and denote the natural isomorphism $\eta : FF^{-1} \cong \mathrm{Id}_{\mathcal{D}}$:

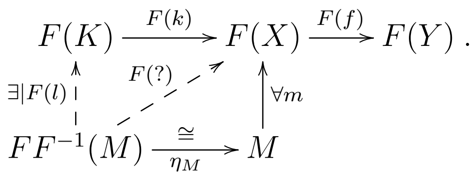


where $? : F^{-1}(M) \to X$ satisfies $F(?) = m\eta_M$ (it exists since F is fully faithful), and $f? = 0$ holds consequently. Then need to find some morphism from M (or $FF^{-1}(M)$) to $F(K)$ making the diagram commute, and show its uniqueness.

The equation $f? = 0$ lead us back to a diagram in $\mathcal{C}$, which is



where $l : F^{-1}(M) \to K$ is the only morphism making the right-side diagram commute. Finally, we act F on this diagram, and obtain the following commuting one in $\mathcal{D}$:



The uniqueness of $F(l)\eta_M$ (or $F(l)$) could be followed by Practice 1.2 (1).

b) ***For cokernels:***

The proof could be similar to a). Another way might be according to the "fact" that: Cokernels in $\mathcal{C}$ are kernels in the dual additive category $\mathcal{C}^\vee$.

(2) Not hard.

Let $\mathcal{C}$ be an abelian category.

2.4. Basic properties on certain morphisms.

(1) Show that if (K, k) is the kernel of some morphism, then k is a monomorphism. Claim a similar property for cokernels;

(2) In a canonical decomposition

$$K \xrightarrow{k} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{c} C$$

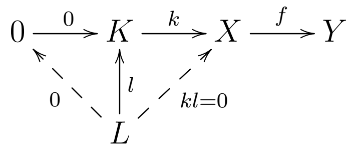
of $\varphi : X \to Y$, try to prove that i must be an epimorphism and j must be a monomorphism;

(3) Show that a morphism is a monomorphism if and only if it is monic, and a morphism is an epimorphism if and only if it is epic.

Use these characteristic property to prove: A subobject of a subobject of Z is also a subobject of Z, as well as a similar property on quotient objects.

Answer. *Due to the duality between kernels and cokernels, as well as monomorphisms and epimorphisms, we only need to prove propositions for kernels or monomorphisms.*

(1) Let (K, k) be the kernel of a morphism $f : X \to Y$. We aim to prove $\mathrm{Ker}(k) = 0$. Suppose $l : L \to K$ satisfies $kl = 0$. Our goal is to show $l = 0 \circ 0$ (i.e. the left-side small triangle commutes), since $0 : L \to 0$ is the unique morphism in $\mathrm{Hom}_{\mathcal{C}}(L, 0)$.

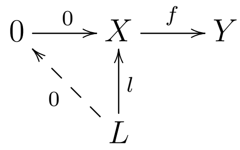


Fist we note that $f(kl) = f0 = 0$. Because (K, k) is the kernel of f, it follows that there should be a unique morphism $? : L \to K$ making the right-side small triangle commute (i.e. k?=kl=0). But l and 0 both satisfy this property. We conclude that $l = 0 = 0 \circ 0$.

(2) It is required from the definition of Abelian categories that (I, j) be the kernel of some morphism (which is c here). Thus j should be a monomorphism according to (1).

(3) The axiom (A1) in the definition of additive categories follows the fact that: A morphism f is monic if and only if $fg = 0$ implies $g = 0$ (as long as gf is defined).

i) *Suppose* $f : X \to Y$ *is monic. We aim to show that* 0 *is the kernel of* f*. Let* $l : L \to X$ *satisfies* $fl = 0$*:*



Since f *is monic, the claim in the last paragraph shows that* $l = 0$*. Then the unique morphism* $0 : L \to 0$ *makes the triangle commute.*

ii) *On the other hand we suppose* $f : X \to Y$ *is a monomorphism. Assume* $g : W \to X$ *satisfies* $fg = 0$*, and we aim to show* $g = 0$*. But the assumption together with* $0 = \mathrm{Ker}(f)$ *imply that: There exists a unique morphism* $0 : W \to 0$ *such that* $g = 0 \circ 0 = 0$*.*

The second claim holds since the composition of monic morphisms are still monic.

