§1.5. Triangle functor (Δ-functor)

Def: 1.5.1

(1) $(\mathcal{C}, T, \mathcal{E})$ & $(\mathcal{C}', T', \mathcal{E}')$: Δ-categories, let $F: \mathcal{C} \to \mathcal{C}'$ be an additive functor, $\varphi: FT \cong T'F$ be a natural isomorphism.
we call $(F, \varphi): (\mathcal{C}, T, \mathcal{E}) \longrightarrow (\mathcal{C}', T', \mathcal{E}')$ a Δ-functor, if it maps distinguished triangle (d-Δ) to d-Δ, i.e.

if $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX \in \mathcal{E}$

then $FX \xrightarrow{Fu} FY \xrightarrow{Fv} FZ \xrightarrow{\varphi_X \circ Fw} T'FX \in \mathcal{E}'$

$$\begin{tikzcd} FZ \arrow[dr, "F(w)"'] \arrow[rr, "\varphi_X \circ Fw"] & & T'FX \\ & FTX \arrow[ur, "\varphi_X"'] & \end{tikzcd}$$

(2) we say (F, φ) is a Δ-equivalence (resp. Δ-isom) if F is an equivalence (resp. isom.).

· $(\mathcal{C}, T, \mathcal{E})$ and $(\mathcal{C}', T', \mathcal{E}')$ are called Δ-equivalent (resp. Δ-isomorphic), if there exists a Δ-equivalence (resp. Δ-isom)- between them.

Remark:

(1) we emphasize: a Δ-functor is a two-tuples.

(2) Δ-functor is also called exact functor.

(3) Let $F: \mathcal{C} \to \mathcal{C}'$ be a Δ-functor, then $\ker F = \{ X \in \mathcal{C} \mid F(X) \cong 0 \}$ is a Δ-subcategory.

proof: · full additive subcategory.
· closed under isom. If $X \cong Y$, then $F(X) \cong F(Y) \cong 0$.
· $T|_{\ker F}$ is a self-isom.
i.e. if $X \in \ker F$, then $F(TX) \cong T'F(X) \cong 0$.
& $F(T^{-1}X) \cong (T')^{-1}(FX) \cong 0$. i.e. $TX, T^{-1}X \in \ker F$.

$FT \cong T'F \qquad T'^{-1}FT \cong F \qquad T'^{-1}F = FT^{-1}$

· closed under extensions: if $X \to Y \to Z \to TX$ is d-Δ in $\ker F$
& $X, Z \in \ker F$, $\underset{\overset{\shortparallel}{0}}{F(X)} \to F(Y) \to \underset{\overset{\shortparallel}{0}}{F(Z)} \to T'F(X) \in \mathcal{E}$.
$\Rightarrow F(Y) \cong 0$. $\Rightarrow Y \in \ker F$.

#.

prop. 1.5.2. Assume $F: \mathcal{C} \to \mathcal{C}'$ is a Δ-functor, and F is fully. If F maps non-zero objects to non-zero objects, then F is also faithful.

proof: Let $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, s.t. $F(f) = 0$.
$X \xrightarrow{f} Y \xrightarrow{g} Z \to TX \in \mathcal{E}$ then $F(X) \xrightarrow{0} FY \xrightarrow{F(g)} FZ \to T'FX \in \mathcal{E}$

$\Rightarrow FY \underset{\underset{Fg'}{\longleftarrow}}{\xrightarrow{Fg}} FZ \longrightarrow T'FX \xrightarrow{0} T'FY \in \mathcal{E}$.

by lem 1.3.3. Fg is split monomorphism. that is, $\exists\ Fg': FZ \to FY$ s.t. $Fg'Fg = \mathrm{id}_{FY}$. where $g' \in \mathrm{Hom}_{\mathcal{C}}(Z, Y)$ by F is fully.

let $Y \xrightarrow{g'g} Y \longrightarrow \overset{0}{\underset{\|\wr}{}}\!W \longrightarrow TY \in \mathcal{E}$. then

$F(Y) \xrightarrow{id_{FY}} F(Y) \longrightarrow F(W) \longrightarrow T'F(Y) \in \mathcal{E}'$

by lem 1.3.7. $F(W) \cong 0$, $\Rightarrow W \cong 0$. $\Rightarrow g'g$ is isom. i.e. g is split mono.
Thus $f=0$, by lem 1.5.5.

