§2.6. $\mathcal{A} \to$ abelian cate.

- left (right) brutal truncation ⇐
- left (right) truncation ⇐ (X^i, d). n

Defi (i) left brutal truncation $\boxed{X_{\le n}}$: $X^i_{\le n} = \begin{cases} X^i & i \le n \\ 0 & i > n \end{cases}$, $d^i_{\le n} = \begin{cases} d^i & i < n \\ 0 & i \ge n \end{cases}$

$\longrightarrow X^{n-2} \xrightarrow{d^{n-2}} X^{n-1} \xrightarrow{d^{n-1}} X^n \to 0 \to 0$ $\quad \underline{\mathcal{C}(\mathcal{A})}$

(ii) $X_{\ge n}$:

$\cdots 0 \to 0 \to X^n \xrightarrow{d^n} X^{n+1} \xrightarrow{d^{n+1}} \cdots$

(iii) left truction $\boxed{\tau_{\le} X}_n$

$\longrightarrow X^{n-2} \xrightarrow{d^{n-2}} X^{n-1} \xrightarrow{d^{n-1}} \ker d^n \to 0 \to \cdots$

(iii) right truction

$\cdots 0 \to 0 \to \operatorname{Im} d^{n-1} \to X^n \xrightarrow{d^n} X^{n+1} \to X^{n+2} \to \cdots$

Remark: $\exists$ $X_{\ge n} \underset{\pi}{\overset{u}{\rightleftarrows}} X \underset{s}{\overset{v}{\rightleftarrows}} X_{<n}$ ☆ chain split s.e.s $\quad \tau_{\le n} X \quad \tau_{\ge n} X$

$$\begin{array}{cccccccc}
X_{\ge n} & \cdots & 0 & \to & X^n & \to & X^{n+1} & \to \cdots \\
\downarrow u & & \downarrow & & \| & & \| & \\
X & \cdots \to & X^{n-1} & \to & X^n & \to & X^{n+1} & \to \cdots \\
\downarrow v & & \| & & & & & \\
X_{<n} & \cdots \to & X^{n-1} & \to & 0 & \to & 0 & \to
\end{array}$$

By Thm 2.5.4 (2)

Cor: $\mathcal{K}(\mathcal{A})$ $\quad X_{\ge n} \xrightarrow{u} X \xrightarrow{v} X_{<n} \xrightarrow{h} X_{\ge n}[1]$

pf: (1) (☆) chain split s.e.s $\quad (vs)=\mathrm{Id} \quad (\pi u)=\mathrm{Id} \quad \pi s=0 \quad u\pi + sv = \mathrm{Id}$

(2) h homotopy invari

$$\begin{array}{ccccc}
X_{\ge n} & \xrightarrow{u} & X & \xrightarrow{v} & X_{<n} \\
\| & & \downarrow \binom{\pi}{v} & & \| \\
X_{\ge n} & \xrightarrow{(u,s)} & X_{\ge n} \oplus X_{<n} & \longrightarrow & X_{<n}
\end{array}$$

$$\begin{cases} u^{i+1} h^i = s^{i+1} d_X^i - d^i s^i & (**) \\ h^n v^n = d_X^n \pi^n - \pi^{n+1} d_X^n & (***) \end{cases}$$

right of $(**)$ = $\begin{cases} 0 & i < n-1 \\ -d_X^{n-1} & i = n-1 \\ 0 & i > n-1 \end{cases} = u^{i+1} h^i$

Recall $\begin{cases} \cdot\text{ direct system} \\ \cdot\text{ direct limit} \end{cases}$ $\quad (I, \le)$. fix $\mathcal{C}$. and $(I, \le)$

$\begin{cases} \text{inverse system} \\ \text{inverse limit} \end{cases}$ $\quad (I, \le)$: $\forall u, v \in I$, $\exists w \in I$, s.t $u, v \le w$

$(X_i, f_{ji}) = \begin{cases} X_i \in \mathrm{obj}\, \mathcal{C}, \ \forall i \in I \\ f_{ii} = \mathrm{id}_{X_i} \\ f_{kj} f_{ji} = f_{ki} \quad \forall i \le j \le k \end{cases}$

$$\begin{array}{ccc}
 & X_i & \\
{}^{f_{ji}}\swarrow & & \searrow^{f_{ki}} \\
X_j & \xleftarrow[f_{kj}]{} & X_k
\end{array}$$

(X_i, f_{ji}):

(i) $\exists f_i : X_i \to X \quad \forall i \in I$. s.t $f_j f_{ji} = f_i \quad \forall i \le j$