§0.3 Practice: Week 03 - The Definition of Monoidal Categories

*3.1. Let A be a (unital) ring, and $\mathcal{C} := A$-**mod** (or simply, $\mathcal{C} := \Bbbk$-**Vec**). Prove that for any $X \in \mathcal{C}$, the (covariant)functor $\mathrm{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \to$ **Abel** is left exact. Try to use this and the dual abelian category $\mathcal{C}^{\vee}$, to explain the contravariant one $\mathrm{Hom}_{\mathcal{C}}(-, Y)$ is also left exact for any $Y \in \mathcal{C}$.

Answer. *See Theorem 3.1 in Jacobson's book "Basic Algebra II".*

3.2. Let $\mathcal{C}$ be a monoidal category. Show that:

(1) For any objects $X, Y \in \mathcal{C}$ and their identities id_X and id_Y, the morphism $\mathrm{id}_X \otimes \mathrm{id}_Y$ is exactly the identity $\mathrm{id}_{X \otimes Y}$ in $\mathrm{Hom}_{\mathcal{C}}(X \otimes Y, X \otimes Y)$;

(2) If f and g are isomorphisms in $\mathcal{C}$, then $f \otimes g$ is an isomorphism, too.

Answer. *(1) This is because that* $(\mathrm{id}_X, \mathrm{id}_Y)$ *is the identity morphism on the object* $(X, Y) \in \mathcal{C} \times \mathcal{C}$*, and that* $\otimes$ *is a bifunctor which keeps identities (i.e.* $\mathrm{id}_X \otimes \mathrm{id}_Y = \mathrm{id}_{X \otimes Y}$*).*

(2) Denote the inverses of f and g by f^{-1} and g^{-1}, respectively. Since $\otimes$ is a bifunctor, we obtain following equations:

$$(f^{-1} \otimes g^{-1})(f \otimes g) = (f^{-1}f) \otimes (g^{-1}g) = \mathrm{id} \otimes \mathrm{id} = \mathrm{id},$$
$$(f \otimes g)(f^{-1} \otimes g^{-1}) = (ff^{-1}) \otimes (gg^{-1}) = \mathrm{id} \otimes \mathrm{id} = \mathrm{id}\,.$$

3.3. Let $\mathcal{C}$ be a monoidal category. Show that the "**2. The unit axiom**" in Definition 2.1.1 holds. That is to say, functors $L_1 := \mathbf{1} \otimes -$ and $R_1 := - \otimes \mathbf{1}$ on $\mathcal{C}$ are (auto)equivalences.

Answer. *It is said in our definition of monoidal categories that there are natural isomorphsms*

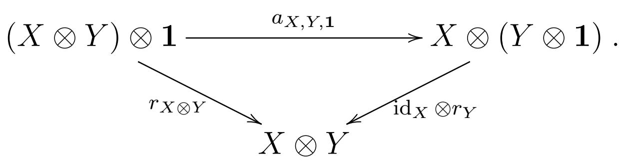
$$l : L_1 \xrightarrow{\sim} \mathrm{Id}_{\mathcal{C}} \quad \textit{and} \quad r : R_1 \xrightarrow{\sim} \mathrm{Id}_{\mathcal{C}},$$

which imply that $\mathrm{Id}_{\mathcal{C}}$ *is the quasi-inverse of* L_1 *as well as* R_1*.*

§0.4 Practice: Week 04 - Properties of the Unit Object

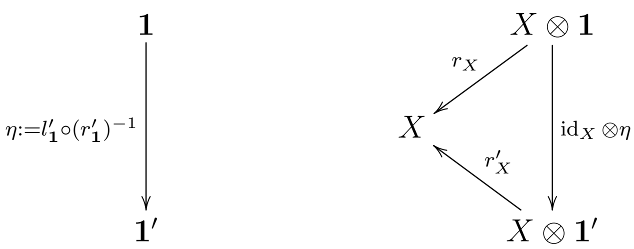
4.1. Let $(\mathcal{C}, \otimes, a, \mathbf{1}, l, r)$ be a monoidal category. Please fill parts in the proof of Proposition 2.2.4 ("the right triangle") and Proposition 2.2.6 (uniqueness of the unit), which are:

