Recall: $\mathcal{C}$. preΔ-cate. $M \in ob(\mathcal{C})$. Then $Hom_{\mathcal{C}}(M, -)$ and $Hom_{\mathcal{C}}(-, M)$ cohomology functor. (Thm 1.2.2)

Cor 1.2.3: $\mathcal{C}$ pre Δ-cate

(1) If the following diag commutes & Both rows are distinguished Δ. if f, g are isom, so is h.

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"] & TX \arrow[d, "Tf"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & TX' \end{tikzcd}$$

(2) For any morphism $u: X \to Y$. There's unique way to embed u into a distinguished Δ.

Pf: (1) Since $Hom_{\mathcal{C}}(Z', -)$ is a cohomology functor $\Rightarrow$

$$\begin{tikzcd} Hom_{\mathcal{C}}(Z', X) \arrow[r] \arrow[d, "\cong"] & Hom_{\mathcal{C}}(Z', Y) \arrow[r] \arrow[d, "\cong"] & Hom_{\mathcal{C}}(Z', Z) \arrow[r] \arrow[d, "h_*"] & Hom_{\mathcal{C}}(Z', TX) \arrow[r] \arrow[d, "\cong"] & Hom_{\mathcal{C}}(Z', TY) \arrow[d, "\cong"] \\ Hom_{\mathcal{C}}(Z', X') \arrow[r] & Hom_{\mathcal{C}}(Z', Y') \arrow[r] & Hom_{\mathcal{C}}(Z', Z') \arrow[r] & Hom_{\mathcal{C}}(Z', TX') \arrow[r] & Hom_{\mathcal{C}}(Z', TY') \end{tikzcd}$$

By Five lemma. $h_*: Hom_{\mathcal{C}}(Z', Z) \xrightarrow{\sim} Hom_{\mathcal{C}}(Z', Z')$ is isom, $\exists\, \varphi \in Hom_{\mathcal{C}}(Z', Z)$ s.t.

$h_*(\varphi) = h\varphi = id_{Z'}$

Similarly $h^*: Hom_{\mathcal{C}}(Z', Z) \xrightarrow{\sim} Hom_{\mathcal{C}}(Z, Z)$ is isom, $\exists\, \psi \in Hom_{\mathcal{C}}(Z', Z)$ s.t.

$h^*(\psi) = \psi h = id_Z \Rightarrow \varphi = \psi$ and h is isom

2). $\Leftarrow$

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, equal] & Z \arrow[r, "w"] \arrow[d, "h"] & TX \arrow[d, equal] \\ X \arrow[r, "u"] & Y \arrow[r, "v'"] & Z' \arrow[r, "w'"] & TX \end{tikzcd}$$

Existence of h by TR3)

h is isom by (1).

Lem 1.2.4:

$X \to Y \to Z \to TX$, $X' \to Y' \to Z' \to TX' \in \Sigma$.

(1) If the following diag commutes, then $\exists$ Δ-morphism (f, g, h). Moreover, if g, h isom, so is f.

$$\begin{tikzcd} Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[d, "h"] \\ Y' \arrow[r, "v'"] & Z' \end{tikzcd}$$

(2) If the following diag commutes, then $\exists$ Δ-morphism (f, g, h). Moreover, if h, Tf isom, so is g

$$\begin{tikzcd} Z \arrow[r, "w"] \arrow[d, "h"'] & TX \arrow[d, "Tf"] \\ Z' \arrow[r, "w'"] & TX' \end{tikzcd}$$

Pf: (1) By TR2) & TR3)

$$\begin{tikzcd} Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & TX \arrow[r, "-Tu"] \arrow[d, dashed, "Tf"] & TY \arrow[d, "Tg"] \\ Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & TX' \arrow[r, "-Tu'"] & TY' \end{tikzcd}$$

$\exists\ Tf: TX \to TX'$ T automorphism.
$\Rightarrow \exists\ f: X \to X'$

By Cor 1.2.3 (1), if g, h isom, so is f.

rmk 1.2.5:

Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX \in \Sigma$. Call u or X Δ-kernel of v
w or TX Δ-cokernel of v

If $\exists\ u': U \to Y$ s.t. $vu' = 0$, then $\exists\ u'': U \to X$. s.t. $uu'' = u'$

Since $\mathrm{Hom}_{\mathcal{C}}(U, X) \xrightarrow{u_*} \mathrm{Hom}_{\mathcal{C}}(U, Y) \xrightarrow{v_*} \mathrm{Hom}_{\mathcal{C}}(U, Z) \longrightarrow \cdots$ is exact.

