1.7 base change and cobase change

$$\begin{tikzcd} TR4 \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & TR4' \\ \text{base change II} \arrow[ur] & \end{tikzcd}$$

base change I $\Downarrow$ TR4'

Prop 1.7.1 $(\mathcal{C}, [1], \mathcal{E})$ pre-Δ cat. equivalent:

(i) TR4

(ii) TR4' $A \xrightarrow{f_1} B \xrightarrow{f_2} C$ in $\mathcal{C}$ $\exists$ comm. diag.

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, equal] & B \arrow[r, "g_1"] \arrow[d, "f_2"] & X \arrow[r, "h_1"] \arrow[d, "\alpha"] & A[1] \arrow[d, equal] \\
A \arrow[r, "f_2f_1"] & C \arrow[r, "g_3"] \arrow[d, "g_2"] & Y \arrow[r, "h_3"] \arrow[d, "\beta"] & A[1] \arrow[d, "f_1[1]"] \\
 & Z \arrow[r, equal] \arrow[d, "h_2"] & Z \arrow[r] \arrow[d, "\gamma"] & B[1] \\
 & B[1] \arrow[r, "g_1[1]"] & X[1] &
\end{tikzcd}$$

C_2, C_3. r_1 r_2 d.o.

pf: TR4 $\Rightarrow$ TR4'

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, equal] & B \arrow[r, "g_1"] \arrow[d, "f_2"] & X \arrow[r, "h_1"] \arrow[d, "\alpha"] & A[1] \\
A \arrow[r, "f_2f_1"] & C \arrow[r, "g_3"] \arrow[d, "g_2"] & Y \arrow[r, "h_3"] \arrow[d, "\beta"] & A[1] \arrow[d, "f_1[1]"] \\
 & Z \arrow[r, equal] \arrow[d, "h_2"] & Z \arrow[r, "h_2"] \arrow[d, "\gamma = g_1[1]h_2"] & B[1] \\
 & B[1] \arrow[r, "g_1[1]"] & X[1] &
\end{tikzcd}$$

C_3 d.o $\circlearrowright$

TR4' $\Rightarrow$ TR4.

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, equal] & B \arrow[r, "g_1'"] \arrow[d, "f_2"] \arrow[dr, phantom, "\circlearrowright"] & X' \arrow[r, "h_1'"] \arrow[d, "\alpha'"] & A[1] \arrow[d, equal] \\
A \arrow[r, "f_2f_1"] & C \arrow[r, "g_3'"] \arrow[d, "g_2'"] & Y' \arrow[r, "h_3'"] \arrow[d, "\beta'"] & A[1] \arrow[d, "f_1[1]"] \\
 & Z' \arrow[r, equal] \arrow[d, "h_2'"] & Z' \arrow[r, "h_2'"] \arrow[d, "\gamma' = g_1'[1]h_2'"] & B[1] \\
 & B[1] \arrow[r, "g_1'[1]"] & X'[1] &
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, equal] & B \arrow[r, "g_1"] \arrow[d, equal] & X \arrow[r, "h_1"] \arrow[d, "a"] & A[1] \arrow[d, equal] \\
A \arrow[r, "f_1"] & B \arrow[r, "g_1'"] & X' \arrow[r, "h_1'"] & A[1]
\end{tikzcd}$$

$g_1' = a g_1$

$\exists$ iso $X \xrightarrow{a} X'$ $\quad Y \xrightarrow{b} Y'$

$Z \xrightarrow{c} Z'$

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "a"] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[r, "\beta"] \arrow[d, "b"] \arrow[dr, phantom, "?"] & Z \arrow[r, "\gamma"] \arrow[d, "c"] \arrow[dr, phantom, "\circlearrowright"] & X[1] \arrow[d, "a[1]"] \\
X' \arrow[r, "\alpha'"] & Y' \arrow[r, "\beta'"] & Z' \arrow[r, "\gamma'"] & X'[1]
\end{tikzcd}$$

let $\alpha' = b\alpha a^{-1}$

$\beta' = c\beta b^{-1}$

$\alpha' g_1' = b\alpha a^{-1} \cdot a g_1 = b\underline{\alpha g_1}$

$g_3' f_2 = b\, \underline{g_3 f_2}$

$\square$.