(1) Show that the following diagram commutes for all $X, Y \in \mathcal{C}$:



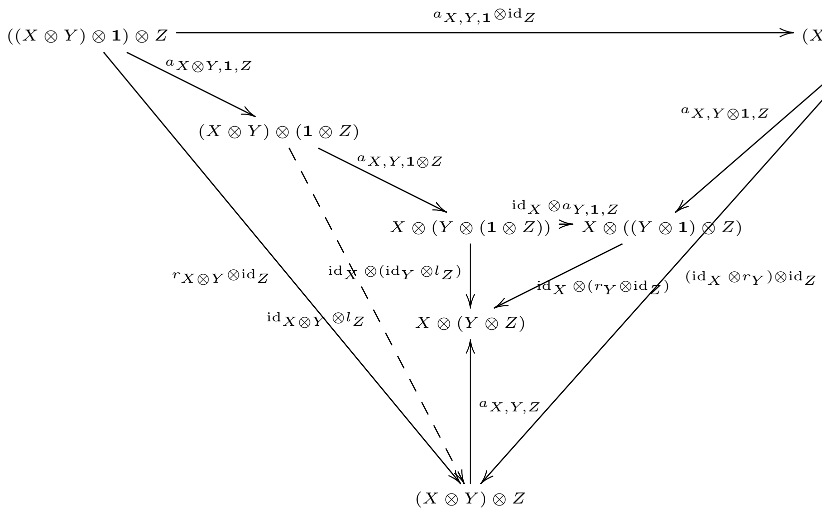
Prove it directly, or use the language of the opposite $\mathcal{C}^{\mathrm{op}}$ (and "the left triangle");

(2) Suppose $(\mathbf{1}', l', r')$ is another unit object. Show that the following diagram commutes for all $X \in \mathcal{C}$:



Prove it directly, and try to explain the difficulty if using the language of $\mathcal{C}^{\mathrm{op}}$ (and that "η keeps the left unity").

Answer. *(1) For any $Z \in \mathcal{C}$, consider the following diagram:*



It is sufficient to establish the commutativity of the outside triangle (which will be equivalent to the proposition when $Z = \mathbf{1}$). The pentagon axiom follows the commutativity of the upper pentagon, while the central and left triangles are commutative by the triangle axiom. The remaining parts are two quadrangles, whose commutativity is ensured by the naturality of $a_{X,-,-}$ in the second and third variables.

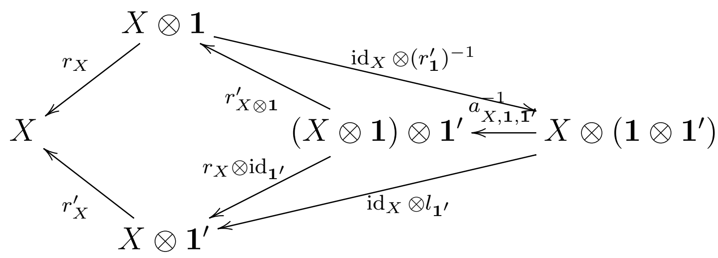
With the help of the opposite monoidal category $\mathcal{C}^{\mathrm{op}} = (\mathcal{C}, \otimes^{\mathrm{op}}, a^{\mathrm{op}}, \mathbf{1}, l^{\mathrm{op}}, r^{\mathrm{op}})$ to $\mathcal{C}$, we first note the following commuting "left" triangle diagram in $\mathcal{C}^{\mathrm{op}}$:

$$\begin{array}{ccc} (\mathbf{1} \otimes^{\mathrm{op}} X) \otimes^{\mathrm{op}} Y & \xrightarrow{a^{\mathrm{op}}_{\mathbf{1},X,Y}} & \mathbf{1} \otimes^{\mathrm{op}} (X \otimes^{\mathrm{op}} Y)\,. \\ & {\scriptstyle l^{\mathrm{op}}_X \otimes^{\mathrm{op}} \mathrm{id}_Y} \searrow \quad \swarrow {\scriptstyle l^{\mathrm{op}}_{X \otimes^{\mathrm{op}} Y}} & \\ & X \otimes^{\mathrm{op}} Y & \end{array}$$