$v_*(u') = vu' = 0 \quad \exists\ u'' \in \mathrm{Hom}_{\mathcal{C}}(U, X) \quad u_*(u'') = uu'' = u'$

Moreover, Δ-kernel / Δ-cokernel is unique under unique isom by Cor 1.2.3 (2).

lem 1.3.1:

$\mathcal{C}$ pre-Δ cate. Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX \in \Sigma$. Then $T^{-1}Z \xrightarrow{-T^{-1}w} X \xrightarrow{u} Y \xrightarrow{v} Z \in \Sigma$.

Pf: Embed map $T^{-1}Z \xrightarrow{-T^{-1}w} X$ into distinguished Δ $T^{-1}Z \xrightarrow{-T^{-1}w} X \xrightarrow{u'} Y' \xrightarrow{v'} Z \in \Sigma$.

$\Rightarrow X \xrightarrow{u'} Y' \xrightarrow{v'} Z \xrightarrow{Tu'} TX \in \Sigma$ By lem 1.2.4 (2), $\exists$ isom of distinguished Δ

$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] & Y' \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, equal] & TX \arrow[d, equal] \\ X \arrow[r] & Y \arrow[r] & Z \arrow[r] & TX \end{tikzcd} \Rightarrow \text{isom} \begin{tikzcd} T^{-1}Z \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, equal] & Y' \arrow[r] \arrow[d, "g"] & Z \arrow[d, equal] \\ T^{-1}Z \arrow[r] & X \arrow[r] & Y \arrow[r] & Z \end{tikzcd}$$

The bottom row is distinguished Δ since the first row is and TR 1)

Cor 1.3.2. $\mathcal{C}$ pre Δ-cate. Then

(1) $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX \in \Sigma. \iff TX \xrightarrow{-Tu} TY \xrightarrow{-Tv} TZ \xrightarrow{-Tw} T^2X \in \Sigma$

$\iff T^{-1}X \xrightarrow{-T^{-1}u} T^{-1}Y \xrightarrow{-T^{-1}v} T^{-1}Z \xrightarrow{-T^{-1}w} X \in \Sigma$.

lem 1.3.3 Direct sum of distinguished Δ is still distinguished Δ.

Pf: By TR1), $\exists$ distinguished Δ:

$$X \oplus X' \xrightarrow{u \oplus u'} Y \oplus Y' \xrightarrow{g} W \xrightarrow{h} T(X \oplus X')$$

By TR3). $\exists$ Δ-morphism $\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, i\right)$ and $\left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, j\right)$

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "\binom{1}{0}"'] & Y \arrow[r, "v"] \arrow[d, "\binom{1}{0}"'] & Z \arrow[r, "w"] \arrow[d, "i"] & TX \arrow[d, "T\binom{1}{0}"] \\ X\oplus X' \arrow[r, "u\oplus u'"] & Y\oplus Y' \arrow[r, "g"] & W \arrow[r, "h"] & T(X\oplus X') \end{tikzcd} \quad ① \quad \& \quad \begin{tikzcd} X' \arrow[r, "u'"] \arrow[d] & Y' \arrow[r, "v'"] \arrow[d] & Z' \arrow[r, "w'"] \arrow[d, "j"] & TX' \arrow[d, "T\binom{0}{1}"] \\ X\oplus X' \arrow[r] & Y\oplus Y' \arrow[r] & W \arrow[r] & T(X\oplus X') \end{tikzcd} \quad ②$$

Combining the above two diag, the following diag is com.

$$\begin{tikzcd} X\oplus X' \arrow[r, "u\oplus u'"] \arrow[d, equal] & Y\oplus Y' \arrow[r, "v\oplus v'"] \arrow[d, equal] & Z\oplus Z' \arrow[r, "w\oplus w'"] \arrow[d, "(i,j)"] & TX\oplus TX' \arrow[d, "(T\binom{1}{0}, T\binom{0}{1})"] \\ X\oplus X' \arrow[r, "u\oplus u'"] & Y\oplus Y' \arrow[r, "g"] & W \arrow[r, "h"] & T(X\oplus X') \end{tikzcd}$$