(iii) (base change I) d.△ $A \to B \to C \to A[1]$ $\varepsilon: C' \to C$ in $\mathcal{C}$

∃ comm. dia.

$$\begin{tikzcd}
 & E \arrow[r, equal] \arrow[d, "\alpha"] & E \arrow[d, "\delta"] & \\
A \arrow[r, "f'"] \arrow[d, equal] & B' \arrow[r, "g'"] \arrow[d, "\beta"] & C' \arrow[r, "h'"] \arrow[d, "\varepsilon"'] & A[1] \arrow[d, equal] \\
A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, "\gamma"] & C \arrow[r, "h"] \arrow[d, "\eta"] & A[1] \arrow[d, "f'[1]"] \\
 & E[1] \arrow[r, equal] & E[1] \arrow[r, "-\alpha[1]"] & B'[1]
\end{tikzcd}$$

s.t. C_2 C_3, r_2, r_3 d.△.

base change I ⟹ TR4'.

pf: $A \xrightarrow{f_1} B \xrightarrow{f_2} C$

∃ d.△. $C[-1] \xrightarrow{-g_2[-1]} Z[-1] \xrightarrow{-h_2[-1]} B \xrightarrow{f_2} C$

$f_1 : A \to B$

by base change I

$$\begin{tikzcd}
 & X[-1] \arrow[r, equal] \arrow[d, "\alpha[-1]"] & X[-1] \arrow[d, "-h_1[-1]"] & \\
C[-1] \arrow[r, "-g_3[-1]"] \arrow[d, equal] & Y[-1] \arrow[r, "h_3[-1]"] \arrow[d, "\beta[-1]"'] & A \arrow[r, "f_2 f_1"] \arrow[d, "f_1"'] & C \arrow[d, equal] \\
C[-1] \arrow[r, "-g_2[-1]"] & Z[-1] \arrow[r, "-h_2[-1]"] \arrow[d, "-\gamma[-1]"'] & B \arrow[r, "f_2"] \arrow[d, "g_1"] & C \arrow[d, "-g_3"] \\
 & X \arrow[r, equal] & X \arrow[r, "-\alpha"] & Y
\end{tikzcd}$$

C_2 C_3 r_2, r_3. d.△.

(base change I)		TR4'
C_3	⟶	r_1
r_2		r_2
r_3		C_2
C_2 (符号法则)		C_3

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, equal] & B \arrow[r, "g_1"] \arrow[d, "f_2"'] & X \arrow[r, "h_1"] \arrow[d, "\alpha"] & A[1] \arrow[d, equal] \\
A \arrow[r, "f_2 f_1"] & C \arrow[r, "g_3"] \arrow[d, "g_2"'] & Y \arrow[r, "h_3"] \arrow[d, "\beta"] & A[1] \arrow[d, "f_1[1]"] \\
 & Z \arrow[r, equal] \arrow[d, "h_2"'] & Z \arrow[r] \arrow[d, "\gamma"] & B[1] \\
 & B[1] \arrow[r, "g_1[1]"] & X[1] &
\end{tikzcd}$$

(iv) base change II. d.△. $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ $\varepsilon: C' \to C$

2 given d.△ $E \xrightarrow{\delta} C' \xrightarrow{\varepsilon} C \xrightarrow{\eta} E[1]$

$$A \xrightarrow{f} B' \xrightarrow{g} C' \xrightarrow{h\varepsilon} A[1]$$

$\exists$ comm. diag.

$$\begin{tikzcd}
 & E \arrow[r, equal] \arrow[d] & E \arrow[d, "\delta"] & \\
A \arrow[r] \arrow[d, equal] & B' \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{I}"] & C' \arrow[r, "h\varepsilon"] \arrow[d, "\varepsilon"] & A[1] \arrow[d, equal] \\
A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d] & C \arrow[r, "h"] \arrow[d, "\eta"] & A[1] \arrow[d, "f'[1]"] \\
 & E[1] \arrow[r, equal] & E[1] \arrow[r] & B'[1]
\end{tikzcd}$$

s.t. C_2, C_3, r_2, r_1 d.o.

