Recall: right fraction (b,s)

$$X \xLeftarrow{s} \cdot \xrightarrow{b} Y.$$

if $(a,r) \sim (b,s)$. Then $\exists$

$$\begin{tikzcd}
 & \bullet \arrow[dl, Rightarrow, "r"'] \arrow[dr, "a"] & \\
X & \cdot \arrow[l, Rightarrow, "u"'] \arrow[r, "u'"] \arrow[u, "i'"] \arrow[d, "h"] & Y \\
 & \cdot \arrow[ul, Rightarrow, "s"] \arrow[ur, "b"'] &
\end{tikzcd}$$

com diag

Denote the equivalent class of (b,s) as b/s.

$$b/s = \{(a,r) \mid (a,r) \sim (b,s)\}.$$

Note there's a com diag

$$\begin{tikzcd}
 & & \cdot \arrow[dl, Rightarrow, "t"'] \arrow[dr, "c"] \arrow[dd, phantom, "\circlearrowright"] & & \\
 & \cdot \arrow[dl, Rightarrow, "r"'] \arrow[dr, "a"] & & \cdot \arrow[dl, Rightarrow, "s"'] \arrow[dr, "b"] & \\
X & & Y & & Z
\end{tikzcd}$$

Existence of the square follows by FR2)

Define $\underline{b/s \circ a/r := bc/rt}$ To show this is well-defined.

1° If $\exists$ com diag

$$\begin{tikzcd}
 & \cdot \arrow[dl, Rightarrow, "t'"'] \arrow[dr, "c'"] & \\
\cdot \arrow[dr, "a"'] & & \cdot \arrow[dl, Rightarrow, "s"] \\
 & \cdot &
\end{tikzcd}$$

then $\dfrac{bc'}{rt'} = \dfrac{bc}{rt}$

2° If $a/r = a'/r'$, then $b/s \circ a/r = b/s \circ a'/r'$

3° If $b/s = b'/s'$, then $b/s \circ a/r = b'/s' \circ a/r$.

Pf: 1° To show

$$\begin{tikzcd}
 & \cdot \arrow[dl, Rightarrow, "rt"'] \arrow[dr, "bc"] & \\
X & \cdot \arrow[l, Rightarrow] \arrow[r] \arrow[u] \arrow[d] & Z \\
 & \cdot \arrow[ul, Rightarrow, "rt'"] \arrow[ur, "bc'"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & \bullet \arrow[dlll, Rightarrow, "rt"'] \arrow[dddrrr, "bc"] & & & \\
 & & & \cdot \arrow[u, Rightarrow, "\check{t}'"'] & & & \\
 & & & & & & \\
X & & \cdot \arrow[ll, Rightarrow, "rt'\check{t}"'] & \cdot \arrow[l, Rightarrow, "w"'] \arrow[r, Rightarrow, "w"] \arrow[uu, Rightarrow, "w"] \arrow[d, Rightarrow, "w"] & \cdot \arrow[rr, "bc'\check{t}"] & & Z \\
 & & & \cdot \arrow[d, Rightarrow, "\tilde{t}"] & & & \\
 & & & \bullet \arrow[uulll, Rightarrow, "rt'"] \arrow[uurrr, "bc'"'] & & &
\end{tikzcd}$$

Note by FR2), the following diag commutes

$$\begin{tikzcd}
\cdot \arrow[r, Rightarrow, "\check{t}'"] \arrow[d, Rightarrow, "\check{t}"'] & \cdot \arrow[d, Rightarrow, "t"] \\
\cdot \arrow[r, Rightarrow, "t'"'] & \cdot
\end{tikzcd}$$

$\Rightarrow at'\check{t} = at\check{t}'$ & $rt\check{t}' = rt'\check{t}$

i.e. $sc'\check{t} = sc\check{t}'$ $\quad s \in S$

By FR3). $\exists\ w \in S$ s.t $c'\check{t}w = c\check{t}'w$. $\Rightarrow$ ① $rt\check{t}'w = rt\check{t}'w$. $\quad rt\check{t}'w \in S$. By FR1)

$= rt'\check{t}w$.