Note T is additive. (preserves finite direct sum) $\Rightarrow (T\binom{1}{0}, T\binom{0}{1})$ is isom (Appendix 12.6.1)

Applying $\mathrm{Hom}_{\mathcal{C}}(W, -)$ & $\mathrm{Hom}_{\mathcal{C}}(-, Z\oplus Z')$ resp to ① & ② and take direct sum

$\Rightarrow \mathrm{Hom}_{\mathcal{C}}(W, Z) \oplus \mathrm{Hom}_{\mathcal{C}}(W, Z') \cong \mathrm{Hom}_{\mathcal{C}}(W, W)$. $\mathrm{Hom}_{\mathcal{C}}(Z, Z\oplus Z') \oplus \mathrm{Hom}_{\mathcal{C}}(Z', Z\oplus Z') \cong \mathrm{Hom}_{\mathcal{C}}(W, Z\oplus Z')$

$\Rightarrow (i, j)$ isom $\Rightarrow$ The top row $\in \Sigma$.

rmk: If $\mathcal{C}$ admits infinite direct sum. Then direct sum of infinite distinguished $\triangle$ is still distinguished $\triangle$

Def: $\mathcal{C}$ add cate. $f: X \to Y$ morphism. f splitting mono. if $\exists f': Y \to X$ s.t $f'f = \mathrm{id}_X$.

f is splitting epi if $\exists f': Y \to X$ s.t $ff' = \mathrm{id}_Y$.

Lem 1.3.5 $\mathcal{C}$ pre-$\triangle$

(1) For any $X, Z \in \mathrm{Ob}(\mathcal{C})$. $\exists$ distinguished $\triangle$ $X \xrightarrow{\binom{1}{0}} X\oplus Z \xrightarrow{(0,1)} Z \xrightarrow{0} TX$

(2) Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ distinguished $\triangle$. TFAE

① $w = 0$

② u splitting mono

③ v splitting epi

④ $\exists\ s: Y \to X$ and $t: Z \to Y$ s.t $\exists$ isom of distinguished $\triangle$

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, "\binom{s}{v}"] & Z \arrow[r, "w"] \arrow[d, "\mathrm{id}"] & TX \arrow[d, equal] \\ X \arrow[r, "\binom{1}{0}"] & X\oplus Z \arrow[r, "(0,1)"] & Z \arrow[r, "0"] & TX \end{tikzcd}$$

Inverse of $\binom{s}{v}: Y \to X\oplus Z$ is $(u, t): X\oplus Z \to Y$

ie $su = \mathrm{Id}_X$ $\quad tv + us = \mathrm{id}_Y$
$vt = \mathrm{Id}_Z$ $\quad st = 0$

Moreover, $T^{-1}X \xrightarrow{0} Z \xrightarrow{u} Y \xrightarrow{v} X'$ is distinguished $\triangle$

Pf: (1) Consider the direct sum of distinguished $\triangle$ of $X \xrightarrow{\mathrm{id}} X \to 0 \to TX$ and
$0 \to Z \xrightarrow{\mathrm{id}_Z} Z \to 0$

(2) ① $\Rightarrow$ ②. Consider:

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r] \arrow[d, dotted] & Z \arrow[r, "0"] \arrow[d, "0"] & TX \arrow[d, equal] \\ X \arrow[r, "\mathrm{Id}_X"] & X \arrow[r] & 0 \arrow[r] & TX \end{tikzcd} \quad \begin{matrix} \in \Sigma \\ \\ \in \Sigma \end{matrix}$$

By Lem 1.2.4 (ii) $\Rightarrow \exists\ u': Y \to X$ s.t $u'u = \mathrm{Id}_X$

② ⇒ ① Consider

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, equal, "id"'] & Y \arrow[r, "v"] \arrow[d, "u'"] & Z \arrow[r, "w"] \arrow[d, "0"] & TX \arrow[d, equal] \\ X \arrow[r] & X \arrow[r] & 0 \arrow[r] & TX \end{tikzcd}$$

$(id_X, u', 0)$ △-morphism of distinguished △

$Id_{TX} \circ w = 0 \Rightarrow w = 0$

① ⟺ ③ is similar.

