3.5 饱和相容乘法系与厚子范畴的一一对应

Recall Thm3.4.2 $\underline{S}$ compatible multi system of Δ cat. $(K, [1])$

then we get $\underline{\underline{\ }}$ Verdier Quotient $S^{-1}K$ which is a Δ cat.

Verdier functor $F: K \longrightarrow S^{-1}K$ is Δ functor.

$\forall s \in S$ $F(s)$ is iso

If S saturated, $F(s)$ iso $\Leftrightarrow$ $s \in S$.

Goal: From Δ-subcat. $\mathcal{B} \longrightarrow S = \varphi(\mathcal{B})$

$\underbrace{K/\mathcal{B} := S^{-1}K}_{\Delta\ \text{cat}}$ $\underbrace{F: K \to K/\mathcal{B}}_{\Delta.\ \text{functor}}$ $\mathcal{B} \subseteq \ker F.$

If $\mathcal{B}$ thick, then $S = \varphi(\mathcal{B})$ saturated. $\mathcal{B} = \ker F$.

(Def 3.5.1) Δ subcat. $\mathcal{B}$ is a thick subcat. if it satisfy (T)

(T) $X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1]$ d.Δ. in K, $Z \in \mathcal{B}$.

$\underline{f = hg}$

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \\ & \underline{W} \arrow[ur, "h"'] & \end{tikzcd}$$

$W \in \mathcal{B}$. then $X \in \mathcal{B}$, $Y \in \mathcal{B}$.

thick $\left\{ \begin{array}{l} \underline{\Delta\ \text{subcat}}. \\ \underline{(T)} \end{array} \right.$

(Prop3.5.2) (1) If $\mathcal{B}$ is full subcat. of Δ cat $(K, [1])$, $\underline{\mathcal{B}\ \text{has}\ 0}$, closed under automorphism. Then $\mathcal{B}$ is thick subcat. iff $[1]$ is $\underline{\text{automorphism}}$ of $\mathcal{B}$ and $\mathcal{B}$ satis. (T).

pf: $\Longrightarrow$ $\mathcal{B}$ thick $\to$ Δ subcat. $\longrightarrow$ $[1]$ is auto. of $\mathcal{B}$.

$\Longleftarrow$ We need to show $\mathcal{B}$ is a Δ-subcat.

- $\mathcal{B}$ is an additive cat. ✓

(i) has 0. ✓

(ii) $\mathrm{Hom}_K(X, Y)$ Abel group. distribution laws ✓. (Δ cat: add. cat. $\mathcal{B}$: full subcat.)

(iii) $\forall X, Y \in \mathcal{B}$, $X \oplus Y \in \mathcal{B}$.

$X \xrightarrow{\binom{1}{0}} X \oplus Y \xrightarrow{(0,1)} \underline{\underline{Y}} \xrightarrow{0} X[1]$ d.Δ. $\underline{Y \in \mathcal{B}}$.

$$\begin{tikzcd} X \arrow[rr, "\binom{1}{0}"] \arrow[d, "id_X"'] & & X \oplus Y \\ \boxed{X} \arrow[urr, "\binom{1}{0}"'] & & \end{tikzcd}$$

$\underline{X \in \mathcal{B}}$. then by (T) $\underline{X \oplus Y \in \mathcal{B}}$.

- $\mathcal{B}$: closed under auto ✓
- $[1]$ auto of $\mathcal{B}$ ✓
- If $X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1]$ d.Δ. $X, Z \in \mathcal{B}$. then $Y \in \mathcal{B}$.

$Z \in \mathcal{B}$.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "id_X"'] & & Y \\ & X \arrow[ur, "f"'] & \end{tikzcd}$$

$Y \in \mathcal{B}$ (by (T))

∵ $\mathcal{B}$ is △ subcat. $\Longrightarrow$ thick

(2) If $\mathcal{B}$ is a △ subcat. of $\mathcal{K}$, Then $\mathcal{B}$ is thick $\Longleftrightarrow$ $\mathcal{B}$ is closed under direct summand
i.e. if $\underline{\boxed{X_1\oplus X_2 \in \mathcal{B}}, \text{ then } X_1\in\mathcal{B},\ X_2\in\mathcal{B}}$.

pf: "$\Longrightarrow$" thick If $X_1\oplus X_2\in\mathcal{B}$,

$X_2 \to X_2\oplus X_1 \to X_1 \xrightarrow{0} X_2[1]$ d.△.