However, the definition of $\mathcal{C}^{\mathrm{op}}$ makes the diagram to be the following (in $\mathcal{C}$):

$$\begin{array}{ccc} Y \otimes (X \otimes \mathbf{1}) & \xrightarrow{a^{-1}_{X,Y,\mathbf{1}}} & (Y \otimes X) \otimes \mathbf{1}\,. \\ & {\scriptstyle \mathrm{id}_Y \otimes r_X} \searrow \quad \swarrow {\scriptstyle r_{Y \otimes X}} & \\ & Y \otimes X & \end{array}$$

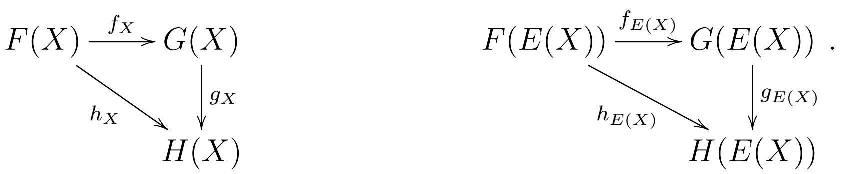
(2) *Recall that we choose* $\eta := l_{\mathbf{1}'} \circ (r'_{\mathbf{1}})^{-1} : \mathbf{1} \xrightarrow{\sim} \mathbf{1}'$ *when proving that* η *keeps the "left unity". Consider the following diagram:*



Our goal is to establish the commutativity of the outside quadrangle because of the definition of η*. The upper triangle commutes by the "right triangle" of* $(\mathbf{1}', r')$*, and the bottom one does by the triangle axiom of* $(\mathbf{1}, l, r)$*. Finally, the naturality of* r' *on the morphism* $r_X : X \otimes \mathbf{1} \to X$ *follows that the remaining small quadrangle is also commutative.*

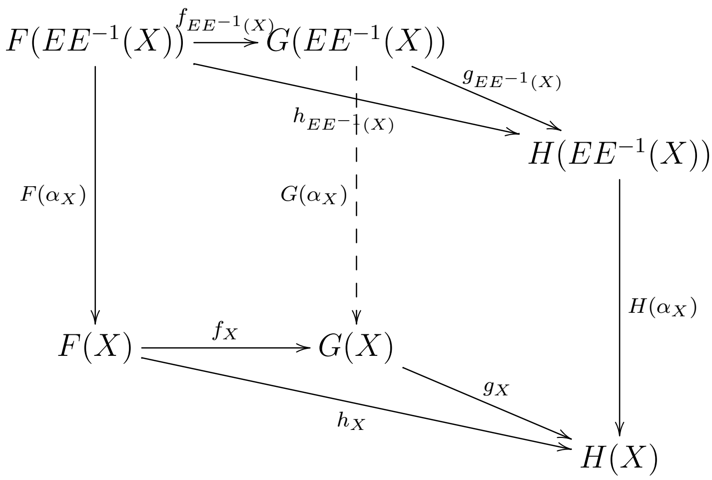
The language of opposite monoidal categories could help if the equation $l_{\mathbf{1}'} \circ (r'_{\mathbf{1}})^{-1} = r_{\mathbf{1}'} \circ (l'_{\mathbf{1}})^{-1}$ *holds.*

4.2. Let $\mathcal{C}$ be category, and $F, G, H : \mathcal{C} \to \mathcal{C}$ be functors. Suppose E is an autoequivalence on $\mathcal{C}$. Show that within the following diagrams, the left one commutes for each $X \in \mathcal{C}$ if and only if the right one commutes for each $X \in \mathcal{C}$:



where f, g and h are all natural. (This might be the last step of the Proof 2.2.3 in the book.)

Answer. *The proof is similar to Practice 1.2. We only prove the "if" part. Denote* *the natural isomorphism $\alpha : EE^{-1} \cong \mathrm{Id}_{\mathcal{C}}$, and consider the following diagram:*



The upper triangle commutes since the diagram in our sufficient condition commutes on the object $E^{-1}(X)$. Other rectangles commute by the naturality of f, g and h.