④ ⇒ ① is obvious. (com of the diag)

① ⇒ ④

$$\begin{tikzcd} Z \arrow[r, "0"] \arrow[d, equal] & TX \arrow[d, equal] \\ Z \arrow[r, "0"] & TX \end{tikzcd}$$

is com. By lem 1.2.4 (ii) $\Rightarrow$ $\exists$ isom $\begin{pmatrix} s \\ v \end{pmatrix} : Y \to X \oplus Z$ s.t

The following diag commutes.

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, "\begin{pmatrix} s \\ v \end{pmatrix}"] & Z \arrow[r, "0"] \arrow[d, equal] & TX \arrow[d, equal] & \in \Sigma \\ X \arrow[r, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] & X \oplus Z \arrow[r, "(0\ 1)"] & Z \arrow[r, "0"] & TX & \in \Sigma. \end{tikzcd}$$

Suppose inverse $\begin{pmatrix} s \\ v \end{pmatrix}$ is $(a\ t) : X \oplus Z \to Y$ then $a = u$ by com of diag

The relation of $\begin{pmatrix} s \\ v \end{pmatrix}$ and $(a\ t)$ follows easily

$T^{-1}X \xrightarrow{0} Z \xrightarrow{t} Y \xrightarrow{s} X$ is distinguished △ since

$$\begin{tikzcd} T^{-1}X \arrow[r, "0"] \arrow[d, equal] & Z \arrow[r, "t"] \arrow[d, equal] & Y \arrow[r, "s"] \arrow[d, "\begin{pmatrix} s \\ v \end{pmatrix}"] & X \arrow[d, equal] \\ T^{-1}X \arrow[r, "0"] & Z \arrow[r, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] & X \oplus Z \arrow[r, "(1,0)"] & X \end{tikzcd}$$

+ TR 1)

lem 1.3.6 : Let $X \xrightarrow{u} Y \xrightarrow{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}} Z_1 \oplus Z_2 \xrightarrow{(w_1, w_2)} TX \in \Sigma$.

i) $w_2 = 0 \Leftrightarrow \exists\ t : Z_2 \to Y$ s.t $v_1 t = 0$ $v_2 t = id_{Z_2}$. In particular. v_2 is splitting epi

ii) $v_2 = 0 \Leftrightarrow \exists\ s : TX \to Z_2$ s.t $s w_1 = 0$ $s w_2 = id_{Z_2}$. In particular, w_2 is splitting mono.

Pf: ii) is dual of i)

i) $\Rightarrow$ $w_2 = 0$. Then.

$$\begin{tikzcd} Z_2 \arrow[r] \arrow[d, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] & 0 \arrow[d] \\ Z_1 \oplus Z_2 \arrow[r, "(w_1, w_2)"] & TX \end{tikzcd}$$

is com $\Rightarrow$ $\exists$ morphism t. s.t. the following diag commutes

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & Z_2 \arrow[r] \arrow[d, "t"] & Z_2 \arrow[r] \arrow[d, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] & 0 \arrow[d] \\ X \arrow[r, "u"] & Y \arrow[r, "\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}"] & Z_1 \oplus Z_2 \arrow[r, "(w_1, 0)"] & TX \end{tikzcd}$$

$\Rightarrow$ $v_1 t = 0$
$v_2 t = id_{Z_2}$

$\Leftarrow$ If $\exists$ $t : Z_2 \to Y$. s.t $v_1 t = 0$.
$v_2 t = id_{Z_2}$.