(TR4')$\Leftrightarrow$(TR4) $\Longrightarrow$ (base change II).

$$\begin{tikzcd}
C' \arrow[r, "\varepsilon"] \arrow[d, equal] & C \arrow[r, "\eta"] \arrow[d] & E[1] \arrow[r] \arrow[d, "\alpha[1]"] & C'[1] \arrow[d, equal] \\
C' \arrow[r] & A[1] \arrow[r, "-f'[1]"] \arrow[d, "f[1]"] & B'[1] \arrow[r] \arrow[d, "-\beta[1]"] \arrow[dr, phantom, "\mathrm{I}"] & C'[1] \arrow[d] \\
 & B[1] \arrow[r, equal] \arrow[d] & B[1] \arrow[r] \arrow[d] & C[1] \\
 & C[1] \arrow[r] & E[2] &
\end{tikzcd}$$

TR4		base change II.
r_1	$\longrightarrow$	C_3
r_2		r_2
C_2		r_3
C_3		C_2 (符号法则).

$\square$.

base change II $\Longrightarrow$ base change I obviously.

(v) (cobase change I) $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ $\alpha: A \to A'$ in $\mathcal{C}$

$$\begin{tikzcd}
 & F \arrow[r, equal] \arrow[d, "\eta"] & F \arrow[d, "\varepsilon"] & \\
C[-1] \arrow[r, "h[-1]"] \arrow[d, equal] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[d, equal] \\
C[-1] \arrow[r, "-h'[-1]"] & A' \arrow[r, "f'"] \arrow[d, equal] & B' \arrow[r, "g'"] \arrow[d, "\delta"] & C \arrow[d, "-h"] \\
 & F[1] \arrow[r, equal] & F[1] \arrow[r] & A[1]
\end{tikzcd}$$

r_2, r_3 C_2, C_3. d.o.

(vi) (cobase change II) $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ $\alpha: A \to A'$ in $\mathcal{C}$.

2 given d.o

$$F \xrightarrow{\eta} A \xrightarrow{\alpha} A' \xrightarrow{r} F[1]$$

$$A' \xrightarrow{f'} B' \xrightarrow{g'} C' \xrightarrow{\alpha[1]h} A'[1]$$

$$\begin{tikzcd}
C \arrow[r, "h"] & A[1] \arrow[d, "\alpha[1]"] \\
 & A'[1]
\end{tikzcd}$$

$\exists$.

$F = F$

$A \to B \to C \to A[1]$ $\quad C \to A[1]$

$\downarrow \alpha[1]$

$A'[1]$

∃.

$$\begin{tikzcd}
 & F \arrow[r, equal] \arrow[d, "\eta"] & F \arrow[d, "\varepsilon"] & \\
C[-1] \arrow[r, "h[-1]"] \arrow[d, equal] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[d, equal] \\
C[-1] \arrow[r, "\alpha h[-1]"] & A' \arrow[r, "f'"] \arrow[d, equal] & B' \arrow[r, "g'"] \arrow[d, "\delta"] & C \arrow[d, "-h"] \\
 & F[1] \arrow[r, equal] & F[1] \arrow[r] & A[1]
\end{tikzcd}$$

base change 拉回公理

cobase change 推出.

2.2.

$\mathcal{A}$ additive cat. $\quad \underline{u}: X \to Y$

Cone(u)

$(\mathrm{Cone}(u))^n := X^{n+1} \oplus Y^n \quad \forall n \in \mathbb{Z}.$

diff. $\begin{pmatrix} -d_X^{n+1} & 0 \\ \underline{u}^{n+1} & d_Y^n \end{pmatrix} : X^{n+1} \oplus Y^n \longrightarrow X^{n+2} \oplus Y^{n+1}.$

$$X \xrightarrow{u} Y \xrightarrow{\binom{0}{1}} \mathrm{Cone}(u) \xrightarrow{(1\ 0)} X[1] \qquad (2.3)$$

$\mathcal{E}$: 是$K(\mathcal{A})$与映射锥诱导的三角(2.3)同构的三角作成的类

Ω: 所有映射锥诱导的三角(2.3)作成的类.

$$\begin{tikzcd}
X' \arrow[r, "u'"] \arrow[d, "f"'] & Y' \arrow[r, "v'"] \arrow[d, "g"] & Z' \arrow[r, "w'"] \arrow[d, "h"] & X'[1] \arrow[d, "f[1]"] \\
X \arrow[r, "u"] & Y \arrow[r, "\binom{0}{1}"] & \mathrm{Cone}(u) \arrow[r, "(1\ 0)"] & X[1]
\end{tikzcd} \qquad (\in \mathcal{E}!)$$