#.

Def 1.5.3. $(\mathcal{C}, T, \mathcal{E})$, $(\mathcal{C}', T', \mathcal{E}')$: $\triangle$-cat.

(1) Assume that $F: \mathcal{C} \to \mathcal{C}'$ is a contravariant addi. functor and $\varphi : T'^{-1}F \cong FT$ is a nat. isom.
we call $(F, \varphi) : (\mathcal{C}, T, \mathcal{E}) \longrightarrow (\mathcal{C}', T', \mathcal{E}')$ a $\triangle$-functor, if

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX \in \mathcal{E}$$

Then

$$\begin{tikzcd}
(T')^{-1}FX \arrow[r, "Fw\circ\varphi_X"] \arrow[dr, "\varphi_X"'] & FZ \arrow[r, "Fv"] & FY \arrow[r, "Fu"] & FX \in \mathcal{E}' \\
& FTX \arrow[u, "F(w)"'] \arrow[u, phantom, "\wr", shift left=3ex]
\end{tikzcd}$$

(2) $F: \mathcal{C} \to \mathcal{C}'$: contra. $\triangle$-functor, F is called $\triangle$-duality if $F: \mathcal{C}^{op} \longrightarrow \mathcal{C}'$ inducing F is an equivalence.

(3) $\mathcal{C}$ and $\mathcal{C}'$ are $\triangle$-dual. if there is a $\triangle$-duality.

§1.6. 伴随中的三角函子.

· Notation : T^n is denoted by $[n]$, T by $[1]$, T^{-1} by $[-1]$, $id_{\mathcal{C}}$ by $[0]$.

· Recall :

(1) Yoneda lemma:

$$\mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(X, -), \mathrm{Hom}_{\mathcal{C}}(Y, -)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Y, X).$$
$$\eta \longmapsto \eta(id_X).$$

η is an natural isom. iff $X \cong Y$ in $\mathcal{C}$

(2) Adjoint pair

① (F, G) is an adjoint pair. if $\mathrm{Hom}_{\mathcal{B}}(FX, Y) \cong \mathrm{Hom}_{\mathcal{A}}(X, GY)$.

② Let (F, G) be an adjoint pair, for any $X \in \mathcal{A}$, define:

$$\eta_X := \eta_{X,FX}(id_{FX}) : X \longrightarrow GFX. \quad \eta = (\eta_X)_{X\in\mathcal{A}} \text{ unit.}$$

where $\eta_{X,FX} : \mathrm{Hom}_{\mathcal{B}}(FX, FX) \cong \mathrm{Hom}_{\mathcal{A}}(X, GFX)$.

and for $Y \in \mathcal{B}$, define :

$$\varepsilon_Y := \eta^{-1}_{GY,Y}(id_{GY}) : FGY \longrightarrow Y. \quad \varepsilon = (\varepsilon_Y)_{Y\in\mathcal{B}} \text{ counit.}$$

where $\eta_{GY,Y} : \mathrm{Hom}_{\mathcal{B}}(FGY, Y) \cong \mathrm{Hom}_{\mathcal{A}}(GY, GY)$

prop 12.8.1 Let (F, G, η) is an adjoint pair, where $F: \mathcal{A} \to \mathcal{B}$, $G: \mathcal{B} \to \mathcal{A}$.

$$\begin{tikzcd} FU \arrow[r, "F\alpha"] \arrow[d, "f"'] & FV \arrow[d, "g"] \\ U' \arrow[r, "\beta"] & V' \end{tikzcd} \quad \text{is commuting iff} \quad \begin{tikzcd} U \arrow[r, "\alpha"] \arrow[d, "\eta_{U,U'}(f)"'] & V \arrow[d, "\eta_{V,V'}(g)"] \\ GU' \arrow[r, "G\beta"] & GV' \end{tikzcd} \quad \text{is commuting}$$