(X, f_i)

$$\begin{array}{ccc}
 & X_i & \\
{}^{f_i}\swarrow & & \searrow^{f_{ji}} \\
X & \xleftarrow[f_j]{} & X_j
\end{array}$$

小 → 大

(ii) $\exists (Y, g_i)$, $g_i : X_i \to Y$. $\forall i \in I$. s.t $g_j f_{ji} = g_i \quad \forall i \le j$

$\exists!\ g : X \to Y$ s.t $g f_i = g_i$

$\varinjlim (X_i) = X \quad (X, f_i)$

claim: ① $\varinjlim_{n\geq 0} \tau_{\leq n} X = X$

$\ulcorner$ $\forall n \in \mathbb{Z}$

$$
\begin{tikzcd}
\tau_{\leq n} X \arrow[d, "u_n"] & \cdots \arrow[r] & X^{n-1} \arrow[r] \arrow[d, equal] & \ker d^n \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \cdots \arrow[d] \\
\tau_{\leq n+1} X \arrow[d, "\varphi_{n+1}"] & \cdots \arrow[r] & X^{n-1} \arrow[r] \arrow[d, equal] & X^n \arrow[r] \arrow[d, equal] & \ker d^{n+1} \arrow[r] \arrow[d] & 0 \arrow[d] \cdots 0 \quad 0 \\
X & \cdots \arrow[r] & X^{n-1} \arrow[r] \arrow[d, "g^{n-1}"] & X^n \arrow[r] \arrow[d, "g^n"] & X^{n+1} \arrow[r] \arrow[d] & X^{n+2} \cdots \arrow[d] \\
Y & & Y^{n-1} \arrow[r] & Y^n \arrow[r] & Y^{n+1} \arrow[r] & Y^{n+2}
\end{tikzcd}
$$

$\lrcorner$

② $\varprojlim_{(n\leq 0)} \tau_{\geq n} X = X$

$\ulcorner$ $n \in \mathbb{Z}$

$$
\begin{tikzcd}
X \arrow[d, "\phi_n"] & \cdots \arrow[r] & X^{n-3} \arrow[r] \arrow[d] & X^{n-2} \arrow[r] \arrow[d] & X^{n-1} \arrow[r, "d^{n-1}"] \arrow[d] & X^n \arrow[r] \arrow[d, equal] & X^{n+1} \arrow[r, "d^{n+1}"] \arrow[d, equal] & \cdots \\
\tau_{\geq n} X \arrow[d, "v_n"] & \cdots & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathrm{Im}\, d^{n-1} \arrow[r] \arrow[d] & X^n \arrow[r] \arrow[d] & X^{n+1} \arrow[r, "d^{n+1}"] \arrow[d, equal] & {} \\
\tau_{\geq n+1} \arrow[r] & & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathrm{im}\, d^n \arrow[r] & X^{n+1} \arrow[r, "d^{n+1}"] & {}
\end{tikzcd}
$$

$\lrcorner$

Defi: $X[n,m] := (\tau_{\leq m} X)_{\geq n} := \tau_{\leq m}(X_{\geq n})$

$$\cdots \quad 0 \to 0 \to X^n \xrightarrow{d^n} \cdots \to X^{m-1} \to \ker d^n \to 0 \to 0 \cdots$$

$[n',m'] \supseteq [n,m]$ $\qquad$ $[n',m'] \succ [n,m]$

$X[n,m] \longrightarrow [n',m']$

$X[n,m] \longrightarrow X$

$\varinjlim_{n\geq 0} X_{[-n,\, n+1]} = X$

§2.7. $\mathrm{Hom}^{\bullet}(-,-)$

$\mathcal{A}$: addi. category. $\quad \forall X, Y \in \mathrm{obje}(\mathcal{A})$

$\mathrm{Hom}^{\bullet}(X,Y)$: $\quad$ cpx. of abolian group

n-component: $\quad \mathrm{Hom}^n(X,Y) := \prod_{p\in\mathbb{Z}} \mathrm{Hom}(X^p, Y^{p+n}) \quad \forall n \in \mathbb{Z}$

$$d^n(f) = (d^n(f)^p)_{p\in\mathbb{Z}} \in \mathrm{Hom}^{n+1}(X,Y)$$

where $d^n(f)^p = \partial_Y^{n+p} f^p \underbrace{+(-1)^{n+1}}_{\text{Koszul sign}} f^{p+1} \partial_X^p \in \mathrm{Hom}_{\mathcal{A}}(X^p, Y^{p+n+1})$

claim: ① $d^{n+1} d^n = 0 \quad \forall n \in \mathbb{Z}$

$\boxed{\ker d^n} = \{ f = (f^p)_{p\in\mathbb{Z}} \in \mathrm{Hom}^n(X,Y) \mid \underline{\partial_Y^{n+p} f^p + (-1)^n f^p \partial_X^p} \ \forall p \in \mathbb{Z} \}$