We have

$$\begin{tikzcd} Z_2 \arrow[r, "id"] \arrow[d, "t"'] \arrow[dr, phantom, "\circlearrowleft"] & Z_2 \arrow[d, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] \\ Y \arrow[r, "\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}"] & Z_1 \oplus Z_2 \end{tikzcd}$$

$\Rightarrow$ com diag

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & Z_2 \arrow[r] \arrow[d, "t"] & Z_2 \arrow[r] \arrow[d, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] \arrow[dr, phantom, "\circlearrowleft"] & 0 \arrow[d] \\ X \arrow[r, "u"] & Y \arrow[r, "\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}"] & Z_1 \oplus Z_2 \arrow[r, "(w_1, w_2)"] & TX \end{tikzcd}$$

$\Rightarrow w_2 = 0$

lem 1.3.7 : $\mathcal{C}$ pre-Δ cate. $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX \in \Sigma$. Then u is isom $\Leftrightarrow$ $Z=0$

$\Rightarrow$
$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "u"'] & Y \arrow[r, "v"] \arrow[d, equal] & Z \arrow[r, "w"] \arrow[d, dashed, "g"] & TX \arrow[d, "Tu"] \\ Y \arrow[r, "id"] & Y \arrow[r] & 0 \arrow[r] & TY \end{tikzcd}$$
By TR3) & u isom $\exists\, g$ isom $g: Z \to 0$.

$\Leftarrow$ $Z=0$.
$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "u"'] & Y \arrow[r, "v"] \arrow[d, equal] & 0 \arrow[r, "w"] \arrow[d, equal] & TX \arrow[d, "Tu"] \\ Y \arrow[r] & Y \arrow[r] & 0 \arrow[r] & TY \end{tikzcd}$$
$X \to Y \to 0 \to TX \in \Sigma$, $Y \to Y \to 0 \to TY \in \Sigma$. com diag between distinguished Δ $\Rightarrow$ u isom.

Cor 1.3.8. Let $X \xrightarrow{u} Y \xrightarrow{\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}} \bigoplus_i Z_i \xrightarrow{(w_1 \cdots w_n)} TX \in \Sigma$. If X, Y, Z_i $(1 \le i \le n)$ indecomposable. u is neither zero nor isom then every v_i and w_i is neither isom nor zero.

Pf: If $v_i = 0$ for some i, by lem 1.3.6 (ii). $w_i: Z_i \longrightarrow TX$ is splitting mono. But Z_i and TX are indecomposable. [By lem 1.3.5 (iv)] $\Rightarrow$ w_i is isom $\Rightarrow$ $(w_1 \cdots w_n)$ splitting epi $\left(\exists \begin{pmatrix} 0 \\ \vdots \\ w_i^{-1} \\ \vdots \\ 0 \end{pmatrix}\right)$

$\Rightarrow u = 0$ by lem 1.3.5.

If v_i is isom for some i, then $\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ is splitting mono $(\exists\ (0, v_i^{-1}, \cdots 0)) \Rightarrow u = 0$

The argument for w_i is dual to v_i

lem 1.3.9 : $\mathcal{C}$ pre-Δ cate. If (X, Y, Z, u, v, w) distinguished Δ. So is $(X, Y, Z, -u, -v, w)$ $(X, Y, Z, u, -v, -w)$ $(X, Y, Z, -u, v, -w)$

Pf:
$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, "-1"] & Z \arrow[r, "w"] \arrow[d, equal] & TX \arrow[d, equal] \\ X \arrow[r, "-u"] & Y \arrow[r, "-v"] & Z \arrow[r, "w"] & TX \end{tikzcd}$$
is Δ-isom

rmk: Example 2.5.5 shows $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{-w} W$ is no longer distinguished Δ

1.4.

Def 1.4.1 $(\mathcal{C}, T, \Sigma)$ is a Δ-cate of

TR4) If $X \xrightarrow{u} Y \xrightarrow{i} Z' \xrightarrow{i'} TX \in \Sigma$, $\Rightarrow$ $\exists\, f, g$

$X \xrightarrow{vu} Z \xrightarrow{k} Y' \xrightarrow{k'} TX \in \Sigma$ s.t. $Z' \xrightarrow{f} Y' \xrightarrow{g} X' \xrightarrow{(Ti)j'} TZ' \in \Sigma$.

$Y \xrightarrow{v} Z \xrightarrow{j} X' \xrightarrow{j'} TY \in \Sigma$.