Th.1.6.1 Assume $F: \mathcal{A} \to \mathcal{B}$ and $G: \mathcal{B} \to \mathcal{A}$ are functor between Δ-categories, and (F, G) is an adjoint pair. Then F is Δ-functor iff G is a Δ-functor.

proof: Denote by η be adjoint isomorphism of (F, G), i.e.

$$\eta_X : \operatorname{Hom}(FX, Y) \xrightarrow{\sim} \operatorname{Hom}(X, GY).$$

"$\Rightarrow$"

let (F, ξ) be a Δ-functor, where $\xi: F\circ[1] \longrightarrow [1]\circ F$ is nat. isom.

claim: G is a Δ-functor.

(1) G is an additive functor by adjoint pair.

(2) $\varphi: G\circ[1] \xrightarrow[\sim]{\text{nat.}} [1]\circ G$.

For any $M \in obj(\mathcal{A})$, $X \in obj(\mathcal{B})$,

$$\boxed{\operatorname{Hom}_{\mathcal{A}}(M[1], G(X[1]))} \overset{\eta^{-1}_{M[1],X[1]}}{\cong} \operatorname{Hom}_{\mathcal{B}}(F(M[1]), X[1]) \xrightarrow[\sim]{(\xi_M^{-1}, X[1])} \operatorname{Hom}_{\mathcal{B}}((FM)[1], X[1])$$

$$\xrightarrow[\sim]{[-1]} \operatorname{Hom}_{\mathcal{B}}(FM, X) \xrightarrow[\sim]{\eta_{M,X}} \operatorname{Hom}_{\mathcal{A}}(M, GX) \overset{[1]}{\cong} \boxed{\operatorname{Hom}_{\mathcal{A}}(M[1], (GX)[1])}$$

which are functorial in M. By Yoneda Lemma, there is.

$$\varphi_X : G(X[1]) \xrightarrow{\sim} (GX)[1] \quad \text{in } \mathcal{A}.$$

s.t. $\operatorname{Hom}_{\mathcal{A}}(M[1], \varphi_X) = [1]\circ\eta_{M,X}\circ\operatorname{Hom}_{\mathcal{B}}(\xi_M^{-1}, X[1])\circ\eta^{-1}_{M[1],X[1]}$ (1.2)

φ_X is functorial in X. Hence we get nat. isom.

$$\varphi: G\circ[1] \longrightarrow [1]\circ G.$$

(3). prepare for proving (4). $\underline{\varphi_X^{-1} = (\qquad\qquad)}$

For any $a \in \operatorname{Hom}_{\mathcal{A}}(M[1], GX[1])$, (1.2) can be rewritted as

$$\begin{aligned}
\varphi_X \circ a &= [1]\circ\eta_{M,X}\circ[-1]\circ\operatorname{Hom}_{\mathcal{B}}(\xi_M^{-1}, X[1])\circ\eta^{-1}_{M[1],X[1]}\circ a.\\
&= [1]\circ\eta_{M,X}\circ[-1]\circ(\eta^{-1}_{M[1],X[1]}(a)\circ\xi_M^{-1})\\
&= [1]\circ\eta_{M,X}\circ(\eta^{-1}_{M[1],X[1]}(a)[-1]\circ\xi_M^{-1}[-1])\\
&= \eta_{M,X}((\eta^{-1}_{M[1],X[1]}(a))[-1]\circ\xi_M^{-1}[-1])[1] \qquad (1.3)
\end{aligned}$$

Let $M = GX$ and $a = \varphi_X^{-1} : (GX)[1] \longrightarrow G(X[1])$ in (1.3).
then we have:

$$\mathrm{id}_{(GX)[1]} = \eta_{GX,X}\left((\eta^{-1}_{(GX)[1],X[1]}(a))[-1] \circ \xi^{-1}_{GX}[-1] \right)[1].$$
$$\shortparallel$$
$$\mathrm{id}_{GX}[1] = [1]\circ(\mathrm{id}_{GX}).$$

i.e. $\mathrm{id}_{GX} = \eta_{GX,X}\left((\eta^{-1}_{(GX)[1],X[1]}(a))[-1] \circ \xi^{-1}_{GX}[-1] \right)$.

