$\mathcal{A}$: abelian cat

Recall : mapping cylinder :

$$X \xrightarrow{u} Y \xrightarrow{\binom{0}{1}} \mathrm{Cone}(u) \xrightarrow{(1\ 0)} X[1]$$

$$\circlearrowleft \quad \underbrace{\mathrm{Cone}(u)[-1] \xrightarrow{(-1\ 0)} X} \longrightarrow \underbrace{\mathrm{Cone}((-1\ 0))}_{\substack{\| \\ \mathrm{Cyl}(u)}} \longrightarrow \mathrm{Cone}(u)$$

Explicitly : $\mathrm{Cyl}(u)^n = X^{n+1} \oplus Y^n \oplus X^n$, $d^n = \begin{pmatrix} d_X^{n+1} & & \\ u^{n+1} & d_Y^n & \\ -1 & & d_X^n \end{pmatrix}$

$\rightsquigarrow \quad 0 \to X \to \mathrm{Cyl}(u) \to \mathrm{Cone}(u) \to 0$ exact in $C(\mathcal{A})$.

Lem 2.3.1

$$\begin{tikzcd}
\underbrace{X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r] \arrow[d, "\simeq"] & \mathrm{Cone}(u)}_{\text{not so good}} \arrow[r] \arrow[d, "\simeq"] & X[1] \arrow[d, equal] \\
\boxed{X \arrow[r] & \mathrm{Cyl}(u) \arrow[r] & \mathrm{Cone}(u)} \arrow[r] & X[1]
\end{tikzcd} \quad \text{in } K(\mathcal{A})$$

s.e.s in $C(\mathcal{A})$.

In addition, u : mono

Prop 2.3.2.

$$X \xrightarrow{u} Y \quad \begin{tikzcd}
\underbrace{X \arrow[r] \arrow[d, equal] & \mathrm{Cyl}(u) \arrow[r] \arrow[d, "(0\ 1\ u)", "\simeq"'] & \mathrm{Cone}(u)} \arrow[r] \arrow[d, "(0\ v)"] & X[1] \quad \text{in } K(\mathcal{A}) \\
0 \to X \arrow[r, "u"] & Y \arrow[r] & Z \arrow[r] & 0. \quad \text{in } C(\mathcal{A})
\end{tikzcd}$$

quasi-isom in $C(\mathcal{A})$

$$\begin{tikzcd}
0 \arrow[r] & X^n \arrow[r, "u^n"] \arrow[d] & Y^n \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Coker}\, u^n \arrow[r] \arrow[d, dashed, "d_Z^n"] & 0 \\
0 \arrow[r] & X^{n+1} \arrow[r, "u^{n+1}"] & Y^{n+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Coker}\, u^{n+1} \arrow[r] \arrow[d, "d_Z^{n+1}"] & 0 \\
 & & Y^{n+2} \arrow[r] & \mathrm{Coker}\, u^{n+2} \arrow[r] & 0
\end{tikzcd}$$

$\mathrm{Him}(u)$,

Def : $X \xrightarrow{u} Y$

$\rightsquigarrow$

$$X \xrightarrow{u} \underbrace{Y \xrightarrow{\binom{0}{1}} \mathrm{Cone}(u)}_{\mathrm{Coker}\, u} \xrightarrow{(1\ 0)} X[1]$$

$\rightsquigarrow$

$$Y \xrightarrow{\binom{0}{1}} \mathrm{Cone}(u) \xrightarrow{\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}} \mathrm{Cone}(\tbinom{0}{1}) \xrightarrow{(1\ 0\ 0)} Y[1]$$

$$\circlearrowleft \quad \underbrace{\mathrm{Cone}(\tbinom{0}{1})[-1]}_{\substack{\| \\ \mathrm{Him}(u)}} \xrightarrow{(-1\ 0\ 0)} \underbrace{Y \xrightarrow{\binom{0}{1}} \mathrm{Cone}(u)}_{\mathrm{Coker}\, u} \xrightarrow{\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}} \mathrm{Cone}(\tbinom{0}{1})$$

Explicitly : $\mathrm{Him}(u)^n = Y^n \oplus X^n \oplus Y^{n-1}$ $\quad d = \begin{pmatrix} d_Y^n & & \\ & d_X^n & \\ -1 & -u^n & -d_Y^{n-1} \end{pmatrix}$.

$\longrightarrow$ D.T. $\mathrm{Cone}(u)[-1] \xrightarrow{\begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}} \mathrm{Him}(u) \xrightarrow{(-1\ 0\ 0)} Y \xrightarrow{\binom{0}{1}} \mathrm{Cone}(u)$

$\xrightarrow{\text{sign}}$ D.T. $\underbrace{\mathrm{Cone}(u)[-1] \xrightarrow{\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}} \mathrm{Him}(u) \xrightarrow{(1\ 0\ 0)} Y}_{\text{s.e.s in } C(\mathcal{A})} \xrightarrow{\binom{0}{1}} \mathrm{Cone}(u)$. induced by Him.