§0.5 Practice: Week 06 - Monoidal Functors and Strictness Theorem

6.1. Let $\mathcal{C}, \mathcal{D}$ be a monoidal category, and $(F, J) : \mathcal{C} \to \mathcal{D}$ be a monoidal functor. Suppose that $\varphi : \mathbf{1} \xrightarrow{\sim} F(\mathbf{1})$ is the canonical isomorphism (in $\mathcal{D}$). Show that:

(1) Diagram (2.25) commutes for each $X \in \mathcal{C}$ if and only if the following diagram commutes for each $X \in \mathcal{C}$:

$$\begin{array}{ccc} \mathbf{1} \otimes F(\mathbf{1} \otimes X) & \xrightarrow{l_{F(\mathbf{1}\otimes X)}} & F(\mathbf{1} \otimes X) \\ {\scriptstyle \varphi \otimes \mathrm{id}_{F(\mathbf{1}\otimes X)}} \downarrow & & \downarrow {\scriptstyle F(l_{\mathbf{1}\otimes X})^{-1}} \\ F(\mathbf{1}) \otimes F(\mathbf{1} \otimes X) & \xrightarrow{J_{\mathbf{1},\mathbf{1}\otimes X}} & F(\mathbf{1} \otimes (\mathbf{1} \otimes X)) \end{array}$$

(2) The endofunctor $- \otimes F(\mathbf{1})$ on $\mathcal{D}$ is an autoequivalence;

(3) Try to prove Prop 2.4.3 then.

Answer. *(1) Since the endofunctor $\mathbf{1} \otimes -$ on $\mathcal{C}$ is an autoequivalence, the claim holds according to Practice 4.2.*

(2) Choose an isomorphism $\psi : \mathbf{1} \xrightarrow{\sim} F(\mathbf{1})$ in $\mathcal{D}$. We could obtain a natural

isomorphism

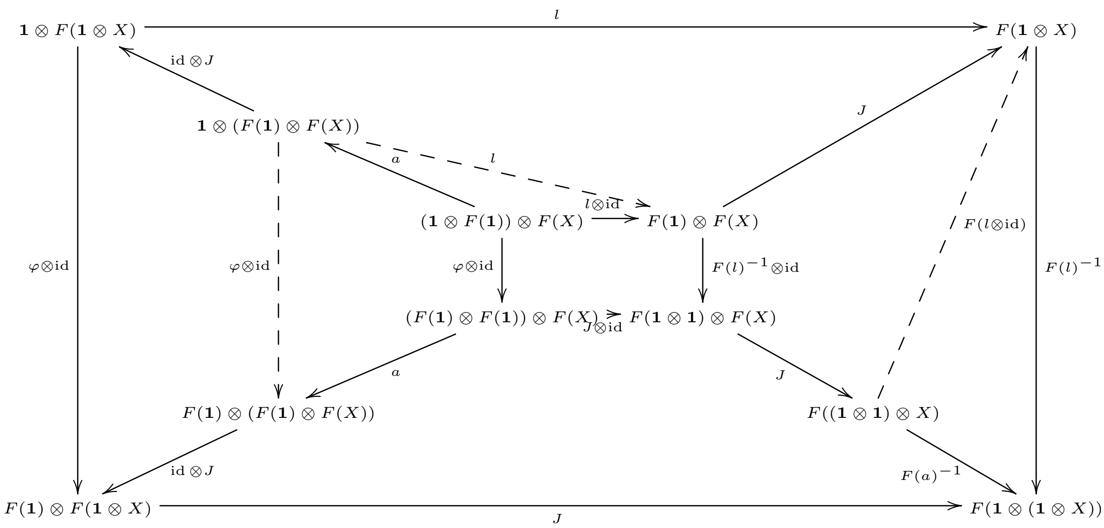
$$r_- \circ (\mathrm{id}_- \otimes \psi^{-1}) : - \otimes F(\mathbf{1}) \cong \mathrm{Id}_{\mathcal{D}} .$$