$\Rightarrow$ ② $bc\check{t}'w = bc'\check{t}w$.

2° $(a,r) \sim (a',r')$ i.e $\exists$ com diag

$$\begin{tikzcd}
 & A \arrow[dl, Rightarrow, "r"'] \arrow[dr, "a"] & \\
X & c \arrow[l, Rightarrow, "u"'] \arrow[r, "v"] \arrow[u, "i"] \arrow[d, "j"] & Y \\
 & B \arrow[ul, Rightarrow, "r'"] \arrow[ur, "a'"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & & \cdot \arrow[dl, Rightarrow, "t"'] \arrow[dr, "c"] & & \\
 & A \arrow[dl, Rightarrow, "r"'] \arrow[dr, "a"] & & \cdot \arrow[dl, Rightarrow, "s"'] \arrow[dr, "b"] & \\
X & & Y & & Z
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & & \cdot \arrow[dl, Rightarrow, "t'"'] \arrow[dr, "c'"] & & \\
 & B \arrow[dl, Rightarrow, "r'"'] \arrow[dr, "a'"] & & \cdot \arrow[dl, Rightarrow, "s"'] \arrow[dr, "b"] & \\
X & & Y & & Z
\end{tikzcd}$$

To show

$$\begin{tikzcd}
 & \cdot \arrow[dl, Rightarrow, "rt"'] \arrow[dr, "bc"] & \\
X & \cdot \arrow[l, Leftarrow, "uk_1w"'] \arrow[u, "i'w"'] \arrow[d, "j'x"] \arrow[r, "bci'w"] & Z \\
 & \cdot \arrow[ul, Rightarrow, "r't'"] \arrow[ur, "bc'"'] &
\end{tikzcd}$$

$\exists$ com diag

$$\begin{tikzcd}
\cdot \arrow[r, "i'"] \arrow[d, Rightarrow, "k_1"'] & \cdot \arrow[d, Rightarrow, "t"] \\
C \arrow[r, "i"] & A
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\cdot \arrow[r, "j'"] \arrow[d, Rightarrow, "k_2"'] & \cdot \arrow[d, Rightarrow, "t'"] \\
\cdot \arrow[r, "j"] & B
\end{tikzcd}
\quad \Rightarrow \exists \text{ com diag} \quad
\begin{tikzcd}
\cdot \arrow[r, "w"] \arrow[d, Rightarrow, "x"'] & \cdot \arrow[d, Rightarrow, "k_1"] \\
\cdot \arrow[r, Rightarrow, "k_2"] & C
\end{tikzcd}$$

$rti'w = rik_1w = uk_1w$ and $r't'j'x = r'jk_2x = uk_2x = uk_1w$

Since $sci'w = ati'w = aik_1w = aik_2x = vk_2x = a'jk_2x = a't'j'x = sc'j'x$

$\overset{FR3)}{\Rightarrow} \exists\ w' \in S$ s.t. $ci'ww' = c'j'xw'$

So $rti'ww' = rik_1ww' = uk_1ww'$

$r't'j'xw' = r'jk_2xw' = uk_2xw' = uk_1ww'$

and $bci'ww' = bc'j'xw'$

3° is similiar.

Def: Let S be multiplicative system of add cate $\mathcal{K}$. Then $\mathrm{Obj}(S^{-1}\mathcal{K}) = \mathrm{Obj}(\mathcal{K})$

$\mathrm{Hom}_{S^{-1}\mathcal{K}}(X, Y)$ = The set of equivalent class of right fraction from X to Y.

$\Rightarrow S^{-1}\mathcal{K}$ is a cate with identity $\mathrm{Id}_X/\mathrm{Id}_X = s/s$, where $s: X \to X \in S$. is arbitary.

rmk: Assume the equivalent class of right fraction from X to Y. is set! not a class.

For any morphism a/r and b/s from X to Y. $\exists\ t \in S$. s.t

$$a/r = a'/t, \quad b/s = b'/t.$$

This follows by

$$\begin{tikzcd}
\cdot \arrow[r, "s'"] \arrow[d, Rightarrow, "r'"'] & \cdot \arrow[d, Rightarrow, "r"] \\
\cdot \arrow[r, Rightarrow, "s"] & X
\end{tikzcd}$$

Let $t = rs' = sr'$.