$Y \xrightarrow{v} Z$

Lem 2.3.3

$$\begin{tikzcd}
\underbrace{\mathrm{Cone}(v)[-1] \arrow[r] \arrow[d, equal] & \mathrm{Him}(v) \arrow[r] \arrow[d, shift left, "(0\ 1\ 0)"] & Z} \arrow[d, equal] \arrow[r] & \mathrm{Cone}(v) \arrow[d, equal] \\
 & \arrow[u, shift left, "\begin{pmatrix} -v \\ -1 \\ 0 \end{pmatrix}"] & &
\end{tikzcd}$$

Lem 2.3.3

$$\begin{tikzcd}
\mathrm{Cone}(v)[-1] \arrow[r] \arrow[d, equal] & \mathrm{Cyl}(v) \arrow[r] \arrow[d, shift left, "(0\ 1\ v)"] & Z \arrow[r] \arrow[d, equal] & \mathrm{Cone}(v) \arrow[d, equal] \\
\mathrm{Cone}(v)[-1] \arrow[r] & Y \arrow[r] \arrow[u, shift left, "\begin{pmatrix} -v \\ -1 \\ 0 \end{pmatrix}"] & Z \arrow[r] & \mathrm{Cone}(v)
\end{tikzcd}$$

Prop 2.3.4. In addition v : epi

$Y \xrightarrow{v} Z$

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, "\begin{pmatrix} u \\ 0 \end{pmatrix}"] & Y \arrow[r] \arrow[d, "\begin{pmatrix} -v \\ -1 \\ 0 \end{pmatrix}"] & Z \arrow[r] \arrow[d, equal] & 0 & \text{in } C(\mathcal{A}) \\
\mathrm{Cone}(v)[-1] \arrow[r] & \mathrm{Cyl}(v) \arrow[r] & Z \arrow[r] & \mathrm{Cone}(v) & & \text{in } K(\mathcal{A})
\end{tikzcd}$$

quasi-isom in $C(\mathcal{A})$

§2.4. Fundamental thm.

Case of $C(\mathcal{A})$

By thm 2.1.1 : $H^n : C(\mathcal{A}) \to \mathcal{A}$ functor,

$0 \to X \to Y \to Z \to 0 \rightsquigarrow H^n(X) \to H^n(Y) \to H^n(Z) \to H^{n+1}(X)$.

Case of $K(\mathcal{A})$.

$\mathrm{obj}\, K(\mathcal{A}) = \mathrm{obj}\, C(\mathcal{A})$ $\quad H^n : \mathrm{obj}\, K(\mathcal{A}) \to \mathrm{obj}\, \mathcal{A}$.

$\mathrm{Hom}_{K(\mathcal{A})}(X, Y) := \mathrm{Hom}(X, Y)/\mathrm{Htp}(X, Y)$

$H^n : \mathrm{Hom}_{K(\mathcal{A})}(X, Y) \to \mathcal{A}$ well-defined

$\quad \bar{f} \mapsto H^n(f)$. $\quad$ (By prop 2.1.2. $f \sim g \Rightarrow H^n(f) = H^n(g)$)

$H^n(\bar{f} \circ \bar{g}) = H^n((f + \underset{\substack{\downarrow \\ \mathrm{Htp}(-,-)}}{H}) \circ (g + H)) = H^n(f \circ g + \underbrace{f \circ H + H \circ g + H \circ H}_{\in H}) = H^n(f \circ g)$

$= H^n(f) \circ H^n(g)$

$= H^n(\bar{f}) \circ H^n(\bar{g})$.

$\Rightarrow H^n : K(\mathcal{A}) \to \mathcal{A}$ functor.

$\forall$ D.T. $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d] & Y \arrow[r, "v"] \arrow[d] & Z \arrow[r, "w"] \arrow[d] & X[1] \arrow[d] & \simeq \text{ in } K(\mathcal{A}) \\
X \arrow[r] & \mathrm{Cyl}(u') \arrow[r] & \mathrm{Cone}(u') \arrow[r] & X[1]
\end{tikzcd}$$

$\Big\downarrow H^n$

$$\begin{tikzcd}
H^n(X) \arrow[r] \arrow[d] & H^n(Y) \arrow[r] \arrow[d] & H^n(Z) \arrow[d] & \simeq \text{ in } \mathcal{A}. \\
H^n(X) \arrow[r] & H^n(\mathrm{Cyl}(u')) \arrow[r] & H^n(\mathrm{Cone}(u'))
\end{tikzcd}$$

exact in $\mathcal{A}$

$\longrightarrow$ $H^n(X) \xrightarrow{H^n(u)} H^n(Y) \xrightarrow{H^n(v)} H^n(Z)$ is exact in $\mathcal{A}$.

Similar : $H^{n-1}(Z) \to H^n(X) \xrightarrow{H^n(u)} H^n(Y)$

$H^n(Y) \xrightarrow{H^n(v)} H^n(Z) \to H^{n+1}(X)$

$\rightsquigarrow \; \to H^n(X) \to H^n(Y) \to H^n(Z) \to H^{n+1}(X) \to \cdots$

Thm 2.4.1

(1) $X \to Y \to Z \to X[1]$ D.T. in $K(\mathcal{A})$.

$\to$ l.e.s. $H^n(X) \to H^n(Y) \to H^n(Z) \to H^{n+1}(X) \to \cdots$

(2)
$$\begin{array}{ccccccc} X & \to & Y & \to & Z & \xrightarrow{w} & X[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X' & \to & Y' & \to & Z' & \xrightarrow{w'} & X'[1] \end{array}$$

$\downarrow H^n$

$$\begin{array}{ccccccccccc} \cdots \to & H^n(X) & \to & H^n(Y) & \to & H^n(Z) & \xrightarrow{H^n(w)} & H^{n+1}(X) & \to & \cdots \\ & \downarrow & \supseteq & \downarrow & \supseteq & \downarrow & \supseteq & \downarrow & & \\ \to & H^n(X') & \to & H^n(Y') & \to & H^n(Z') & \xrightarrow{H^n(w')} & H^{n+1}(X') & \to & \cdots \end{array}$$

(3) $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ D.T.

u is a quasi-isom iff Z is acyclic.