Hence $\underbrace{\eta^{-1}_{GX,X}(\mathrm{id}_{GX})}_{=\varepsilon_X} = (\eta^{-1}_{GX[1],X[1]}(a) \circ \xi^{-1}_{GX})[-1]$.

$\Rightarrow \quad \varepsilon_X[1] = \eta^{-1}_{(GX)[1],X[1]}(a) \circ \xi^{-1}_{GX}$

i.e. $\varepsilon_X[1] \circ \xi_{GX} = \eta^{-1}_{(GX)[1],X[1]}(a)$.

Thus, $\varphi_X^{-1} = \eta_{(GX)[1],X[1]}(\varepsilon_X[1] \circ \xi_{GX})$ (1.4).

(4) (G, φ) is a Δ-functor.

Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1] \in \mathcal{E}_{\mathcal{B}}$.

we want: $GX \xrightarrow{Gf} GY \xrightarrow{Gg} GZ \xrightarrow{\varphi_X \circ Gh} (GX)[1] \in \mathcal{E}_{\mathcal{A}}$.

we have: $GX \xrightarrow{Gf} GY \xrightarrow{\alpha} C \xrightarrow{\beta} (GX)[1] \in \mathcal{E}_{\mathcal{A}}$

Considering counit $\varepsilon : FG \longrightarrow \mathrm{id}_{\mathcal{B}}$. we have:

$$\begin{tikzcd}
FGX \arrow[r, "FGf"] \arrow[d, "\varepsilon_X"] \arrow[dr, phantom, "\circlearrowright"] & FGY \arrow[r, "F\alpha"] \arrow[d, "\varepsilon_Y"] \arrow[dr, phantom, "\circlearrowright"] & FC \arrow[r, "\xi_{GX}\circ F\beta"] \arrow[d, dashed, "\exists \Phi"] \arrow[dr, phantom, "\circlearrowright"] & (FGX)[1] \arrow[d, "\varepsilon_X[1]"] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & X[1]
\end{tikzcd} \qquad (1.5)$$

we have:

$$\begin{tikzcd}
FC \arrow[r, "F\beta"] \arrow[d, "\Phi"] \arrow[dr, phantom, "\circlearrowright"] & F((GX)[1]) \arrow[d, "\varepsilon_X[1]\circ\xi_{GX}"] \\
Z \arrow[r, "h"] & X[1]
\end{tikzcd}$$

By prop 12.8.1 (viii) we have.

$$
\begin{tikzcd}
GX \arrow[r, "Gf"] \arrow[d, equal] & GY \arrow[r, "\alpha"] \arrow[d, equal, "id_{GY}"] & C \arrow[r, "\beta"] \arrow[d, "\eta_{C,Z}(\Phi)"] & (GX)[1] \arrow[d, "\eta_{(GX)[1],X[1]}(\varepsilon_{X[1]}\circ\xi_{GX}) \,=\, \varphi_X^{-1}"] \\
GX \arrow[r, "Gf"] & GY \arrow[r, "Gg"] & GZ \arrow[r, "Gh"] & G(X[1])
\end{tikzcd}
$$

By (1.4), we have.

$$
\begin{tikzcd}
GX \arrow[r, "Gf"] \arrow[d, equal] & GY \arrow[r, "\alpha"] \arrow[d, equal, "id_{GY}"] & C \arrow[r, "\beta"] \arrow[d, "\eta_{C,Z}(\Phi)"] & (GX)[1] \arrow[d, equal] \\
GX \arrow[r, "Gf"] & GY \arrow[r, "Gg"] & GZ \arrow[r, "\varphi_X\circ Gh"] & G(X[1])
\end{tikzcd}
$$