$[1]$ 向左平移一步

$X: \quad \cdots \to X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \to X^n \xrightarrow{d^n} X^{n+1} \to \cdots$

$X[1]: \cdots \to X^1 \xrightarrow{-d^1} X^2 \to \cdots \xrightarrow{-d^n} X^{n+1} \xrightarrow{-d^{n+1}} X^{n+2}$

$f: X \to Y \qquad f[1]: X[1] \to Y[1]. \qquad (f[1])^n = f^{n+1}$

Next we show $(K(\mathcal{A}), [1], \underline{\mathcal{E}})$ $\triangle$. cat (by lemma 1.4.4)

Lemma 2.2.1

$$\begin{tikzcd}
X \arrow[r, "id_X"] \arrow[d, "id_X"'] \arrow[dr, phantom, "\wr"] & X \arrow[r, "0"] \arrow[d, "id_X"] \arrow[dr, phantom, "?"] & 0 \arrow[rr, "0"] \arrow[d, "\binom{0}{0}"] \arrow[drr, phantom, "\wr"] & & X[1] \arrow[d, "id_{X[1]}"] \\
X \arrow[r, "id_X"] & X \arrow[r, "\binom{0}{1}"] & \mathrm{Cone}(id_X) = X[1] \oplus X \arrow[rr, "(1\ 0)"] & & X[1]
\end{tikzcd} \quad \in \mathcal{E}$$

tr1 ✓

$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ 同伦等价

(逆 $(1\ 0)$)

Lemma 2.2.2 $\forall u: X \to Y$

we have $Y \xrightarrow{\binom{0}{1}} \mathrm{Cone}(u) = X[1] \oplus Y \xrightarrow{(1\ 0)} X[1] \xrightarrow{-u[1]} Y[1] \quad \in \mathcal{E}$ $\quad \begin{pmatrix} u[1] & 0 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

we have

$$\begin{array}{ccccccc}
Y & \xrightarrow{\binom{0}{1}} & \mathrm{Cone}(u)=X[1]\oplus Y & \xrightarrow{(1\ 0)} & X[1] & \xrightarrow{-u[1]} & Y[1] \quad \in \mathcal{E} \\
\| & \circlearrowright & \| & & \downarrow \begin{pmatrix} -u[1] \\ 1 \\ 0 \end{pmatrix} & \circlearrowright & \| \\
Y & \xrightarrow{\binom{0}{1}} & \mathrm{Cone}(u)=X[1]\oplus Y & \xrightarrow{\begin{pmatrix} 0&0\\ 1&0 \\ 0&1\end{pmatrix}} & \mathrm{Cone}\binom{0}{1} & \xrightarrow{(1\ 0\ 0)} & Y[1] \\
 & & & & =Y[1]\oplus\mathrm{Cone}(u) & &
\end{array}$$

$$\begin{pmatrix} u[1] & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 0&0\\0&0\\0&0 \end{pmatrix}$$

$\begin{pmatrix} -u[1] \\ 1 \\ 0 \end{pmatrix}$ 同伦等价 $\downarrow$ 序 $(0\ 1\ 0)$

tr 2

Lemma. 2.2.3 $\begin{array}{ccc} X & \xrightarrow{u} & Y \\ f\downarrow & & \downarrow g \\ X' & \xrightarrow{u'} & Y' \end{array}$ $\exists\, h: \mathrm{Cone}(u)=X[1]\oplus Y \longrightarrow \mathrm{Cone}(u')=X'[1]\oplus Y'$

s.t. comm. diag.

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{\binom{0}{1}} & \mathrm{Cone}(u) & \xrightarrow{(1\ 0)} & X[1] \\
f\downarrow & & \downarrow g & & \downarrow h & & \downarrow f[1] \\
X' & \xrightarrow{u'} & Y' & \longrightarrow & \mathrm{Cone}(u') & \longrightarrow & X'[1]
\end{array}$$

pf: let $h=\begin{pmatrix} f[1] & 0 \\ 0 & g \end{pmatrix}$.

tr3.