Diagram :

$$
\begin{tikzcd}
X \arrow[r, "u"] \arrow[dr] & Y \arrow[r, "i"] \arrow[d] & Z' \arrow[r] \arrow[d, dashed, "\exists f"] & TX \arrow[ddd, equal] \\
 & Z \arrow[r] \arrow[dd] & Y' \arrow[dl, dashed, "\exists g"'] \arrow[ddr] & \\
 & & & \\
 & X' \arrow[dl, "(Ti)j'"'] \arrow[d, "j'"] & & TX \arrow[dll, "Tu"] \\
TZ' & TY \arrow[l, "Ti"] & &
\end{tikzcd}
$$

Def 1.4.3 : $\mathcal{D}$: additive full subcate of Δ-cat $\mathcal{C} := (\mathcal{C}, T, \Sigma)$ is called Δ-subcate, if:

1) $\mathcal{D}$ is closed under automorphism.

2) T is automorphism of $\mathcal{D}$.

3) If $X \to Y \to Z \to TX \in \Sigma$ in $\mathcal{C}$. and $X, Z \in \mathcal{D} \Rightarrow Y \in \mathcal{D}$.

If 1), 2) holds

3) $\Leftrightarrow$ If $X \to Y \to Z \to TX \in \Sigma$ in $\mathcal{C}$ & $X, Y \in \mathcal{D} \Rightarrow Z \in \mathcal{D} \Leftrightarrow X \to Y \to Z \to TX \in \Sigma$ in $\mathcal{C}$ & $Y, Z \in \mathcal{D} \Rightarrow X \in \mathcal{D}$

It's always difficult to prove TR4). The next lem reduces the verification of TR4) to standard Δ

lem 1.4.4

$\mathcal{C}$ add cate. $T: \mathcal{C} \to \mathcal{C}$ automorphism. Ω : class of Δ in $\mathcal{C}$. Let Σ be the class of Δ which is isom to those in Ω. if Ω and Σ satisfying tr1) – tr4), Then $(\mathcal{C}, T, \Sigma)$ is Δ-cate.

tr1) : i) for any morphism $u: X \to Y$. $\exists\ \Delta \in \Omega$. $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX \in \Omega$.

ii) $X \xrightarrow{Id_X} X \longrightarrow 0 \longrightarrow TX \in \Sigma$.

tr2) : If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX \in \Omega$. $\Rightarrow Y \xrightarrow{v} Z \xrightarrow{w} TX \xrightarrow{-Tu} TY \in \Sigma$.

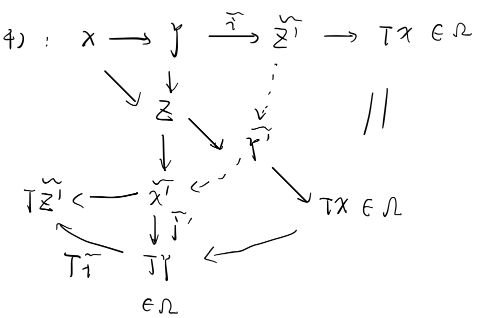
tr3) : If $(X, Y, Z, u, v, w) \in \Omega$ & $(X', Y', Z', u', v', w') \in \Omega$.

$$
\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "g"] \\
X' \arrow[r, "u'"] & Y'
\end{tikzcd}
$$

$\Rightarrow \exists\ (f, g, h)$ Δ-map. i.e $\exists\ h : Z \to Z'$ s.t the following diag commutes

$$
\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & TX \arrow[d, "Tf"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & TX'
\end{tikzcd}
$$

tr4) :



$\exists\ \tilde{f}, \tilde{g}$ s.t.

$\tilde{Z}' \xrightarrow{\tilde{f}} \tilde{Y}' \xrightarrow{\tilde{g}} \tilde{X}' \xrightarrow{(T\tilde{i})\tilde{j}'} T\tilde{Z}' \in \Sigma.$

Pf: TR1): i) $\forall\ \Delta$, if $\Delta \cong \Delta'$, $\Delta' \in \Sigma \Rightarrow \Delta \in \Sigma$. (✓ since $\Delta \cong \Delta''$, where $\Delta'' \in \Omega$)

ii), iii) is just tr1)

TR2) : If $X \to Y \to Z \to TX \in \Sigma$. $\exists$ Δ-isom $X' \to Y' \to Z' \to TX' \in \Omega$.