In particular, $X \xrightarrow{u} Y \to \mathrm{Cone}(u) \to X[1]$

u is a quasi-isom iff $\mathrm{Cone}(u)$ is acyclic.

$\langle \to \underset{0}{\underset{\|}{H^{n-1}(Z)}} \to \underline{H^n(X) \xrightarrow{H^n(u)} H^n(Y)} \to \underset{0}{\underset{\|}{H^n(Z)}} \to \cdots \rangle$

Rmk: $H^n(X) = H^0(X[n])$

$\Rightarrow$ Thm 2.4.1 (1) & (2) $\Leftrightarrow$

$$\begin{array}{ccccccc} X & \to & Y & \to & Z & \to & X[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X' & \to & Y' & \to & Z' & \to & X'[1] \end{array}$$

$\downarrow$

$H^0(X[n]) \to H^0(Y[n]) \to H^0(Z[n]) \to H^0(X[n+1])$

$\downarrow \qquad \downarrow$

$\Leftrightarrow$ H^0 is a cohom functor in the sense of Def 1.2.1 $\quad H^0(Z'[n]) \to H^0(X'[n+1])$.

Cor 2.4.2 $K^{-,b}(\mathcal{A})$ & $K^{+,b}(\mathcal{A})$ is tri-subcat of $K(\mathcal{A})$.

$K^{-,b}(\mathcal{A}) :=$ a full subcat of $K^-(\mathcal{A})$.

$\mathrm{obj}\, K^{-,b}(\mathcal{A}) := \{X \in K^-(\mathcal{A}) \mid H^n(X) = 0 \text{ a.e. } n \in \mathbb{Z}\}$.

(for only fin many n, $H^n \neq 0$)

Pf: only need to show

(i) $X' \simeq X$ in $K(\mathcal{A})$, $X \in K^{-,b}$, $\Rightarrow X' \in K^{-,b}$. ✓.

(ii) $X \in K^{-,b} \Rightarrow X[1], X[-1] \in K^{-,b}$. ✓

(iii) $X \to Y \to Z \to X[1]$, D.T. $X, Z \in K^{-,b}(\mathcal{A})$ ✓.

$\Rightarrow Y \in K^{-,b}(\mathcal{A})$.

$\langle H^n(X) \to H^n(Y) \to H^n(Z)$, exact.

$H^n(Y) \neq 0 \Rightarrow H^n(X) \neq 0$ or $H^n(Z) \neq 0 \rangle$. □

§2.5.

Def: (split s.e.s)

§2.5.

Def: (split s.e.s)

$X \xrightarrow{u} Y \xrightarrow{v} Z$ in $\mathcal{A}$ is called a split s.e.s, if

$$\exists \quad \begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\ X' \arrow[r, "\binom{1}{0}"] & X' \oplus Z' \arrow[r, "(0\ 1)"] & Z' \end{tikzcd} \quad \simeq \text{ in } \mathcal{A}.$$

or $0 \to X \xrightarrow{u} Y \underset{s}{\overset{v}{\rightleftarrows}} Z \to 0$ exact, & $\exists\, s : Z \to Y$ s.t. $vs = id_Z$.

If such a s exists,

$\overset{Hom(Y,-)}{\longrightarrow}$ $\quad 0 \to Hom(Y,X) \xrightarrow{u_*} Hom(Y,Y) \xrightarrow{v_*} Hom(Y,Z) \qquad id_Z$

$\exists!\, \pi \quad \mapsto \quad 1_Y - sv \quad \mapsto \quad v(1-sv) = v - \overset{\shortparallel}{(vs)}v$
$= 0.$

i.e. $\exists\, \pi : Y \to X$ s.t. $1_Y = u\pi + sv$.

$\overset{Hom(X,-)}{\longrightarrow}$ $\quad 0 \to Hom(X,X) \xrightarrow{u_* - u^*} Hom(Y,X)$

$\pi u \quad \mapsto \quad u(\pi u) = (u\pi)u$
$= (1_Y - sv)u$
$= u - s\underbrace{(vu)}_{=0}$
$= u.$

$id_X \quad \mapsto \quad u$

$\Rightarrow \pi u = id_X$.

$(u\pi)s = (1_Y - sv)s = s - s\underbrace{(vs)}_{id_Z} = 0$ & u - mono
$\Rightarrow \pi s = 0$.

i.e. $X \xrightarrow{u} Y \xrightarrow{v} Z$ split s.e.s $\Leftrightarrow$ $0 \to X \underset{\pi}{\overset{u}{\rightleftarrows}} Y \underset{s}{\overset{v}{\rightleftarrows}} Z \to 0$
& $\exists\, \pi, s$ s.t. $\boxed{\begin{array}{l} \pi u = id_X,\ vs = id_Z \\ 1_Y = u\pi + sv \\ \pi s = 0. \qquad (*) \end{array}}$

Def. (chain split s.e.s —(c.s.s.e.s.))

$C \xrightarrow{f} D \xrightarrow{g} E$ in $C(\mathcal{A})$ is called c.s.s.e.s., if

$\forall n \in \mathbb{Z}$, $C^n \xrightarrow{f^n} D^n \xrightarrow{g^n} E^n$ is a split s.e.s. in $\mathcal{A}$.

or $\forall n \in \mathbb{Z}$, $\exists\, \underline{s^n, \pi^n}$, $(*)$ holds.
not need to be chain maps.