(Lemma 2.2.4) 设 $u: X\to Y$, $v: Y\to Z$ 则下图是链映射图，且它是 $\mathcal{E}$ 中的 $\triangle$。

$$\begin{array}{ccc}
 & Y[1]\oplus Z & \\
 & \| & \\
 & \mathrm{Cone}(v) & \\
\begin{pmatrix} 0&0\\1&0\end{pmatrix} \swarrow & & \nwarrow \begin{pmatrix} u[1] & 0 \\ 0 & 1 \end{pmatrix} \\
\mathrm{Cone}(u) & \xrightarrow{V=\begin{pmatrix} 1&0\\0&v\end{pmatrix}} & \mathrm{Cone}(vu) \\
\| & & \| \\
X[1]\oplus Y & & X[1]\oplus Z.
\end{array}$$

pf:

$$\begin{array}{ccc}
X^{n+1}\oplus Y^{n} & \xrightarrow{\begin{pmatrix} -d_X^{n+1} & 0 \\ u^{n+1} & d_Y^n \end{pmatrix}} & X^{n+2}\oplus Y^{n+1} \\
\begin{pmatrix} 1&0\\0&v^n\end{pmatrix}\downarrow & & \downarrow \begin{pmatrix} 1&0\\0&v^{n+1}\end{pmatrix} \\
X^{n+1}\oplus Z^{n} & \xrightarrow{\begin{pmatrix} -d_X^{n+1} & 0 \\ v^{n+1}u^{n+1} & d_Z^n \end{pmatrix}} & X^{n+2}\oplus Z^{n+1}
\end{array}$$

$$\begin{pmatrix} 1&0\\0&v^{n+1}\end{pmatrix}\begin{pmatrix} -d_X^{n+1} & 0 \\ u^{n+1} & d_Y^n \end{pmatrix} = \begin{pmatrix} -d_X^{n+1} & 0 \\ v^{n+1}u^{n+1} & \underline{v^{n+1}d_Y^n} \end{pmatrix}$$

$$\begin{pmatrix} -d_X^{n+1} & 0 \\ v^{n+1}u^{n+1} & d_Z^n \end{pmatrix}\begin{pmatrix} 1&0\\0&v^{n}\end{pmatrix} = \begin{pmatrix} -d_X^{n+1} & 0 \\ v^{n+1}u^{n+1} & \underline{d_Z^n v^n} \end{pmatrix}$$

$\Leftarrow d_Y^n$

$$\begin{array}{ccc}
Y^n & \xrightarrow{v^n} & Z^n \\
\downarrow & \circlearrowright & \downarrow d_Z^n \\
Y^{n+1} & \xrightarrow{v^{n+1}} & Z^{n+1}
\end{array}$$

$\therefore \begin{pmatrix} 1&0\\0&v\end{pmatrix}$ 链映射。

$v(1\ 0)$ $\quad (u[1]\ 0)$ $\quad (0\ 0) \longrightarrow$ 链映射

$\begin{pmatrix} & \\ 0 & v \end{pmatrix}$ 证 ...

$$\begin{tikzcd}
\mathrm{Cone}(u) \arrow[r, "V=\begin{pmatrix}1&0\\0&v\end{pmatrix}"] \arrow[d, equal] & \mathrm{Cone}(vu) \arrow[r, "\begin{pmatrix}u[1]&0\\0&1\end{pmatrix}"] \arrow[d, equal] & \mathrm{Cone}(v) \arrow[r, "\begin{pmatrix}0&0\\1&0\end{pmatrix}"] \arrow[d, "\begin{pmatrix}0&0\\1&0\\0&0\\0&1\end{pmatrix}"] & \mathrm{Cone}\,u[1] \arrow[d, equal] \\
\mathrm{Cone}(u) \arrow[r, "V=\begin{pmatrix}1&0\\0&v\end{pmatrix}"'] & \mathrm{Cone}(vu) \arrow[r, "\begin{pmatrix}0&0\\0&0\\1&0\\0&1\end{pmatrix}"'] & \mathrm{Cone}(V) \arrow[r, "\begin{pmatrix}1&0&0&0\\0&1&0&0\end{pmatrix}"'] & \mathrm{Cone}\,u[1]
\end{tikzcd}$$