In fact, this is deduced from the following commuting diagram in $\mathcal{D}$:

$$\begin{array}{ccccccc} X & \quad & X \otimes F(\mathbf{1}) & \xrightarrow{\mathrm{id}_X \otimes \psi^{-1}} & X \otimes \mathbf{1} & \xrightarrow{r_X} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f \otimes \mathrm{id}_{F(\mathbf{1})}} & & \downarrow{\scriptstyle f \otimes \mathbf{1}} & & \downarrow{\scriptstyle f} \\ Y & & Y \otimes F(\mathbf{1}) & \xrightarrow{\mathrm{id}_Y \otimes \psi^{-1}} & Y \otimes \mathbf{1} & \xrightarrow{r_Y} & Y \end{array}$$

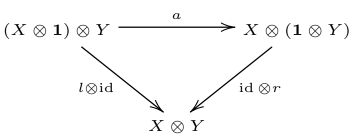
where $f : X \to Y$ is an arbitrary morphism in $\mathcal{D}$.

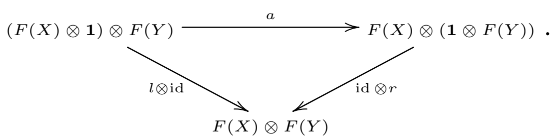
(3) *Here we prove Proposition 2.4.3(2.25) for example. By (1), it is sufficient to establish the commutativity of the following diagram:*



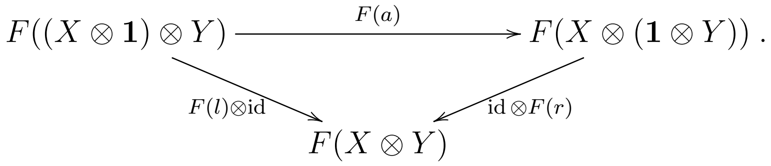
Another way to prove the commutativity of Diagram (2.26) is to show that φ is also canonical for r, which is however complicated as well.

6.2. Let $\mathcal{C}, \mathcal{D}$ be a monoidal category, and $(F, J) : \mathcal{C} \to \mathcal{D}$ be an equivalence of monoidal categories. Show that the left-side diagram commutes in $\mathcal{C}$ if and only if the right side diagram commutes in $\mathcal{D}$:





Answer. *We only prove the sufficiency. According to Practice 1.2(1), we just need to show that the following diagram commutes:*



But the diagram might be too large...

§0.6 Practice: Week 07 - Duals

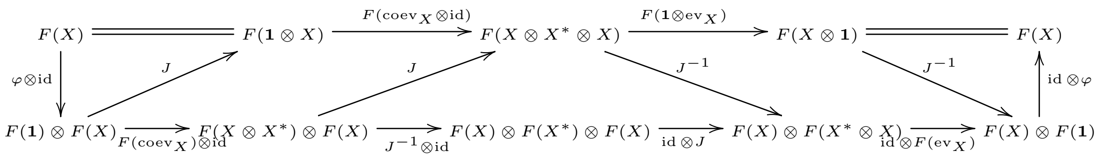
7.1. Let $\mathcal{C}$, $\mathcal{D}$ be monoidal categories, and $F = (F, J, \varphi) : \mathcal{C} \to \mathcal{D}$ be a monoidal functor. Let X be an object in $\mathcal{C}$ with a left dual X^*. Prove that $F(X^*)$ is a left dual of $F(X)$ with the evaluation and coevaluation given by

$$\mathrm{ev}_{F(X)} : F(X^*) \otimes F(X) \xrightarrow{J_{X^*,X}} F(X^* \otimes X) \xrightarrow{F(\mathrm{ev}_X)} F(\mathbf{1}) \xrightarrow{\varphi^{-1}} \mathbf{1},$$
$$\mathrm{coev}_{F(X)} : \mathbf{1} \xrightarrow{\varphi} F(\mathbf{1}) \xrightarrow{F(\mathrm{coev}_X)} F(X \otimes X^*) \xrightarrow{J^{-1}_{X,X^*}} F(X) \otimes F(X^*).$$

Answer. *We will* ***suppress*** *some of the associativity and unit constraints, and we only show the equation*

$$(\mathrm{id}_{F(X)} \otimes \mathrm{ev}_{F(X)}) \circ (\mathrm{coev}_{F(X)} \otimes \mathrm{id}_{F(X)}) = \mathrm{id}_{F(X)} .$$