$\Rightarrow Y' \to Z' \to TX' \to TY' \in \Sigma.$

$\Rightarrow Y \to Z \to TX \to TY \in \Sigma.$

TR3) If
$$\begin{array}{ccccccc} X & \to & Y & \to & Z & \to & TX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & & & \\ X' & \to & Y' & \to & Z' & \to & TX' \end{array} \in \Sigma$$
satisfy the condition of TR3)

$\exists$ Δ-isom $\bar{X} \to \bar{Y} \to \bar{Z} \to T\bar{X} \in \Omega$

$\bar{X}' \to \bar{Y}' \to \bar{Z}' \to T\bar{X}' \in \Omega.$

$$\begin{tikzcd} \bar{X} \arrow[r] \arrow[d, "\alpha^{-1}"] & \bar{Y} \arrow[d, "\beta^{-1}"] \\ X \arrow[r] \arrow[d, "f"] & Y \arrow[d, "g"] \\ X' \arrow[r] \arrow[d, "\bar{\alpha}"] & Y' \arrow[d, "\bar{\beta}"] \\ \bar{X}' \arrow[r] & \bar{Y}' \end{tikzcd}$$
commutes

$\Rightarrow \exists\ \bar{h}: \bar{Z} \to \bar{Z}'$

Since $\gamma: Z \to \bar{Z}$

$\bar{\gamma}: Z' \to \bar{Z}'$

$h := \bar{\gamma}^{-1}\bar{h}\gamma : Z \to Z'$

making the original diag commutes

TR4) Suppose. $X \xrightarrow{u} Y \xrightarrow{i} Z' \xrightarrow{i'} TX \in \Sigma$

$X \xrightarrow{vu} Z \xrightarrow{k} Y' \xrightarrow{k'} TX \in \Sigma$

$Y \xrightarrow{v} Z \xrightarrow{j} X' \xrightarrow{j'} TY \in \Sigma.$

$\exists$ isom of Δ

$X \xrightarrow{u} Y \xrightarrow{\alpha i} \tilde{Z}' \xrightarrow{i'\alpha^{-1}} TX \in \Omega.$

$X \xrightarrow{vu} Z \xrightarrow{\gamma k} \tilde{Y}' \xrightarrow{k'\gamma^{-1}} TX \in \Omega$

$Y \xrightarrow{v} Z \xrightarrow{\beta j} \tilde{X}' \xrightarrow{j'\beta^{-1}} TY \in \Omega.$

where $\alpha: Z' \longrightarrow \tilde{Z}'$

$\beta: X' \longrightarrow \tilde{X}'$

$\gamma: Y' \longrightarrow \tilde{Y}'$

tr4)
$\Rightarrow \tilde{Z}' \xrightarrow{\tilde{f}} \tilde{Y}' \xrightarrow{\tilde{g}} \tilde{X}' \xrightarrow{T(\alpha i)j'\beta^{-1}} T\tilde{Z}' \in \Sigma.$

Let $f = \gamma^{-1}\tilde{f}\alpha : Z' \to Y'$ $g = \beta^{-1}\tilde{g}\gamma : Y' \to X'$ $\Rightarrow Z' \xrightarrow{f} Y' \xrightarrow{g} X' \longrightarrow TZ' \in \Sigma.$

Lem 1.4.5:

Let $\mathcal{C}$ add cate. $T: \mathcal{C} \to \mathcal{C}$ automorphism. Ω: class of Δ in $\mathcal{C}$. Σ: class of Δ isom to those in Ω

If Ω, Σ satisfy tr2) – tr4) and tr1') $\Rightarrow$ $(\mathcal{C}, T, \Sigma)$ Δ-cate

tr1') : ① $\forall\, u: X \to Y$, $\exists$ isom of Δ

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "\alpha"'] & Y \arrow[r, "v"] \arrow[d, equal] & Z \arrow[r, "w"] \arrow[d, equal] & TX \arrow[d, "T\alpha"] \\ X' \arrow[r, "u'"] & Y \arrow[r, "v"] & Z \arrow[r, "w'"] & TX' \end{tikzcd}$$

s.t the second row lies in Ω.

$\Downarrow$

$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX \in \Sigma$.

tr2') $X \xrightarrow{Id_X} X \longrightarrow 0 \longrightarrow TX \in \Sigma$.