Rmk: split s.e.s. in $C(\mathcal{A})$ = c.s.s.e.s in $C(\mathcal{A})$ + $d_D^n = d_C^n \oplus d_E^n$.
or s, π are chain maps.

Every c.s.s.e.s in $C(\mathcal{A})$ can induce a D.T. in $K(\mathcal{A})$.

Lem 2.5.1 ($X \to Y \to Z \dashrightarrow X[1]$).

$X \xrightarrow{u} Y \underset{s}{\overset{v}{\rightleftarrows}} Z$ c.s.s.e.s

(1) $s^n : Z^n \to Y^n$ s.t. $v^n s^n = id_{Z^n}$, $\forall n \in \mathbb{Z}$.

$X \xrightarrow{u} Y \underset{s}{\overset{v}{\rightleftarrows}} Z$ c.s.s.e.s

(1) $s^n : Z^n \to Y^n$ s.t. $v^n s^n = id_{Z^n}$, $\forall n \in \mathbb{Z}$.

$\Rightarrow$ chain map $\underline{h : Z \to X[1]}$ s.t.

$$\underline{u^{n+1} h^n = s^{n+1} d_Z^n - d_Y^n s^n : Z^n \to Y^{n+1},\ \forall n \in \mathbb{Z}.} \quad (1)$$

h is independent from s up to htp. i.e. $\exists s'^n$ - - -.
$k : Z \to X[1]$ s.t. (1) holds $\Rightarrow h \sim k$.

(2). If $\exists\ \tilde{\tau}^n$, $(*)$ holds, then $\quad$ (1) $\Rightarrow$ (2)

$$\underline{h^n v^n = d_X^n \tilde{\tau}^n - \tilde{\tau}^{n+1} d_Y^n : Y^n \to X^{n+1},} \quad \forall n \in \mathbb{Z} \quad (2)$$

$$\begin{tikzcd} X^n \arrow[r, "u^n"] \arrow[d] & Y^n \arrow[r, "v^n"] \arrow[d] \arrow[dl] & Z^n \arrow[d] \arrow[dl, "h^n"] \arrow[l, bend left, "s^n"] \\ X^{n+1} \arrow[r, "u^{n+1}"'] & Y^{n+1} \arrow[r, "v^{n+1}"'] & Z^{n+1} \end{tikzcd} \quad \begin{matrix}(4)\\(4)\end{matrix}$$

$$\begin{tikzcd} Z \arrow[r, "h"] & X[1] \\ Y \arrow[u] \arrow[r, "hv"] & X[1] \end{tikzcd}$$

$$\begin{tikzcd} Y^{n-1} \arrow[r] & Y^n \arrow[r, "d_Y^n"] \arrow[dl, "\tilde{\tau}^n"'] \arrow[d, "hv^n"] & Y^{n+1} \arrow[dl, "\tilde{\tau}^{n+1}"] \\ X^n \arrow[r, "-d_X^n"'] & X^{n+1} \arrow[r] & X^{n+2} \end{tikzcd}$$

(2) $\Leftrightarrow -hv \overset{\tilde{\tau}}{\sim} 0$

Pf: (1) (existence of h)

$\overset{\mathrm{Hom}(Z^n, -)}{\rightsquigarrow}$ $\quad 0 \to \mathrm{Hom}(Z^n, X^{n+1}) \xrightarrow{u_*^{n+1}} \mathrm{Hom}(Z^n, Y^{n+1}) \xrightarrow{v_*^{n+1}} \mathrm{Hom}(Z^n, Z^{n+1})$

$\exists\ h^n \quad \mapsto \quad s^{n+1} d_Z^n - d_Y^n s^n \mapsto v^{n+1}(s^{n+1} d_Z^n - d_Y^n s^n)$

$$= \underbrace{v^{n+1} s^{n+1}}_{1} d_Z^n - \underline{v^{n+1} d_Y^n} s^n$$

$$= d_Z^n - d_Z^n \underbrace{v^n s^n}_{1}$$

$$= 0.$$

i.e. $\exists h$ s.t. $\underline{u^{n+1} h^n = s^{n+1} d_Z^n - d_Y^n s^n}$. (1)

⟨ h is a chain map ⟩.

$$\Leftrightarrow \begin{tikzcd} Z^n \arrow[r, "d_Z^n"] \arrow[d, "h^n"'] \arrow[dr, phantom, "\circlearrowright"] & Z^{n+1} \arrow[d, "h^{n+1}"] \\ X^{n+1} \arrow[r, "-d_X^{n+1}"'] & X^{n+2} \end{tikzcd} \Leftrightarrow h^{n+1} d_Z^n + d_X^{n+1} h^n = 0.$$

$$u^{n+2}(h^{n+1} d_Z^n + d_X^{n+1} h^n) = (s^{n+2} \underline{d_Z^{n+1}} - d_Y^{n+1} s^{n+1}) \underline{d_Z^n} + d_Y^{n+1} \underline{u^{n+1} h^n}$$

$$= - d_Y^{n+1} s^{n+1} d_Z^n + \underline{d_Y^{n+1}}(s^{n+1} d_Z^n - \underline{d_Y^n} s^n)$$

$$= 0.$$

u^{n+2} is mono $\Rightarrow h^{n+1} d_Z^n + d_X^{n+1} h^n = 0$. i.e. h is chain map.

⟨uniqueness of h⟩.