链映射

$$\begin{pmatrix}0&0\\1&0\\0&0\\0&1\end{pmatrix}\begin{pmatrix}u[1]&0\\0&1\end{pmatrix} \sim \begin{pmatrix}0&0\\0&0\\1&0\\0&1\end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix}0&0\\u[1]&0\\-1&0\\0&0\end{pmatrix} \sim 0$$

$$\begin{tikzcd}
X^{n+1}\oplus Z^{n} \arrow[r, "\begin{pmatrix}-d_X^{n+1}&0\\v^{n+1}u^{n+1}&d_Z^{n}\end{pmatrix}"] \arrow[d, "\begin{pmatrix}-1&0\\0&0\\0&0\\0&0\end{pmatrix}"'] \arrow[dr, "\begin{pmatrix}0&0\\u^{n+1}&0\\-1&0\\0&0\end{pmatrix}"] & X^{n+2}\oplus Z^{n+1} = (\mathrm{Cone}(vu))^{n+1} \arrow[d, "\begin{pmatrix}-1&0\\0&0\\0&0\\0&0\end{pmatrix}"] \\
X^{n+1}\oplus Y^{n}\oplus X^{n}\oplus Z^{n-1} \arrow[r, "\begin{pmatrix}d_X^{n+1}&0&0&0\\-u^{n+1}&d_Y^{n}&0&0\\1&0&-d_X^{n}&0\\0&v^{n}&v^nu^n&d_Z^{n-1}\end{pmatrix}"'] & X^{n+2}\oplus Y^{n+1}\oplus X^{n+1}\oplus Z^{n}
\end{tikzcd}$$

$= (\mathrm{Cone}\,V)^{n+1}$

$$\begin{pmatrix}-1&0\\0&0\\0&0\\0&0\end{pmatrix}\begin{pmatrix}-d_X^{n+1}&0\\v^{n+1}u^{n+1}&d_Z^{n}\end{pmatrix} + \begin{pmatrix}d_X^{n+1}&0&0&0\\-u^{n+1}&d_Y^{n}&0&0\\1&0&-d_X^{n}&0\\0&v^{n}&v^nu^n&d_Z^{n-1}\end{pmatrix}\begin{pmatrix}-1&0\\0&0\\0&0\\0&0\end{pmatrix}$$

$$=\begin{pmatrix}0&0\\u^{n+1}&0\\-1&0\\0&0\end{pmatrix}$$

再证 $\begin{pmatrix}0&0\\1&0\\0&0\\0&1\end{pmatrix}$ 是同伦等价

$$\begin{tikzcd}
\mathrm{Cone}(v) \arrow[r, shift left, "\begin{pmatrix}0&0\\1&0\\0&0\\0&1\end{pmatrix}"] & \mathrm{Cone}(V) \arrow[l, shift left, "\begin{pmatrix}0&1&u&0\\0&0&0&1\end{pmatrix}"]
\end{tikzcd}$$

由 $(0\ 1\ u\ 0)$ 链映射.

$$\Phi\begin{pmatrix} 1 & 0 & 0 & \\ 0 & 1 & u & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ 链映射.

$$\begin{pmatrix} 0 & 1 & u & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & u & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & u & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

即证 $\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim 0$ $\qquad \mathrm{Cone}(V) \to \mathrm{Cone}(V)$

$$\begin{tikzcd}
& (\mathrm{Cone}(V))^{n} \arrow[r, "{\begin{pmatrix} dx & 0 & 0 & 0 \\ -u & -dy & 0 & 0 \\ 1 & 0 & -dx & 0 \\ 0 & v & -vu & dz \end{pmatrix}}"] \arrow[dl, "s"'] \arrow[d, "{\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}"] & (\mathrm{Cone}(V))^{n+1} \arrow[dl, "{\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = s}"] \\
(\mathrm{Cone}(V))^{n-1} \arrow[r, "{\begin{pmatrix} \ \end{pmatrix}_1}"'] & (\mathrm{Cone}(V))^{n} &
\end{tikzcd}$$

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} dx & 0 & 0 & 0 \\ -u & -dy & 0 & 0 \\ 1 & 0 & -dx & 0 \\ 0 & v & -vu & dz \end{pmatrix} + \begin{pmatrix} \ \end{pmatrix}_1 s = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

囗.