$a' = as'$

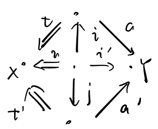
$b' = br'$

$$\begin{tikzcd}
 & \cdot \arrow[dl, Rightarrow, "r"'] \arrow[dr, "a"] & \\
X & \cdot \arrow[l, Leftarrow] \arrow[u, "s'"'] \arrow[d, "\mathrm{Id}"] \arrow[r] & Y \\
 & \cdot \arrow[ul, Rightarrow, "t=sr"] \arrow[ur, "a'=as'"'] &
\end{tikzcd}$$

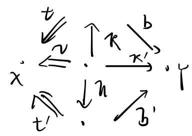
Define $a/r + b/s := \dfrac{a'+b'}{t}$. To show it's well-defined

lem: If $(a,t) \sim (a',t')$, $(b,t) \sim (b',t')$ Then $(a+b, t) \sim (a'+b', t')$

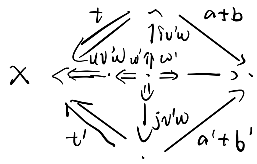
Pf: $\exists$ com diag



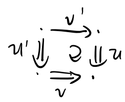
&



Goal :



By FR2)

$\exists$  $\Rightarrow$ $uv' = vu'$

$tiv' = tku'$

$\overset{FR3)}{\Rightarrow}$ $\exists w$ s.t $iv'w = ku'w$. $\Rightarrow$ $tiv'w = tku'w$ i.e $uv'w = vu'w$

$\Rightarrow t'jv'w = t'hu'w$.

$tiv'w = uv'w$

$= t'jv'w$

$\overset{FR3)}{\Rightarrow}$ $\exists w'$ $jv'ww' = hu'ww'$ $(iv'ww' = ku'ww')$

Then

$$
\begin{aligned}
(a+b)\,iv'ww' &= aiv'ww' + biv'ww' \\
&= i'v'ww' + bku'ww' \\
&= i'v'ww' + k'u'ww' \\
&= a'j'v'ww' + b'hu'ww' \\
&= a'+b'\,(jv'ww')
\end{aligned}
$$

Some Fact :

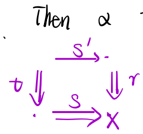
Let S be multiplicative system of add cate K, Then the quotient cate $S^{-1}K$ is an add cate.

① By addivity of right fraction. $\mathrm{Hom}_{S^{-1}K}(X, Y)$ is an abelian group, with zero element $0/t = 0/s$

where $t: Z \to X$ and $s: Z' \to X$ are any morphism in S

$\frac{a}{r} + \frac{b}{s} = \frac{a'+b'}{t} = \frac{b'+a'}{t} = \frac{b}{s} + \frac{a}{r}$. $\frac{a}{r} + \frac{0}{s} = \frac{a'+0}{t} = \frac{0}{s} + \frac{a}{r}$ $\quad \frac{a'}{t} \overset{H}{=} \frac{a}{r}$

let $\alpha = \frac{a}{s} \in \mathrm{Hom}_{S^{-1}K}(X, Y)$, where $s \in S$. Then α is zero map $\Leftrightarrow$ $\exists\, t \in S$ s.t $st \in S$ and at is the zero map in K.



$\Rightarrow$ α is zero map $\frac{a}{s} + \frac{b}{r} = \frac{at}{st} + \frac{bs'}{rs'} = \frac{at + bs'}{st} = \frac{bs'}{rs'}$ $\Rightarrow \exists\, t \in S$ s.t $st \in S$

$at \in K$ zero map

$\Leftarrow$ $\frac{a}{s} + \frac{b}{t} = \frac{at + bs'}{st} = \frac{b}{t}$ $\Rightarrow$ $\frac{a}{s}$ zero map

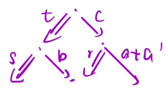
② With the condition of composing being admitted. We have.