It is sufficient to consider the following commuting diagram:



The proof of the other equation $(\mathrm{ev}_{F(X)} \otimes \mathrm{id}_{F(X^*)}) \circ (\mathrm{id}_{F(X^*)} \otimes \mathrm{coev}_{F(X)} = \mathrm{id}_{F(X^*)}$ *could be left as an Practice.*

7.2. Let $\mathcal{C}$ be a monoidal category, and let

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

be a diagram in C. Suppose X, Y, Z have left duals. Prove that:

*(1) $(g \circ f)^* = f^* \circ g^*$;

(2) $Y^* \otimes X^*$ is a left dual of $X \otimes Y$.

§0.7 Practice: Week 08 - Tensor Categories

8.1. (1) Show that additive equivalences between abelian categories are exact. (This is exactly Practice 2.3 (2).)

(2) Suppose F and G are natural isomorphic additive functors. Show that

$$\begin{aligned} & F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \quad \text{is exact} \\ \iff \; & G(X) \xrightarrow{G(f)} G(Y) \xrightarrow{G(g)} G(Z) \quad \text{is exact.} \end{aligned}$$

Answer. *(1) Let $E : \mathcal{C} \to \mathcal{D}$ be an additive equivalence between abelian categories. It follows from Practice 2.3 (1) that E keeps kernels and cokernels. Now suppose that*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is an arbitrary exact sequence in $\mathcal{C}$, we aim to obtain the exactness of

$$E(X) \xrightarrow{E(f)} E(Y) \xrightarrow{E(g)} E(Z)$$

in $\mathcal{D}$, which means that $\mathrm{Im}(E(f)) = \mathrm{Ker}(E(g))$. Then it is sufficient to show that E keeps images as well.

For this purpose, we act E onto the canonical decomposition of f:

$$\mathrm{Ker}(f) \xrightarrow{k} X \xrightarrow{i} \mathrm{Im}(f) \xrightarrow{j} Y \xrightarrow{c} \mathrm{Coker}(f).$$

We write the result as follows:

$$E(\mathrm{Ker}(f)) \xrightarrow{E(k)} E(X) \xrightarrow{E(i)} E(\mathrm{Im}(f)) \xrightarrow{E(j)} E(Y) \xrightarrow{E(c)} E(\mathrm{Coker}(f)).$$

In fact, this is exactly a canonical decomposition of $E(f)$, since E keeps kernels and cokernels. As a consequence, $\underline{E(\mathrm{Im}(f)) = \mathrm{Im}(E(f))}$.

Therefore, we know that

$$\mathrm{Im}(E(f)) = E(\mathrm{Im}(f)) = E(\mathrm{Ker}(g)) = \mathrm{Ker}(E(g)).$$

(2) Suppose $\eta : F \cong G$ is a natural isomorphism. Then we have following equations

$$G(f) = \eta_Y \circ F(f) \circ \eta_X,\ G(g) = \eta_Z \circ F(g) \circ \eta_Y.$$

Since η_X, η_Y and η_Z are all isomorphisms, it could be shown that

$$\mathrm{Im}(G(f)) = \mathrm{Im}(F(f)),\ \mathrm{Ker}(G(g)) = \mathrm{Ker}(F(g)).$$

8.2. Let $\mathcal{C}$ be an abelian category.

*(1) For any $V \in \mathcal{C}$, functors $\mathrm{Hom}_{\mathcal{C}}(V, -)$ and $\mathrm{Hom}_{\mathcal{C}}(-, V)$ are both left exact;

***(2) Left/right adjoint functors are right/left exact;

(Hint: Use Mitchell Theorem, Practice 2.3 (1) and 3.1.)

Answer. *(1) We only prove $\mathrm{Hom}_{\mathcal{C}}(V, -)$ is left exact for example.*

*Mitchell Theorem establishes that there exists a ring A such that $\mathcal{C}$ is additively equivalent to a full abelian subcategory $\mathcal{D}$ of A-**Mod**. Suppose $E : \mathcal{C} \to \mathcal{D}$ is an additive equivalence. Recall that Hom functors from $\mathcal{D}$ are left exact. Thus*

$$\mathrm{Hom}_{\mathcal{D}}(E(V), E(-)) = \mathrm{Hom}_{\mathcal{D}}(E(V), -) \circ E : \mathcal{C} \to \mathbf{Ab}$$

is also left exact.