(Lemma 2, 2.5) $\qquad$ <u>tr4</u> $\qquad$ d.o.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}"] \arrow[d, "v"'] & \mathrm{Cone}(u) \arrow[r] \arrow[d, "{\begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}}"] & X[1] \arrow[d, equal] \\
X \arrow[r, "vu"] & Z \arrow[r, "{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}"] \arrow[d, "{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}"'] & \mathrm{Cone}(vu) \arrow[r, "{(1\ 0)}"] \arrow[d, "{\begin{pmatrix} u[1] & 0 \\ 0 & 1 \end{pmatrix}}"] & X[1] \arrow[d, "{u[1]}"] \\
& \mathrm{Cone}(v) \arrow[r, equal] \arrow[d, "{(1\ 0)}"'] & \mathrm{Cone}(v) \arrow[r, "{(1\ 0)}"] \arrow[d, "{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}"] & Y[1] \\
& Y[1] \arrow[r, "{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}"] & \mathrm{Cone}(u)[1] &
\end{tikzcd}$$

$r_1, r_2, C_2 \in \Omega$.

$\Rightarrow C_3 \in \mathcal{E}$ $\Longrightarrow$ tr4.

$\therefore (k(\mathcal{A}), [1], \mathcal{E})$: Δ. cat.

2.3 作为同伦核的映射筒.

$u: X \to Y$

$$X \xrightarrow{u} Y \xrightarrow{\binom{0}{1}} \mathrm{Cone}(u) \xrightarrow{(1\ 0)} X[1]$$

$k(\mathcal{A})$ 三角核 $\Rightarrow$ 同伦核

…余核 $\Rightarrow$ ‥余核

u 的同伦核 $(-1\ 0): \mathrm{Cone}(u)[-1] \to X$

define: u 的映射筒 $\mathrm{Cyl}(u)$: $(-1\ 0)$ 的映射锥 $\mathrm{Cone}(-1\ 0)$

$$(\mathrm{Cyl}(u))^n := \underline{X^{n+1} \oplus Y^n} \oplus X^n.$$

$$\begin{pmatrix} -d_X^{n+1} & 0 & 0 \\ u^{n+1} & d_Y^n & 0 \\ -\mathrm{Id}_{X^{n+1}} & 0 & d_X^n \end{pmatrix} : X^{n+1} \oplus Y^n \oplus X^n \longrightarrow X^{n+2} \oplus Y^{n+1} \oplus X^{n+1}$$

d.Δ: $\mathrm{Cone}(u)[-1] \xrightarrow{(-1\ 0)} X \xrightarrow{\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)} \mathrm{Cyl}(u) \xrightarrow{\left(\begin{smallmatrix}1&0&0\\0&1&0\end{smallmatrix}\right)} \mathrm{Cone}(u)$

$\circlearrowright$ $\underline{X \xrightarrow{\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)} \mathrm{Cyl}(u) \xrightarrow{\left(\begin{smallmatrix}1&0&0\\0&1&0\end{smallmatrix}\right)} \mathrm{Cone}(u) \xrightarrow{(1\ 0)} X[1]}$ (2.5节中:前3项链可列短正合列)

(2.6)

由映射筒诱导的 d.Δ.

$$X \xrightarrow{u} Y \xrightarrow{\binom{0}{1}} \mathrm{Cone}(u) \xrightarrow{(1\ 0)} X[1]$$

$X \xrightarrow{u} Y$ $\qquad X^{n+1} \oplus Y^n$

Lemma 2.3.1 $\mathcal{A}$ 是加法 cat. $u: X \to Y$, then

$$X \xrightarrow{\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)} \mathrm{Cyl}(u) \xrightarrow{\left(\begin{smallmatrix}1&0&0\\0&1&0\end{smallmatrix}\right)} \mathrm{Cone}(u) \xrightarrow{(1\ 0)} X[1] \quad \text{d.}\Delta.$$

有 同伦交换图.

$$
\begin{tikzcd}
X \arrow[r, "\begin{pmatrix}0\\0\\1\end{pmatrix}"] \arrow[d, equal, "②"'] & Cyl(u) \arrow[r, "\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}"] \arrow[d, shift left, "① \begin{pmatrix}0&1&u\end{pmatrix}"] & Cone(u) \arrow[r, "\begin{pmatrix}1&0\end{pmatrix}"] \arrow[d, equal] & X[1] \arrow[d, equal] \\
X \arrow[r, "u"] & Y \arrow[r, "\begin{pmatrix}0\\1\end{pmatrix}"] \arrow[u, shift left, "② \begin{pmatrix}0\\1\\0\end{pmatrix}"] & Cone(u) \arrow[r, "\begin{pmatrix}1&0\end{pmatrix}"] & X[1]
\end{tikzcd}
$$

pf: $\begin{pmatrix}0\\1\\0\end{pmatrix}$ $\begin{pmatrix}0&1&u\end{pmatrix}$ 链映射.