$\overset{\mathrm{Hom}(Z^n, -)}{\longrightarrow}$ $\quad 0 \to \mathrm{Hom}(Z^n, X^n) \xrightarrow{u_*^n} \mathrm{Hom}(Z^n, Y^n) \xrightarrow{v_*^n} \mathrm{Hom}(Z^n, Z^n)$

$\exists\ f^n \quad \mapsto \quad s^n - s'^n \mapsto id_Z - id_Z = 0.$

i.e. $\exists f^n$ s.t. $\underline{s^n - s'^n = u^n f^n}$.

$u^{n+1} h^n = s^{n+1} d_Z^n - d_Y^n s^n$, $\quad$ (1)

$u^{n+1} k^n = s'^{n+1} d_Z^n - d_Y^n s'^n$, $\quad$ (1')

$\Rightarrow \cancel{u^{n+1}}(h^n - k^n) = u^{n+1} f^{n+1} d_Z^n - d_Y^n u^n f^n$

$$u^{n+1}k^n = s'^{n+1}d_Z^n - d_Y^n s'^n, \quad (1')$$

$$\Rightarrow \cancel{u^{n+1}}(h^n - k^n) = u^{n+1}f^{n+1}d_Z^n - d_Y^n \underline{u^n} f^n$$

$$= u^{n+1}f^{n+1}d_Z^n - u^{n+1}d_X^n f^n = \cancel{u^{n+1}}(f^{n+1}d_Z^n - d_X^n f^n)$$

$$\Rightarrow h \overset{f}{\sim} k. \quad \square$$

(2) $\cancel{u^{n+1}}(h^n v^n) = (s^{n+1}d_Z^n - d_Y^n s^n)v^n$

$= s^{n+1}\underline{v^{n+1}d_Y^n} - d_Y^n \underline{s^n v^n}$

$= (\cancel{1} - u^{n+1}\tilde{u}^{n+1})d_Y^n - d_Y^n(\cancel{1} - u^n\tilde{u}^n)$

$= d_Y^n u^n \tilde{u}^n - u^{n+1}\tilde{u}^{n+1}d_Y^n$

$= u^{n+1}d_X^n \tilde{u}^n - u^{n+1}\tilde{u}^{n+1}d_Y^n$

$= \cancel{u^{n+1}}(d_X^n \tilde{u}^n - \tilde{u}^{n+1}d_Y^n)$

$\Rightarrow$ (2). $\square$

Def 2.5.2. $X \underset{\tilde{u}}{\overset{u}{\rightleftarrows}} Y \underset{s}{\overset{v}{\rightleftarrows}} Z$ c.s.s.e.s. (*) holds.

$h: Z \to X[1]$ is called a homotopy invariant (htp invar) if (1) & (2) holds.

Rmk: Lem 2.5.1 $\Rightarrow$ existence of htp invar.

htp invar has some important properties in $C(\mathcal{A})$.

Prop: 2.5.3

(1) $X \underset{\tilde{u}}{\overset{u}{\rightleftarrows}} Y \underset{s}{\overset{v}{\rightleftarrows}} Z$ c.s.s.e.s. (*) holds.

h corresponding htp invar. Then:

$$\begin{tikzcd}
Z[-1] \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d, "\binom{v}{\tilde{u}}"] & Z \arrow[d, equal] \\
Z[-1] \arrow[r] & X \arrow[r, "\binom{0}{1}"] & \mathrm{Cone}(-h[-1]) \arrow[r, "(1\ 0)"] & Z
\end{tikzcd}$$

, $X \to Y \to Z \to Z[-1]$, in $C(\mathcal{A})$.

$Z^n \oplus X^n$, $d^n = \begin{pmatrix} d_Z^n & \\ -h^n & d_X^n \end{pmatrix}$

$\binom{v}{\tilde{u}}$ is a isom of cpx, with inverse (s, u).

In par, $\forall$ c.s.s.e.s. can be induced by a mapping cone in $C(\mathcal{A})$

(2). Condition in (1) holds, $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ is split $\Leftrightarrow h \sim 0$.

(3). $h: Z \to X[1]$ is the htp invar of

$$0 \to X \xrightarrow{\binom{0}{1}} \mathrm{Cone}(-h[-1]) \xrightarrow{(1\ 0)} Z \to 0$$

(4). $f: X \to Y$, $f \sim 0 \Leftrightarrow$

$$0 \to Y \xrightarrow{\binom{0}{1}} \mathrm{Cone}(f) \xrightarrow{(1\ 0)} X[1] \to 0 \text{ is split in } C(\mathcal{A}).$$

$f - h[1]$

$$0 \to Y \xrightarrow{\binom{0}{1}} \mathrm{Cone}(f) \xrightarrow{(1\ 0)} X[1] \to 0 \quad \text{is split in } C(\mathcal{A}).$$

$$f = -h[-1] \iff h = -f[1].$$

Pf: (1) comm is trivial.