$= \mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(X, Y[n])$, in particular $(n=1)$ $\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(X,Y)$

$\mathrm{Im}\, d^{n-1} = \{ f = (f^p)_{p\in\mathbb{Z}} \in \mathrm{Hom}^n(X,Y) \mid \exists\, s = (s^p)_{p\in\mathbb{Z}} \in \mathrm{Hom}^{n-1}(X,Y)$

s.t $\underline{f^p = (d^{n-1}s)^p = \partial_Y^{n-1+p} s^p + (-1)^n s^{p+1} \partial_X^p}$ $\}$

$= \mathrm{Htp}(X, Y[n])$

prop(2.7.1).

$$H^n \operatorname{Hom}^{\cdot}_{\mathcal{C}(A)}(X,Y) = \operatorname{Hom}_{K(A)}(X, Y[n]) \quad \forall n \in \mathbb{Z}$$

☆ $\left.\begin{array}{l}\operatorname{Hom}^{\cdot}(X, -) \\ \operatorname{Hom}^{\cdot}(-, X)\end{array}\right\} : K(A) \to K(Ab)$

① $f : Y \to Y'$

$\boxed{\operatorname{Hom}^{\cdot}(X,f)} : \operatorname{Hom}^{\cdot}(X,Y) \longrightarrow \operatorname{Hom}^{\cdot}(X,Y')$

where $(\operatorname{Hom}^n(X,f))_{n\in\mathbb{Z}}$

$$\operatorname{Hom}^n(X,f) : \prod_{p\in\mathbb{Z}} \operatorname{Hom}(X^p, Y^{p+n}) \longrightarrow \prod_{p\in\mathbb{Z}} \operatorname{Hom}(X^p, Y'^{p+n})$$
$$(g^p)_{p\in\mathbb{Z}} \longmapsto (f^{p+n} g^p)_{p\in\mathbb{Z}}$$

Γ

$$\begin{array}{ccc}
H^n(X,Y) = \prod_{p\in\mathbb{Z}} \operatorname{Hom}(X^p, Y^{p+n}) & \longrightarrow & H^{n+1}(X,Y) = \prod_{p\in\mathbb{Z}} \operatorname{Hom}(X^p, Y^{p+n+1}) \\
\downarrow & & \downarrow \\
H^n(X,Y') = \prod_{p\in\mathbb{Z}} \operatorname{Hom}(X^p, Y'^{p+n}) & \longrightarrow & H^{n+1}(X,Y') = \prod_{p\in\mathbb{Z}} \operatorname{Hom}(X^p, Y'^{p+n+1})
\end{array}$$

$g = (g^p)_{p\in\mathbb{Z}} \longmapsto \partial_Y^{n+p} g^p + (-1)^{n+1} g^{p+1} \partial_X^p$

$\mapsto f^{n+p+1} \partial_Y^{n+p} g^p + (-1) f^{n+p+1} g^{p+1} \partial_X^p$

$(f^{p+n} g^p)_{p\in\mathbb{Z}} \longmapsto \partial_{Y'}^{n+p} [f^{p+n} g^p] + (-1)^{n+1} [f^{p+n+1} g^{p+1}] \partial_X^p$

② $\operatorname{Hom}^{\cdot}(X, -)$ 加法性ある:

③ $s : f \sim 0$ $\qquad f = \partial_Y s^p + s^{p+1} \partial_Y$

$\operatorname{Hom}^{\cdot}(X, f) \sim 0$

Γ $\operatorname{Hom}^n(X,f) : \prod_{p\in\mathbb{Z}} \operatorname{Hom}(X^p, Y^{p+n}) \to \prod_{p\in\mathbb{Z}} \operatorname{Hom}(X^p, Y'^{p+n})$

$(g^p)_{p\in\mathbb{Z}} \mapsto (f^{p+n} g^p)_{p\in\mathbb{Z}}$

$S = Sg$

$$(f^{p+n} g^p) = \partial_{Y'} s^p g^p + s^{p+1} \partial_Y g^p$$
$$= \partial_{Y'} s^p g^p + s^{p+1} g^{p+1} \partial_Y$$