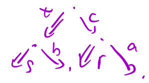
$b/s \circ (a/r + a'/r) = b/s \circ a/r + b/s \circ a'/r$

$(a/r + a'/r) \circ b/s = a/r \circ b/s + a'/r \circ b/s$

Pf : 2° LHS $= \frac{a+a'}{r} \circ \frac{b}{s} = \frac{(a+a')c}{st}$ RHS $= \frac{ac}{st} + \frac{a'c}{st}$

$= \frac{(a+a')c}{st}$

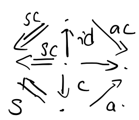




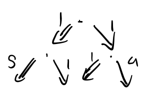
---

iii) 允许合成,

$\frac{ac}{sc} = \frac{a}{s}$ $\forall s \in S.\ c \in S.$

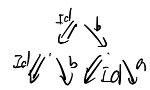


$a/1 \circ 1/s = a/s \quad \Leftarrow$



But $a/1 \circ 1/s$ and $1/s \circ a/1$ can't be defined simultaneously.

$a/\mathrm{Id} \circ b/\mathrm{Id} = ab/\mathrm{Id}. \Leftarrow$



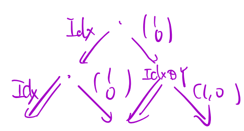
iv) For $X, Y \in S^{-1}\mathcal{K}$, $X \oplus Y$ is just the direct sum in $\mathcal{K}$.

In fact. $\exists$ morphism in $S^{-1}\mathcal{K}$. $X \underset{p_1}{\overset{\sigma_1}{\rightleftarrows}} X \oplus Y \underset{\sigma_2}{\overset{p_2}{\rightleftarrows}} Y$. where $\sigma_1 = \binom{1}{0}/\mathrm{Id}_X$ $\sigma_2 = \binom{0}{1}/\mathrm{Id}_Y$.

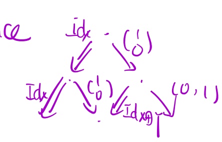
$p_1 = (1,0)/\mathrm{Id}_{X\oplus Y}$ $p_2 = (0\ 1)/\mathrm{Id}_{X\oplus Y}$

s.t $p_1\sigma_1 = \mathrm{Id}_X/\mathrm{Id}_X$ $p_2\sigma_2 = \mathrm{Id}_Y/\mathrm{Id}_Y$ $p_2\sigma_1 = p_1\sigma_2 = 0$ $\sigma_1 p_1 + \sigma_2 p_2 = \mathrm{Id}_{X\oplus Y}/\mathrm{Id}_{X\oplus Y}$

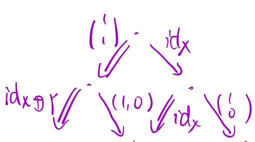
$p_1\sigma_1 = (1,0)/\mathrm{Id}_{X\oplus Y} \circ \binom{1}{0}/\mathrm{Id}_X = (1,0)\binom{1}{0}/\mathrm{Id}_X\,\mathrm{Id}_X = \mathrm{Id}_X/\mathrm{Id}_X.$



$p_2\sigma_1 = 0$ since



$\sigma_1 p_1 + \sigma_2 p_2 = \binom{1}{0}\mathrm{id}_X/\mathrm{id}_{X\oplus Y}(1\ 1) + \binom{0}{1}\mathrm{id}_Y/\mathrm{id}_{X\oplus Y}(1\ 1) = \binom{1}{1}/(1\ 1) = \mathrm{Id}_{X\oplus Y}/\mathrm{Id}_{X\oplus Y}$

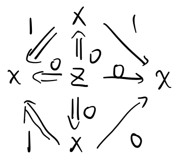


v) Let $f \in \mathrm{Hom}_{\mathcal{K}}(X, Y)$. $f \in S$. Then $f/\mathrm{Id}_X \in \mathrm{Hom}_{S^{-1}\mathcal{K}}(X, Y)$ is isom in $S^{-1}\mathcal{K}$ with inverse Id_X/f

$f/Id_X \circ Id_X/f = f/f = Id_Y/Id_Y$.

ii) For $X \in Obj(S^{-1}K)$ $\quad X \simeq 0 \Leftrightarrow \exists$ obj Z s.t. $0: Z \to X \in S$.