< $\binom{\nu}{\bar\nu}$, $(s\ u)$ are chain maps >

For $\binom{\nu}{\bar\nu}$:

$$\begin{tikzcd} Y^n \arrow[r, "d_Y^n"] \arrow[d, "\binom{\nu^n}{\bar\nu^n}"'] & Y^{n+1} \arrow[d, "\binom{\nu^{n+1}}{\bar\nu^{n+1}}"] \\ Z^n \oplus X^n \arrow[r, "\begin{pmatrix} d_Z^n & \\ -h & d_X^n \end{pmatrix}"'] & Z^{n+1} \oplus X^{n+1} \end{tikzcd}$$

$$\begin{pmatrix} d_Z^n & \\ -h & d_X^n \end{pmatrix}\begin{pmatrix} \nu^n \\ \bar\nu^n \end{pmatrix} = \begin{pmatrix} d_Z^n \nu^n \\ -h^n\nu^n + d_X^n \bar\nu^n \end{pmatrix}$$

$$\begin{pmatrix} \nu^{n+1} \\ \bar\nu^{n+1} \end{pmatrix} d_Y^n = \begin{pmatrix} \nu^{n+1} d_Y^n \\ \underbrace{\bar\nu^{n+1} d_Y^n}_{(2)} \end{pmatrix} = \begin{pmatrix} d_Z^n \nu^n \\ -h^n\nu^n + d_X^n \bar\nu^n \end{pmatrix}$$

$\Rightarrow \binom{\nu}{\bar\nu}$ chain map.

Similar to $(s\ u)$.

< invertible >

$$(s^n\ u^n)\begin{pmatrix} \nu^n \\ \bar\nu^n \end{pmatrix} = s^n\nu^n + u^n\bar\nu^n = 1.$$

$$\begin{pmatrix} \nu^n \\ \bar\nu^n \end{pmatrix}(s^n\ u^n) = \begin{pmatrix} \nu^n s^n & \nu^n u^n \\ \bar\nu^n s^n & \bar\nu^n u^n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad \square$$

(2). $0 \to X \to Y \to Z \to 0$ split

$\iff 0 \to X \xrightarrow{\binom{0}{1}} \mathrm{Cone}(-h[-1]) \underset{s}{\overset{(1\ 0)}{\rightleftarrows}} Z \to 0$ split.

$\iff \exists s : Z \to \mathrm{Cone}(-h[-1])$ $\left(= Z^n \oplus X^n\right)$ is a chain map & $(1\ 0)s = 1$.

$s = \binom{1}{t_2}$

$s = \binom{t_1}{t_2}$

$(1\ 0)s = 1$

$\iff (1\ 0)\binom{t_1}{t_2} = 1.$

$\iff t_1 = 1.$

$\iff s = \binom{1}{t_2}.$

$$\iff \begin{tikzcd} Z^n \arrow[r, "d_Z^n"] \arrow[d, "\binom{1}{t_2^n}"'] \arrow[dr, phantom, "\circlearrowright"] & Z^{n+1} \arrow[d, "\binom{1}{t_2^{n+1}}"] \\ Z^n \oplus X^n \arrow[r, "\begin{pmatrix} d_Z^n & \\ -h^n & d_X^n \end{pmatrix}"'] & Z^{n+1} \oplus X^{n+1} \end{tikzcd}$$

$$\iff \begin{pmatrix} 1 \\ t_2^{n+1} \end{pmatrix} d_Z^n = \begin{pmatrix} d_Z^n & \\ -h^n & d_X^n \end{pmatrix}\begin{pmatrix} 1 \\ t_2^n \end{pmatrix}$$

$$\iff \begin{pmatrix} d_Z^n \\ t_2^{n+1} d_Z^n \end{pmatrix} = \begin{pmatrix} d_Z^n \\ -h^n + d_X^n t_2^n \end{pmatrix}.$$

$$\iff h^n = d_X^n t_2^n - t_2^{n+1} d_Z^n$$

$$\iff h \overset{-t_2}{\sim} 0. \quad \square$$

$$\begin{tikzcd} Z^{n-1} \arrow[r] \arrow[d, phantom] & Z^n \arrow[r, "d_Z^n"] \arrow[dl, "-t_2^n"'] \arrow[d, "h^n"] & Z^{n+1} \arrow[dl, "-t_2^{n+1}"] \\ X^n \arrow[r, "-d_X^n"'] & X^{n+1} \arrow[r] & X^{n+2} \end{tikzcd}$$

(3). $0 \to X \underset{\bar u = (0\ 1)}{\overset{u = \binom{0}{1}}{\rightleftarrows}} \mathrm{Cone}(-h[-1]) \underset{s = \binom{1}{0}}{\overset{\nu = (1\ 0)}{\rightleftarrows}} Z \to 0.$

$$u^{n+1}h^n = \begin{pmatrix} 0 \\ 1 \end{pmatrix} h^n = \begin{pmatrix} 0 \\ h^n \end{pmatrix}$$

$$s^{n+1}d_Z^n - d_Y^n s^n = \begin{pmatrix} 1 \\ 0 \end{pmatrix} d_Z^n - \begin{pmatrix} d_Z^n & \\ -h & d_X^n \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} d_Z^n \\ 0 \end{pmatrix} - \begin{pmatrix} d_Z^n \\ -h^n \end{pmatrix} = \begin{pmatrix} 0 \\ h^n \end{pmatrix} = u^{n+1}h^n$$

(1) holds $\Rightarrow$ h htp invar. $\square$

(4) By (3), split $\iff h = -f[1] \sim 0 \iff f \sim 0$ $\square$

Thm 25.4 (B. Iversen) Let $\mathcal{A}$ be an additive cat.

(1). Every D.T. can be induced by a C.S.S.E.S.