$\operatorname{Hom}^{\cdot}(X,f) : \mathcal{C}(A) \to \mathcal{C}(Ab) \rightsquigarrow K(A) \to K(Ab)$

prop 2.7.2. A: $X \in \mathcal{C}(A)$, $f : Y \to Y'$

$\left\{\begin{array}{ll}\operatorname{Hom}^{\cdot}(X, Y[1]) = \operatorname{Hom}^{\cdot}(X,Y)[1] & (1) \checkmark \\ \operatorname{Hom}^{\cdot}(X, \operatorname{Cone} f) \cong \operatorname{Cone} \operatorname{Hom}^{\cdot}(X, f) & (2) \checkmark\end{array}\right.$ $\qquad Y \to Y' \to \operatorname{Cone}(f) \to Y[1]$

$$\begin{array}{ccccccc}
\operatorname{Hom}^{\cdot}(X,Y) & \xrightarrow{\operatorname{Hom}(X,f)} & \operatorname{Hom}(X,Y') & \longrightarrow & \operatorname{Hom}(X, \operatorname{Cone}(f)) & \longrightarrow & \operatorname{Hom}(X,Y)[1] \\
\| & & \| & & \downarrow \wr & & \| \\
\operatorname{Hom}^{\cdot}(X,Y) & \xrightarrow{\operatorname{Hom}^{\cdot}(X,f)} & \operatorname{Hom}(X,Y') & \longrightarrow & \underset{\sim\sim}{\operatorname{Cone} \operatorname{Hom}^{\cdot}(X,f)} & \longrightarrow & \operatorname{Hom}(X,Y)[1] \in \mathcal{E}
\end{array}$$

For (1) $\operatorname{Hom}^{\cdot}(X, Y[1]) = \left\{\begin{array}{l}\operatorname{Hom}^n(X, Y[1]) = \prod_{p\in\mathbb{Z}} \operatorname{Hom}_A(X^p, \underline{Y^{p+n+1}}) \\ (d^n(f))^p = -\partial_Y^{n+p+1} f^p + (-1)^{n+2} f^{p+1} \partial_X^p\end{array}\right.$

$\updownarrow$

$\underset{\sim\sim}{\operatorname{Hom}^{\cdot}(X,Y)[1]} =$

For (2)

$Hom^\cdot(X, Cone(f)) \Leftarrow Hom^n(X, Cone(f)) = \prod_{p\in\mathbb{Z}} Hom(X^p, Y^{p+n+1} \oplus Y'^{p+n})$

$\downarrow \cong$

$Cone(Hom^\cdot(X,Y) \to Hom^\cdot(X,Y')) \Leftarrow \prod_{p\in\mathbb{Z}} Hom(X^p, Y^{p+n+1}) \oplus \prod_{p\in\mathbb{Z}} Hom(X^p, Y'^{p+n})$

Chapter 3: Quotient of catego

$\mathcal{C}(A) \xrightarrow{\text{加性商}} K(A) \xrightarrow{\text{Verdier商}} D(A)$

$\boxed{\text{拟同构}} \Rightarrow$ 同构

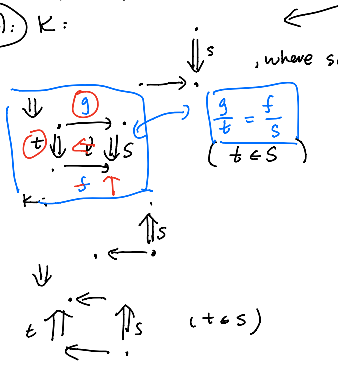
Defi: K, $\boxed{S} \ni s$ $\Rightarrow$

S is called multiplicative systerm. If (FR1–FR3)

(FR1). $\forall f, g \in S \Rightarrow f\circ g \in S$

$id_X \in S$, $\forall X \in K$.

(FR2): K: , where $s \in S$.



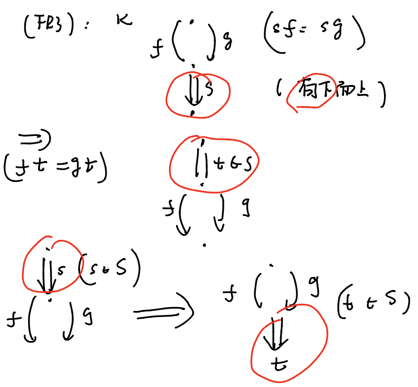
$\frac{g}{t} = \frac{f}{s}$ ($t \in S$)

($t \in S$)

(FR3): K ($sf = sg$) (右下向上)

$\Rightarrow$ ($ft = gt$) $t \in S$

($s \in S$) $\Longrightarrow$ ($t \in S$)



(右上向下)

prop: (拟同构集合是一个乘法系).