① 一, 三 可交换

$$\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix} \sim \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0&1&u\end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix}1&0&0\\0&0&-u\end{pmatrix} \sim 0. \quad : Cyl(u) \longrightarrow Cone(u)$$

$$
\begin{tikzcd}
 & X^{n+1}\oplus Y^n \oplus X^n \arrow[r, "\begin{pmatrix}-dx&0&0\\u&dy&0\\-1&0&dx\end{pmatrix}"] \arrow[dl, "\begin{pmatrix}0&0&-1\\0&0&0\end{pmatrix}"'] \arrow[d, "\begin{pmatrix}1&0&0\\0&0&-u\end{pmatrix}"] & X^{n+2}\oplus Y^{n+1}\oplus X^{n+1} \arrow[dl, "\begin{pmatrix}0&0&-1\\0&0&0\end{pmatrix}"] \\
X^n \oplus Y^{n+1} \arrow[r, "\begin{pmatrix}-dx&0\\u&dy\end{pmatrix}"] & X^{n+1}\oplus Y^{n+2} &
\end{tikzcd}
$$

$$\begin{pmatrix}0&0&-1\\0&0&0\end{pmatrix}\begin{pmatrix}-dx&0&0\\u&dy&0\\-1&0&dx\end{pmatrix} + \begin{pmatrix}-dx&0\\u&dy\end{pmatrix}\begin{pmatrix}0&0&-1\\0&0&0\end{pmatrix} = \begin{pmatrix}1&0&0\\0&0&-u\end{pmatrix}$$

<u>下证 $(0\ 1\ u)$ 同伦等价</u> $\quad \begin{pmatrix}0&1&u\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = 1$

$$\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0&1&u\end{pmatrix} = \begin{pmatrix}0&0&0\\0&1&u\\0&0&0\end{pmatrix} \sim \begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix}-1&0&0\\0&0&u\\0&0&-1\end{pmatrix} \sim 0: Cyl(u) \longrightarrow Cyl(u)$$

$$S = \begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}$$

(Prop 2.3.2) $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ exact in $\mathcal{A}$ (Abel)

then <u>拟同构</u> $(0\ v) : \mathrm{Cone}(u) \to \mathrm{Coker}(u) = Z$.

($f: X \to Y$ $H^n(f) : H^n(X) \cong H^n(Y)$, 称 f 拟同构)

和复形短正合列的交换图.

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\begin{pmatrix}0\\0\\1\end{pmatrix}"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Cyl}(u) \arrow[r, "\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}"] \arrow[d, "(0\ 1\ u)"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Cone}(u) \arrow[r] \arrow[d, "(0\ v)"] & 0 \\
0 \arrow[r] & X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r] & 0
\end{tikzcd}$$

$(0\ 1\ u)\begin{pmatrix}0\\0\\1\end{pmatrix} = u$

$(0\ v)\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix} = (0\ v\ 0)$

其中 $(0\ 1\ u)$ 是同伦等价，其逆为 $\begin{pmatrix}0\\1\\0\end{pmatrix}$，而且上行是可裂短正合列

pf: by lemma 2.3.1 $(0, 1, u)$ 是同伦等价 $\Longrightarrow$ 拟同构.

$$\begin{tikzcd}
\arrow[r] & H^n(X) \arrow[r] \arrow[d, equal] & H^n(\mathrm{Cyl}(u)) \arrow[r] \arrow[d, "\cong"] & H^n(\mathrm{Cone}(u)) \arrow[r] \arrow[d] & H^{n+1}(X) \arrow[r] \arrow[d, equal] & H^{n+1}(\mathrm{Cyl}(u)) \arrow[r] \arrow[d, equal] & \cdots \\
\arrow[r] & H^n(X) \arrow[r] & H^n(Y) \arrow[r] & H^n(Z) \arrow[r] & H^{n+1}(X) \arrow[r] & H^{n+1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

$H^n(\mathrm{Cone}(u)) \cong H^n(Z)$

$(0, v)$ 拟同构. $\square$