(Every D.T. $\simeq \underline{P \xrightarrow{\left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)} \mathrm{Cyl}(f) \xrightarrow{\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right)} \mathrm{Cone}(f) \xrightarrow{(1\ 0)} P[1]}.$)

$$0 \to P \underset{\bar u = (0\ 0\ 1)}{\overset{u = \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)}{\rightleftarrows}} \mathrm{Cyl}(f) \underset{s = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)}{\overset{\nu = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right)}{\rightleftarrows}} \mathrm{Cone}(f) \to 0$$

$$0 \to P \rightleftarrows \mathrm{Cyl}(f) \underset{}{\overset{(\ \ 1\ )}{\rightleftarrows}} \mathrm{Cone}(f) \to 0$$

(Every D.T. $\simeq \underbrace{P \xrightarrow{\begin{pmatrix}0\\0\\1\end{pmatrix}} \mathrm{Cyl}(f) \xrightarrow{\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}} \mathrm{Cone}(f) \xrightarrow{(1\ 0)} P[1]}_{\text{c.s.s.e.s.}}$.) $\quad \pi := (0\ 0\ 1) \quad s = \begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}$

$\Leftrightarrow \underbrace{(1\ 0)}_{=:h}$ is the htp inver of $0 \to P \to \mathrm{Cyl}(f) \to \mathrm{Cone}(f) \to 0$.

(2) $P \xrightarrow{f} Q \xrightarrow{g} R$ c.s.s.e.s., h, htp inver.

$$\begin{tikzcd}
P \arrow[r, "f"] \arrow[d, equal] & Q \arrow[r] \arrow[d, equal] & \mathrm{Cone}(f) \arrow[r] \arrow[d, "(0\ g)"', "\simeq"] & P[1] \arrow[d, equal] \\
P \arrow[r] & Q \arrow[r, "\text{trivial}"] & R \arrow[u, bend left, "\binom{h}{g}"] \arrow[r] & P[1]
\end{tikzcd}$$

Pf.: (1) $u^{n+1}h^n = \begin{pmatrix}0\\0\\1\end{pmatrix}(1\ 0) = \begin{pmatrix}0&0\\0&0\\1&0\end{pmatrix}$.

$$s^{n+1}d_Z^n = \begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}\begin{pmatrix}-d_P^{n+1} & \\ f^{n+1} & d_Q^n\end{pmatrix} - \begin{pmatrix}-d_P^{n+1} & & \\ f^{n+1} & d_Q^n & \\ -1 & & d_P^n\end{pmatrix}\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}$$

$$= \begin{pmatrix}-d_P^{n+1} & 0\\ f^{n+1} & d_Q^n\\ 0 & 0\end{pmatrix} - \begin{pmatrix}-d_P^{n+1} & 0\\ f^{n+1} & d_Q^n\\ -1 & 0\end{pmatrix} = \begin{pmatrix}0&0\\0&0\\1&0\end{pmatrix} = u^{n+1}h^n.$$

$\Rightarrow$ (1) holds.

$\Rightarrow$ h is a htp inver. $\square$

(2) <(0 g) is a chain map>.

$$\begin{tikzcd}
P^{n+1}\oplus Q^n \arrow[r, "\begin{pmatrix}-d_P^{n+1} & \\ f & d_Q^n\end{pmatrix}"] \arrow[d, "(0\ g^n)"] & P^{n+2}\oplus Q^{n+1} \arrow[d, "(0\ g^{n+1})"] \\
R^n \arrow[r, "d_R"'] & R^{n+1}
\end{tikzcd}$$

$$(0\ g^{n+1})\begin{pmatrix}-d_P^{n+1} & \\ f^{n+1} & d_Q^n\end{pmatrix} = (g^{n+1}f^{n+1} \quad g^{n+1}d_Q^n)$$

$$d_R^n(0\ g^n) \qquad = (0 \quad g^{n+1}d_Q^n) \quad ✓$$

< ? >: Since h is htp inver, $\exists\, s, \pi$ s.t. (*) holds. & (1), (2) holds.

$$\begin{tikzcd}
\mathrm{Cone}(f) \arrow[r, "(1\ 0)"] \arrow[d, "(0\ g)"'] & P[1] \arrow[d, equal] \\
R \arrow[r, "h"'] & P[1]
\end{tikzcd}$$

$\Leftrightarrow (1\ 0) \sim (0\ hg)$

$\Leftrightarrow \underline{(-1\ \ hg) \sim 0}$.

$$\begin{tikzcd}
 & P^{n+1}\oplus Q^n \arrow[r, "\begin{pmatrix}-d_P^{n+1} & \\ f^{n+1} & d_Q^n\end{pmatrix}"] \arrow[dl, "(0\ -\pi^n)"'] \arrow[d, "(-1\ h^n g^n)"] & P^{n+2}\oplus Q^{n+1} \arrow[dl, "(0\ -\pi^{n+1})"] \\
P^n \arrow[r, "-d_P^n"'] & P^{n+1} &
\end{tikzcd}$$

$$(0\ \ -\pi^{n+1})\begin{pmatrix}-d_P^{n+1} & \\ f^{n+1} & d_Q^n\end{pmatrix} - d_P^n(0\ \ -\pi^n)$$

$$= (-\underline{\pi^{n+1}f^{n+1}} \quad -\underline{\pi^{n+1}d_Q^n}) + (0 \quad \underline{d_P^n\pi^n})$$

$$= (-1 \quad hg). \quad ✓$$

< invertible >

Consider $\binom{h}{g} : R \to \mathrm{Cone}(f)$ is chain map ✓.

$$\begin{tikzcd}
R^n \arrow[r, "d_R^n"] \arrow[d, "\binom{h^n}{g^n}"] & R^{n+1} \arrow[d, "\binom{h^{n+1}}{g^{n+1}}"] \\
 &
\end{tikzcd}$$