Defi: $K \longleftarrow S$ multi sys. $\Rightarrow$ S is a saturated multi systerm.

$fh \in S$, $kf \in S \Rightarrow f \in S$

[illegible]. $C' \overset{s}{\Longleftarrow} \cdot \longrightarrow D'$ f/s g/t

$f: X' \to Y'$ $H^n(f): H^n(X) \cong H^n(Y)$

$C' \xrightarrow{f} D' \longrightarrow E'$

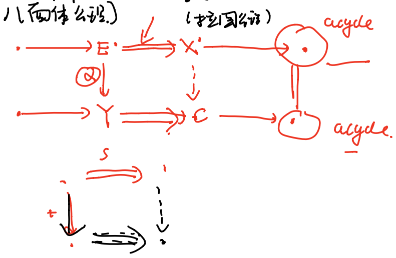
f 为拟同构 $\Longleftrightarrow$ E' 是 acycle

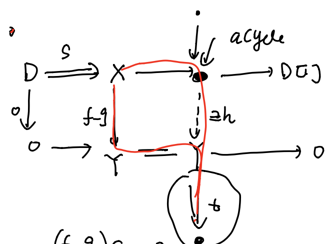
$H^i(E') = 0$

$H^n(f\circ g) = H^n(f) H^n(g)$

TR4 $\longrightarrow$ [illegible]

(八面体公理) ([illegible])





$(f-g)s = 0$

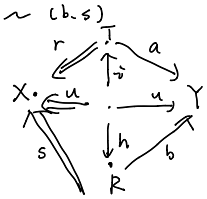
$fs = gs$

$\Longrightarrow t\circ(f-g) = 0$

K 中 $X \to Y$ 右分式 (b,s) $\qquad X \xleftarrow{s} \cdot \xrightarrow{b} Y$. $s\in S$.

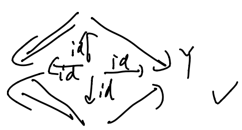
(right roof) ⇝ $\boxed{b/s}$

$(a,r) \sim (b,s)$



Cor 3.2.1 $\sim$ 为右分式作成的类上的等价关系.

Pf: 自反性✓

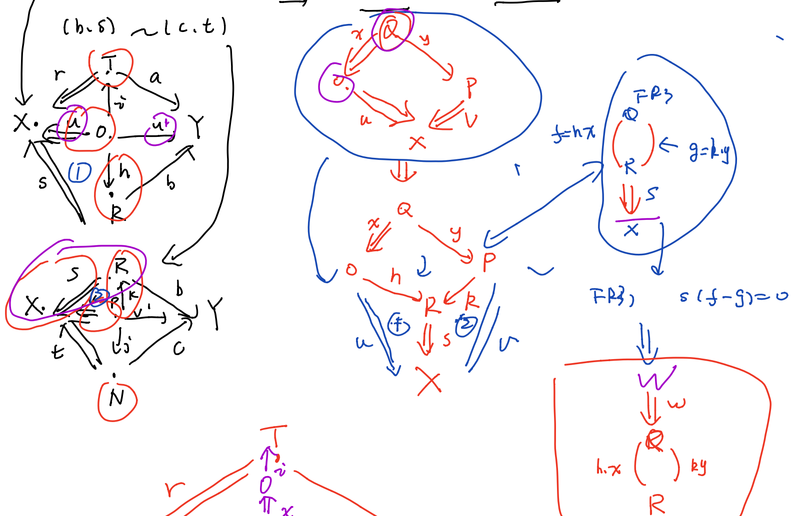


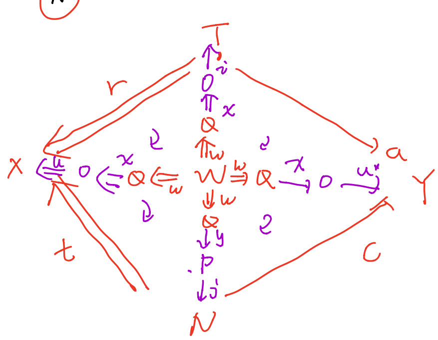
对称性✓

传递性:

$(a,r)\sim(b,s)$ $\qquad \Longrightarrow \quad (a,r) \sim (c,t)$

$(b,s) \sim (c,t)$





$\Rightarrow$ $(a,r)\sim(c,t)$

(b,s) . $s\in S$ $\boxed{b/s}$