$X_1[-1] \xrightarrow{0} X_2 \to X_2\oplus X_1 \to X_1$ d.△.

$\underline{X_2\oplus X_1\in\mathcal{B}}$.

$$\begin{tikzcd}
X_1[-1] \arrow[rr, "0"] \arrow[dr, "0"'] & & X_2 \\
& 0 \arrow[ur, "0"'] &
\end{tikzcd}$$

$0\in\mathcal{B}$. by (T) $X_1[-1],\ X_2\in\mathcal{B}$.

$\Longrightarrow X_1, X_2\in\mathcal{B}$.

"$\Longleftarrow$" If $X\xrightarrow{f} Y\to Z\to X[1]$ is a d.△.

$f=hg$,

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \\
& W \arrow[ur, "h"'] &
\end{tikzcd}$$

, $W\in\mathcal{B}$. $Z\in\mathcal{B}$. we show $X, Y\in\mathcal{B}$

by (TR4)

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, equal] & W \arrow[r, "j"] \arrow[d, "h"] & L \arrow[r, "k"] \arrow[d, dashed] & X[1] \arrow[d, equal] \\
X \arrow[r, "f"] & Y \arrow[r] \arrow[d] & Z \arrow[r, "i"] \arrow[d, dashed] \arrow[dr, dashed] & X[1] \arrow[d, "{g[1]}"] \\
& M \arrow[r, equal] \arrow[d] & M \arrow[r] \arrow[d] \arrow[ur, phantom, "\circlearrowleft"] & W[1] \\
& W[1] \arrow[r] & L[1] &
\end{tikzcd}$$

$M[-1] \to L \to Z \to M$ is d.△.

(TR4)

$$\begin{tikzcd}
Z \arrow[r, "i"] \arrow[d, equal] & X[1] \arrow[r, "{-f[1]}"] \arrow[d, "{g[1]}"] & Y[1] \arrow[r] \arrow[d, dashed] & Z[1] \arrow[d] \\
Z \arrow[r] & W[1] \arrow[r] \arrow[d, "{j[1]}"] & N \arrow[r] \arrow[d, dashed] & Z[1] \\
& L[1] \arrow[r, equal] \arrow[d, "{-k[1]}"] & L[1] \arrow[r] \arrow[d, "\boxed{v}"] & X[2] \\
& X[2] \arrow[r, "{-f[2]}"] \arrow[ur, phantom, "\circlearrowleft"] & Y[2] &
\end{tikzcd}$$

$$\begin{tikzcd}
L[1] \arrow[r, equal] \arrow[d, "{-k[1]}"'] & L[1] \arrow[d, "v"] \\
X[2] \arrow[r, "{-f[2]}"] \arrow[d, "{g[2]}"'] & Y[2] \\
W[2] \arrow[ur, "{-h[2]}"'] &
\end{tikzcd}$$

$L \xrightarrow{v[-1]} Y[1] \to N \to L[1]$ is d.△.

$$\begin{tikzcd}
L \arrow[r, "{v[-1]}"] \arrow[d, "-k"'] & Y[1] & \\
X[1] \arrow[ur, "{-f[1]}"] \arrow[rr, "{g[1]}"'] & & W[1] \arrow[ul, bend right, "{-h[1]}"']
\end{tikzcd}$$

$(f=hg)$

∵ $L \xrightarrow{k} X[1] \xrightarrow{-g[1]} W[1] \xrightarrow{-j[1]} L[1]$ is a d.△.

$-g[1]k=0 \qquad v[-1] = -h[1]\, g[1]\, (-k) = 0$

$-g[1]k=0$ $\quad v[-1] = -h[1]\,\underrightarrow{g[1](-k)} = 0$

by (Lemma 1.3.5) (2) (i)$\Leftrightarrow$(iii)

$\underline{\left(X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX \text{ d.o.} \quad \boxed{w=0} \Leftrightarrow Y = X \oplus Z. \right)}$

$v[-1]=0 \quad v=0. \quad \Rightarrow \quad N \cong \underline{L[1] \oplus Y[1]}.$

$\because Z, W \in \mathcal{B}, \quad W[1], Z[1] \in \mathcal{B}. \Rightarrow N \in \mathcal{B}. \Rightarrow L[1] \in \mathcal{B}, Y[1] \in \mathcal{B}. \Rightarrow Y \in \mathcal{B}.$

$\therefore X \in \mathcal{B}.$

(Lemma 3.5.3) If S is a saturated compatible multi. sys. of $\triangle$ cat $(K, [1])$

Let $\Psi(S)$ be the full subcat.