$-d_P^{n+1} \quad h^n \quad d^{n+1}h^n \quad ''$

$$\begin{tikzcd} R^n \arrow[r, "d_R"] \arrow[d, "\binom{h^n}{s^n}"'] \arrow[dr, phantom, "\circlearrowright"] & R^{n+1} \arrow[d, "\binom{h^{n+1}}{s^{n+1}}"] \\ P^{n+1}\oplus Q^n \arrow[r, "\begin{pmatrix} -d_P^{n+1} & \\ f^{n+1} & d_Q^n \end{pmatrix}"'] & P^{n+1}\oplus Q^{n+1} \end{tikzcd}$$

$$\begin{pmatrix} -d_P^{n+1} & \\ f^{n+1} & d_Q^n \end{pmatrix}\begin{pmatrix} h^n \\ s^n \end{pmatrix} = \begin{pmatrix} -d_P^{n+1}h^n \\ f^{n+1}h^n + d_Q^n s^n \end{pmatrix} = \begin{pmatrix} h^{n+1}d_R^n \\ s^{n+1}d_R^n \end{pmatrix} = \begin{pmatrix} h^{n+1} \\ s^{n+1} \end{pmatrix} d_R^n$$

$(0\ g)\binom{h}{s} = gs = id_R$ ✓.

$\binom{h}{s}(0\ g) = \begin{pmatrix} 0 & hg \\ 0 & sg \end{pmatrix} \underset{\text{to show}}{\sim} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & hg \\ 0 & 1-sg \end{pmatrix} \sim 0.$

$$\begin{tikzcd} & P^{n+1}\oplus Q^n \arrow[r, "\begin{pmatrix} -d_P^{n+1} & \\ f^{n+1} & d_Q^n \end{pmatrix}"] \arrow[dl, "\begin{pmatrix} 0 & \pi^n \\ 0 & 0 \end{pmatrix}"'] \arrow[d, "\begin{pmatrix} 1 & -hg \\ & 1-sg \end{pmatrix}"] & P^{n+2}\oplus Q^{n+1} \arrow[dl, "\begin{pmatrix} 0 & \pi^{n+1} \\ 0 & 0 \end{pmatrix}"] \\ P^n\oplus Q^{n-1} \arrow[r, "\begin{pmatrix} -d_P^n & \\ f^n & d_Q^{n-1} \end{pmatrix}"'] & P^{n+1}\oplus Q^n & \end{tikzcd}$$

$$\begin{pmatrix} 0 & \pi^{n+1} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -d_P^{n+1} & \\ f^{n+1} & d_Q^n \end{pmatrix} + \begin{pmatrix} -d_P^n & \\ f^n & d_Q^{n-1} \end{pmatrix}\begin{pmatrix} 0 & \pi^n \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \pi^{n+1}f^{n+1} & \pi^{n+1}d_Q^n \\ & \end{pmatrix} + \begin{pmatrix} 0 & -d_P^n\pi^n \\ 0 & f^n\pi^n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -hg \\ & 1-sg \end{pmatrix}. \qquad \square$$

Ex. ch1. 8.

$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ D.T.

& $\underline{Hom(X[1], Z) = 0}$.

$\Rightarrow w$ is the unique mor s.t.

$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is D.T.

Pf: Suppose $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w'} X[1]$ D.T.

$$\Rightarrow \begin{tikzcd} X \arrow[r, "u"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[r, "v"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & Z \arrow[r, "w"] \arrow[d, "h", "\text{(by TR3)}"'] \arrow[dr, phantom, "\circlearrowright"] & X[1] \arrow[d, equal] \\ X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r, "w'"] & X[1] \end{tikzcd}$$

$\xrightarrow{Hom(-,Z)}$ $Hom(X[1], Z) \xrightarrow{w^*} Hom(Z, Z) \xrightarrow{v^* \text{— inj}} Hom(Y, Z)$

$Hom(X[1], Z) = 0$

$h \mapsto vh = v$

$id_Z \mapsto v.$

$\Rightarrow h = id_Z$

$\Rightarrow w = w'.$ □

E.g. 2.5.5.

$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d, equal] & Z \arrow[r, "w"] \arrow[d, equal] & X[1] \arrow[d, equal] \\ \vdots \arrow[d] & & & \\ 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & Z \arrow[d] \\ Z \arrow[r, "\binom{1}{-1}"] \arrow[d, "3"'] & Z\oplus Z \arrow[r, "(1\ 1)"] \arrow[d, "\begin{pmatrix} 3 & \\ & 3 \end{pmatrix}"] & Z \arrow[r, "-1"] \arrow[d, "3"] & Z \arrow[d] \\ Z \arrow[r, "\binom{1}{-1}"] \arrow[d] & Z\oplus Z \arrow[r, "(1\ 1)"] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \vdots \\ \vdots & \vdots & \vdots & \end{tikzcd}$$

c.s.s.e.s.

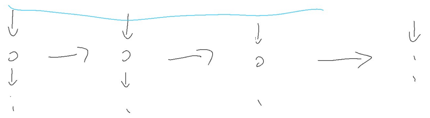
$\rightsquigarrow$ its htp invar

$\rightsquigarrow$ D.T.

$\Rightarrow Hom(X[1], Z) = 0.$

$\Rightarrow X \to Y \to Z \xrightarrow{-w} X[1]$ is NOT D.T.

$\rightsquigarrow \neq 0.$



1 DT.