*Now let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be an arbitrary short exact sequence in $\mathcal{C}$. We consider following **commuting** diagram:*

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(V,X) & \xrightarrow{f_*} & \mathrm{Hom}_{\mathcal{C}}(V,Y) & \xrightarrow{g_*} & \mathrm{Hom}_{\mathcal{C}}(V,Z) \\
& \downarrow E & & \downarrow E & & \downarrow E \\
0 \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(E(V),E(X)) & \xrightarrow{E(f)_*} & \mathrm{Hom}_{\mathcal{D}}(E(V),E(Y)) & \xrightarrow{E(g)_*} & \mathrm{Hom}_{\mathcal{D}}(E(V),E(Z))
\end{array}$$

where the second row is exact, while columns are all isomorphisms between abelian groups (since E is an equivalence). It is then not hard to know that the first row is also exact.

**8.3. (Proposition 2.2.1) Let $\mathcal{C}$ be multitensor category and $V \in \mathcal{C}$.

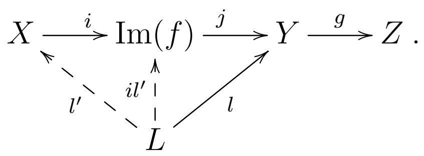
(1) Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be an arbitrary sequence in C. Show that if

$$\mathrm{Hom}_{\mathcal{C}}(U, X) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{C}}(U, Y) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{C}}(U, Z)$$

is exact for each $U \in \mathcal{C}$, then $X \xrightarrow{f} Y \xrightarrow{g} Z$ is also exact.

(2) Prove that $V \otimes -$ is left exact for example.

Answer. *(1) Consider the following diagram in $\mathcal{C}$, in which i is an epimorphism and j is a monomorphism:*



Our goal is to show that $(\mathrm{Im}(f), j)$ is the kernel of g (by definitions).

(i) Choose $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$. *By the exactness of*

$$\mathrm{Hom}_{\mathcal{C}}(X, X) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{C}}(X, Z)\ ,$$

we obtain that

$$gji = gf = (g_* \circ f_*)(\mathrm{id}_X) = 0(\mathrm{id}_X) = 0.$$

Note that i is epic. Thus $gj = 0$ holds.

(ii) Suppose that $l : L \to Y$ satisfies $gl = 0$. By $l \in \mathrm{Ker}(g_*)$ and the exactness of

$$\mathrm{Hom}_{\mathcal{C}}(L, X) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{C}}(L, Y) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{C}}(L, Z)\ ,$$

there exists a morphism $l' : L \to X$ such that

$$l = f_*(l') = fl' = j(il').$$

That is to say, $il' : L \to \mathrm{Im}(f)$ makes the right triangle commutes. The uniqueness condition could be satisfied by i as a monomorphism.

(2) Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be an arbitrary short exact sequence in $\mathcal{C}$. We aim to show that

$$0 \to V \otimes X \xrightarrow{\mathrm{id} \otimes f} V \otimes Y \xrightarrow{\mathrm{id} \otimes g} V \otimes Z$$

is exact.

By (1), it is sufficient to show that, the functor $\mathrm{Hom}_C(U, V \otimes -)$ is left exact for each $U \in \mathcal{C}$. This is true since $\mathrm{Hom}_C(U, V \otimes -)$ is natural isomorphic to a left exact functor $\mathrm{Hom}_{\mathcal{C}}(V^* \otimes U, -)$ (Practice 8.2 (1)).

§0.8 Practice: Week 0? - Exactness and Semisimplicity

*9.1. Show that an abelian category could be defined as follows:

(1) Every morphism has kernels and cokernels;

(2) ......

(3) Any morphism $f : X \to Y$ could be decomposed into $f = ji$ such that following sequences are exact:

$$X \xrightarrow{i} I \longrightarrow 0 \qquad \text{and} \qquad I \xrightarrow{j} Y \longrightarrow 0.$$