$\Leftarrow$

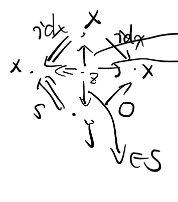


$\Rightarrow \frac{0}{1} = \frac{1}{1} \in Hom_{S^{-1}K}(X, X)$

$\Rightarrow \{X \xleftarrow{1} X \xrightarrow{1} X\} \sim \{X \xleftarrow{1} X \xrightarrow{0} X\} \Rightarrow X \sim 0$.

$\Rightarrow X \simeq 0$. Then $Id_X/Id_X = 0/s$.

want to show $0 \in S$



$\Rightarrow Z \to X \in S$.

$\Rightarrow Z \to X \in S$.

$\Rightarrow 0: Y \to X \in S$.

Define. localization functor $F: K \to S^{-1}K$ as: F keeps objs. For any $f \in Hom_K(X, Y)$. Define.

$F(f) = f/Id_X \in Hom_{S^{-1}K}(X, Y)$ By def $F(f+g) = F(f) + F(g)$. $\Rightarrow F$ additive functor.

Prop 3.2.7: let S be a multiplicative sys in add cate K. Then $F: S \to S^{-1}K$ is additive.

which making morphism S into isoms.

Universal property. If $H: K \to \mathcal{C}$. add functor makes morphism in S into isom in $\mathcal{C}$.

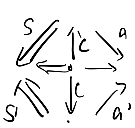
Then there's unique add functor $G: S^{-1}K \to \mathcal{C}$. s.t $H = GF$.

$$\begin{tikzcd} K \arrow[rr, "H"] \arrow[dr, "F"'] & & \mathcal{C} \\ & S^{-1}K \arrow[ur, "G"'] & \end{tikzcd}$$

For any $a/s \in Hom_{S^{-1}K}(X, Y)$. Define $G(a/s) = H(a) H(s)^{-1}$, This is well-defined. Since.

if $a/s \sim a'/s'$, Then $G(a/s) = G(ac/sc)$ Since $H(a) H(s)^{-1} = H(ac) H(sc)^{-1}$

$= H(a) H(c) (H(s) H(c))^{-1}$



The diag is com since. for any morphism f.

$c \in S$ since s and sc is

$= H(a) H(s)^{-1}$

$GF(f) = G(f/Id_X) = H(f) H(Id_X)^{-1}$

$= H(f)$.

G is unique, if $\exists\ G'$ s.t. $G'F = H$. Then

$$G'(a/s) = G'(a/\mathrm{Id})\, G'(\mathrm{Id}/s) = G'F(a) \cdot G'(s/\mathrm{Id})^{-1} = H(a)H(s)^{-1} = G(a/s)$$

§3.3.

We need left fraction and right fraction just as proj module and inj module

S multiplicative system in K, $X, Y \in K$. Define left fraction $(s,b): X \to Y$ as

$$X \cdot \xrightarrow{b} \cdot \overset{s}{\Leftarrow} Y.$$

$(S,b) \underset{l}{\sim} (r,a)$ if the following diag commutes

$$\begin{tikzcd}[row sep=small] & \cdot \arrow[d] & \\ \cdot \arrow[ur, "b"] \arrow[r] \arrow[dr, "a"'] & \cdot & Y \arrow[ul, Rightarrow, "s"'] \arrow[l, Rightarrow, "t"'] \arrow[dl, Rightarrow, "r"] \\ & \cdot \arrow[u] & \end{tikzcd}$$ where $t \in S$.