· prove $\eta_{C,Z}(\Phi): C \longrightarrow GZ$ is isom.

since φ is a nat. isom. we have.

$$
\begin{tikzcd}
G(X[1]) \arrow[r, "\varphi_X"] \arrow[d, "G(f[1])"] \arrow[dr, phantom, "\daleth"] & (GX)[1] \arrow[d, "(Gf)[1]"] \\
G(Y[1]) \arrow[r, "\varphi_Y"] & (GY)[1]
\end{tikzcd}
$$

$\Rightarrow$

$$
\begin{tikzcd}
(M, G(X[1])) \arrow[r, "(M,\varphi_X)"] \arrow[d] \arrow[dr, phantom, "\daleth"] & (M, (GX)[1]) \arrow[d] \\
(M, G(Y[1])) \arrow[r, "(M,\varphi_Y)"] & (M, (GY)[1])
\end{tikzcd}
\quad (1.6).
$$

Using $\mathrm{Hom}_{\mathcal{Y}}(FM, -)$ to $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$
we have exact sequence:

$$(FM, X) \xrightarrow{(FM,f)} (FM,Y) \xrightarrow{(FM,g)} (FM,Z) \xrightarrow{(FM,h)} (FM, X[1]) \xrightarrow{(FM,f[1])} (FM, Y[1])$$

$\Rightarrow$

$$(M,GX) \xrightarrow{(M,Gf)} (M,GY) \xrightarrow{(M,Gg)} (M,GZ) \xrightarrow{(M,Gh)} (M,G(X[1])) \xrightarrow{(M,G(f[1]))} (M,G(Y[1]))$$

Using (1.6) we have exact sequence:

$$(M,GX) \xrightarrow{(M,Gf)} (M,GY) \xrightarrow{(M,Gg)} (M,GZ) \xrightarrow{(M,\varphi_X Gh)} (M,(GX)[1]) \xrightarrow{(M,(Gf)[1])} (M,(GY)[1])$$

· since we have following comm. diagram:

$$\begin{tikzcd}
(M, GZ)) \arrow[r, "{(M, Gh)}"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & (M, (GX)[1]) \arrow[r, "{(M, G(f[1]))}"] \arrow[d, "{(M, \varphi_X)}"] \arrow[dr, phantom, "\circlearrowright \text{ by } (1.6)."] & (M, G(Y[1])) \arrow[d, "{(M, \varphi_Y)}"] \\
(M, GZ)) \arrow[r, "{(M, \varphi_X \circ Gh)}"] & (M, (GY)[1]) \arrow[r, "{(M, (Gf)[1])}"] & (M, (GY)[1])
\end{tikzcd}$$

$$\Rightarrow (Gf)[1] \circ \varphi_X \circ Gh = \varphi_Y \circ G(f[1])\, Gh = \varphi_Y \circ G(f[1] \cdot h) = 0.$$

· $\mathrm{Hom}_A(M, -)$ $\quad GX \xrightarrow{Gf} GY \xrightarrow{\alpha} C \xrightarrow{\beta} (GX)[1]$

we have exact seq.

$$(M, GX) \xrightarrow{(M, Gf)} (M, GY) \xrightarrow{(M, \alpha)} (M, C) \xrightarrow{(M, \beta)} (M, (GX)[1]) \xrightarrow{(M, (Gf)[1])} (M, (GY)[1])$$

and commutative diagram:

$$\begin{tikzcd}
(M, GX) \arrow[r, "{(M, Gf)}"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & (M, GY) \arrow[r, "{(M, Gg)}"] \arrow[d, equal] & (M, GZ) \arrow[r, "{(M, Gh)}"] \arrow[d, "{(M, y_{C,Z}(\Phi))}"] & (M, G(X[1])) \arrow[r, "{(M, G(f[1]))}"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & (M, G(Y[1])) \arrow[d, equal] \\
(M, GX) \arrow[r, "{(M, Gf)}"] & (M, GY) \arrow[r, "{(M, \alpha)}"] & (M, C) \arrow[r, "{(M, \beta)}"] & (M, (GX)[1]) \arrow[r, "{(M, (Gf)[1])}"] & (M, (GY)[1])
\end{tikzcd}$$

by Five lemma, $\mathrm{Hom}_A(M, y_{C,Z}(\Phi))$ is an isom.