It's an equivalent relation. Denote $r\backslash a: X \to Y$ left fraction $s\backslash b$ right fraction

Define. $s\backslash b \circ r\backslash a = ts\backslash ca$, where t, c are given by

$$\begin{tikzcd}[column sep=small] & & \cdot & & \\ & \cdot \arrow[ur, "c"] & & \cdot \arrow[ul, Rightarrow, "t"'] & \\ \cdot \arrow[ur, "a"] & & \cdot \arrow[ul, Rightarrow, "r"'] \arrow[ur, "b"] & & \cdot \arrow[ul, Rightarrow, "s"'] \end{tikzcd}$$

$$\begin{tikzcd} \cdot & \cdot \arrow[l, Rightarrow, "r"'] \\ \cdot \arrow[u, Rightarrow, "s"] & \cdot \arrow[u, Rightarrow, "s'"'] \arrow[l, "r'"] \end{tikzcd}$$

This is well-defined. The additivity is similar. $s\backslash b + r\backslash a = sr'\backslash br' + rs'\backslash as' = rs'\backslash br' + as'$

Define $LS^{-1}K$ as below: $\mathrm{Obj}(LS^{-1}K) = \mathrm{obj}(K)$

$\mathrm{Hom}_{LS^{-1}K}(X, Y) = \{$equivalent class of left fraction from X to $Y\}$.

$\mathrm{Id}_X\backslash\mathrm{Id}_X = s\backslash s \in \mathrm{Hom}_{LS^{-1}K}(X, X)$ $\quad s: X \to Y$ any morphism.

$LS^{-1}K$ is an add cate. $X \simeq 0$ in $LS^{-1}K \Leftrightarrow \exists\ Z \in \mathrm{Obj}(LS^{-1}K)$ s.t. $0: X \to Z \in S$.

$\alpha = s\backslash a \in \mathrm{Hom}_{LS^{-1}K}(X, Y)$, $s \in S$ $\quad \alpha = 0 \Leftrightarrow \exists\ t' \in S$ s.t. $t's \in S$ & $t'a = 0$ in K.

Let $f \in \mathrm{Hom}_K(X, Y)$, $f \in S$. Then $\mathrm{Id}_Y\backslash f \in \mathrm{Hom}_{LS^{-1}K}(X, Y)$ is isom with inverse $f\backslash \mathrm{Id}_Y$.

Define localization functor. $F': K \to LS^{-1}K$. F' preserves objs. $f \in \mathrm{Hom}_K(X, Y)$ $F'(f) = \mathrm{Id}_Y\backslash f$.

F' is add functor. We have the universal property

If $H: K \to L$ makes morphism S into isom in L. Then there's unique add functor $G: LS^{-1}K \to L$ s.t. $H = GF'$.

Prop 3.3.2: There's unique cate isom $\eta : S^{-1}R \to LS^{-1}R$. s.t $F' = \eta F$, where η preserves objs. For any morphism $f : X \to Y$. $\eta(f/Id_X) = Id_Y \backslash f$. And $\eta(a/s) = t\backslash b$ where $t \in S$. s.t the diag commutes

$$\begin{tikzcd}
 & Z \arrow[dl, Rightarrow, "s"'] \arrow[dr, "a"] & \\
X \arrow[dr, "b"'] & & Y \arrow[dl, Rightarrow, "t"] \\
 & \cdot &
\end{tikzcd}$$

Pf: By universal property, there's unique cate isom $\eta : S^{-1}R \to LS^{-1}R$. $F' = \eta F$.

For any morphism $f : X \to Y$ $\eta(f/Id_X) = F'(f)F'(Id_X)^{-1} = Id_Y \backslash f$. Moreover

$$\eta(a/s) = \eta(a/Id_Z \circ Id_Z/s) = \eta(a/Id_Z) \circ \eta(Id_Z/s) = Id_Y\backslash a \circ (Id_Y \backslash s)^{-1}$$
$$= Id_Y \backslash a \circ s\backslash Id_Y$$

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & \cdot & & \\
 & \cdot \arrow[ur, "b"] & & \cdot \arrow[ul, Rightarrow, "t"'] & \\
\cdot \arrow[ur, "Id_Y"] & & \cdot \arrow[ul, Rightarrow, "s"] \arrow[ur, "a"'] & & \cdot \arrow[ul, Rightarrow, "Id_Y"']
\end{tikzcd} \longrightarrow = t\backslash b.$$

Advantage: Avoiding to check well-definedness of η.