Yoneda lemma $\Rightarrow$ $y_{C,Z}(\Phi)$ is an isom.

what is G is Δ-functor.

#_

$$J_{X,Y} : F(X) \otimes F(Y) \longrightarrow F(X \otimes Y).$$

F is a tensor functor

$G(k') \overset{?}{\cong} k$ $\qquad \mathrm{Hom}(M, G(k')) \cong \mathrm{Hom}(M, k).$

$F(k) \cong k'$ $\qquad GF(k) \cong G(k') \xrightarrow{y^{-1}} k.$

$\mathrm{id} \longrightarrow GF.$

$FG \longrightarrow \mathrm{id}$

$k \xrightarrow{y} G(k')$

§1.8. (4×4 lemma).

lemma 1.8.1 (4×4 lemma).

Let $(\mathcal{C}, [1])$ be a Δ-cat. Given any com. diagram in $\mathcal{C}$.

$$\begin{tikzcd}
A_1 \arrow[r, "x_1"] \arrow[d, "a_1"] & B_1 \arrow[d, "b_1"] \\
A_2 \arrow[r, "x_2"] & B_2
\end{tikzcd}$$

and d-Δ in $\mathcal{C}$:

$$A_1 \xrightarrow{x_1} B_1 \xrightarrow{y_1} C_1 \xrightarrow{z_1} A_1[1]$$

$$A_2 \xrightarrow{x_2} B_2 \xrightarrow{y_2} C_2 \xrightarrow{z_2} A_2[1]$$

$$A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} A_3 \xrightarrow{a_3} A_1[1]$$

$$B_1 \xrightarrow{b_1} B_2 \xrightarrow{b_2} B_3 \xrightarrow{b_3} B_1[1]$$

we have following diagram:

$$\begin{tikzcd}
A_1 \arrow[r, "x_1"] \arrow[d, "a_1"] & B_1 \arrow[r, "y_1"] \arrow[d, "b_1"] \arrow[dr, phantom, "1"] & C_1 \arrow[r, "z_1"] \arrow[d, "c_1"] \arrow[dr, phantom, "2"] & A_1[1] \arrow[d, "a_1[1]"] \\
A_2 \arrow[r, "x_2"] \arrow[d, "a_2"] \arrow[dr, phantom, "3"] & B_2 \arrow[r, "y_2"] \arrow[d, "b_2"] \arrow[dr, phantom, "4"] & C_2 \arrow[r, "z_2"] \arrow[d, "c_2"] \arrow[dr, phantom, "5"] & A_2[1] \arrow[d, "a_2[1]"] \\
A_3 \arrow[r, "a_1"] \arrow[d, "a_3"] \arrow[dr, phantom, "6"] & B_3 \arrow[r, "a_2"] \arrow[d, "b_3"] \arrow[dr, phantom, "7"] & \boxed{C_3} \arrow[r, "a_3"] \arrow[d, "c_3"] \arrow[dr, phantom, "8"] & A_3[1] \arrow[d, "a_3[1]"] \\
A_1[1] \arrow[r, "b_1"] & B_1[1] \arrow[r, "b_2"] & C_1[1] \arrow[r, "b_3"] & A_1[2]
\end{tikzcd}$$

proof: By (TR4) we have following commutative diagrams:

$$\begin{tikzcd}
A_1 \arrow[r, "x_1"] \arrow[d, equal] & B_1 \arrow[r, "y_1"] \arrow[d, "b_1"] & C_1 \arrow[r, "z_1"] \arrow[d, dashed, "h"] & A_1[1] \arrow[d, equal] \\
A_1 \arrow[r, "x_2 a_1"] & B_2 \arrow[r, "f"] \arrow[d, "b_2"] & D \arrow[r, "g"] \arrow[d, dashed, "j"] & A_1[1] \arrow[d, "x_1[1]"] \\
 & B_3 \arrow[r, equal] \arrow[d, "b_3"] & B_3 \arrow[r] \arrow[d, dashed, "y_1[1] \circ b_3"] & B_1[1] \\
 & B_1[1] \arrow[r, "y_1[1]"] & C_1[1] &
\end{tikzcd}$$

$\in \mathcal{E}$ (*)

$$
\begin{tikzcd}
A_1 \arrow[r, "a_1"] \arrow[d, equal] & A_2 \arrow[r, "a_2"] \arrow[d, "x_2"] & A_3 \arrow[r, "a_3"] \arrow[d, dashed, "h"] & A_1[1] \arrow[d, equal] \\
A_1 \arrow[r, "b_1 x_1"] & B_2 \arrow[r, "f"] \arrow[d, "y_2"] & D \arrow[r, "g"] \arrow[d, dashed, "i"] & A_1[1] \arrow[d, "a_1[1]"] \\
 & C_2 \arrow[r, equal] \arrow[d, "z_2"] & C_2 \arrow[r, "z_2"] \arrow[d, dashed, "a_2[1]\circ z_2"] & A_2[1] \\
 & A_2[1] \arrow[r, "a_2[1]"] & A_3[1] &
\end{tikzcd}
$$

$\in \mathcal{E}$ (**)

$$
\begin{tikzcd}
A_3 \arrow[r, "h"] \arrow[d, equal] & D \arrow[r, "i"] \arrow[d, "j"] & C_2 \arrow[r, "a_2[1]\circ z_2"] \arrow[d, dashed, "c_2"] & A_3[1] \arrow[d, equal] \\
A_3 \arrow[r, "jh = x_3"] & B_3 \arrow[r] \arrow[d, "y_1[1]\circ b_3"] & C_3 \arrow[r] \arrow[d, dashed, "c_3"] & A_3[1] \arrow[d, "h[1]"] \\
 & C_1[1] \arrow[r, equal] \arrow[d, "-k[1]"] & C_1[1] \arrow[r] \arrow[d, dashed, "-c_1[1] = -(ik)[1]"] & D[1] \\
 & D[1] \arrow[r, "i[1]"] & C_2[1] &
\end{tikzcd}
$$

$\in \mathcal{E}$ (***)

let $x_3 = jh$. $\quad c_1 = ik$.

① $c_1 y_1 = i k y_1 \overset{(*)}{=} i f b_1 \overset{(**)}{=} y_2 b_1$

② $z_2 c_1 = z_2 \circ ik \overset{(**)}{=} a_1[1]\circ g \circ k \overset{(*)}{=} a_1[1]\circ z_1$.

③ $x_3 a_2 = jh a_2 \overset{(**)}{=} j f x_2 \overset{(*)}{=} b_2 x_2$.

④ $y_2 b_2 \overset{(*)}{=} y_2 j f \overset{(***)}{=} c_2 i f \overset{(**)}{=} c_2 y_2$.

⑤ $z_3 \circ c_2 \overset{(***)}{=} a_1[1]\circ z_2$

⑥ $b_3 x_3 = b_3 j h \overset{(*)}{=} x[1]\circ g \circ h \overset{(**)}{=} x_1[1]\circ a_3$.

⑦ $c_3 y_3 \overset{(***)}{=} y_1[1]\circ b_3$.

⑧ $-z_1[1]\circ c_3 \overset{(*)}{=} -(gk)[1]\circ c_3 = -g[1]\circ k[1]\circ c_3 \overset{(***)}{=} g[1]\circ h[1]\circ z_3$

$= (gh)[1]\circ z_3 \overset{(**)}{=} a_1[1]\circ z_3$.

#