$\underline{\Psi(S)} := \{ Z \in K \mid \exists \text{ d.o. } X \xrightarrow{f} Y \to \underset{\sim}{Z} \to X[1] \text{ in } K, \text{ s.t. } \underline{f \in S} \}.$

then $\Psi(S)$ is a thick subcat.

pf: (Prop 3.5.2(v))

$X \xrightarrow{id_X} X \to \underline{0} \to X[1]$ d.o. $\quad id_X \in S, \quad \therefore 0 \in \Psi(S)$

$\forall Z \in \Psi(S), \; Z' \cong Z$

$\exists$ d.o. $\quad \underline{f \in S.}$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d, "\wr"] & X[1] \arrow[d, equal] \\
X \arrow[r, "f"] & Y \arrow[r] & Z' \arrow[r] & X[1]
\end{tikzcd}$$

$\therefore Z' \in \Psi(S)$

$\Psi(S)$ is closed under auto.

Let $Z \in \Psi(S)$ $\exists$ d.o. $X \xrightarrow{f} Y \to Z \to X[1]. \quad f \in S.$

$X[1] \xrightarrow{f[1]} Y[1] \to Z[1] \to X[2] \quad \underline{(FR4)} \Rightarrow f[1] \in S.$

$\therefore Z[1] \in \Psi(S)$

similar $Z[-1] \in \Psi(S) \quad \therefore [1]$ is auto of $\mathcal{B}$.

- $\Psi(S)$ satisfy (T)

Suppose $X \xrightarrow{f} Y \to Z \to X[1]$ d.o.,

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \\
& W \arrow[ur, "h"'] &
\end{tikzcd}$$

$Z, W \in \Psi(S)$

next we show $X, Y \in \Psi(S)$

By (Coro 3.4.3) In $\underline{S^{-1}K}$, $\underline{W = 0 = Z}$.

$$\begin{tikzcd}
F(X) \arrow[rr, "F(f)"] \arrow[dr] & & F(Y) \\
& F(W) \arrow[ur] \arrow[d, equal] & \\
& 0 &
\end{tikzcd}$$

$F(f)=0$ in $S^{-1}K$, where $F: K \to S^{-1}K$.

By lemma 1.3.7. $F(X) \xrightarrow{\boxed{F(f)}} F(Y) \to \underline{F(Z)} \to F(X[1]) \quad \because F(Z)=0 \quad \therefore F(f)$ is iso.

$\therefore$ in $S^{-1}K$ $\quad F(X)=F(Y)=0 \quad X=Y=0$ in $\underline{S^{-1}K}$.

$\underline{\text{Then (Coro 3.4.3)} \quad X, Y \in \Psi(S)}$

$\therefore \Psi(S)$ thick.

Rm without saturated, we can't prove $\varphi(S)$ thick subcat. in general

Lemma. 3.5.5. If $\mathcal{B}$ is a △ subcat. of △ cat $(\mathcal{K}, [1])$
Then $\varphi(\mathcal{B}) = \{ f : X \to Y \mid \exists$ d.△. $X \xrightarrow{f} Y \to Z \to X[1]$, s.t. $Z \in \mathcal{B} \}$. is a compatible multi. system of $\mathcal{K}$ which contains iso. $\mathcal{B} \subseteq \ker F$. $F : \mathcal{K} \to S^{-1}\mathcal{K}$. $S = \varphi(\mathcal{B})$

Moreover, if $\mathcal{B}$ is thick, $\varphi(\mathcal{B})$ saturated. $\mathcal{B} = \ker F$.

pf: (1) $\forall$ iso $f : X \to Y$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d, equal] & TX \arrow[d, "F(f)"] \\
Y \arrow[r, "id_Y"] & Y \arrow[r] & 0 \arrow[r] & TY
\end{tikzcd}$$

$0 \in \mathcal{B}$ $f \in \varphi(\mathcal{B})$ $\varphi(\mathcal{B})$ contains iso.

(FR1) $X \xrightarrow{id_X} X \to 0 \to X[1]$ $0 \in \mathcal{B}$. $id_X \in \varphi(\mathcal{B})$ $\forall X$.

$f : X \to Y$, $g : Y \to Z$. $f, g \in \varphi(\mathcal{B})$ we show $gf \in \varphi(\mathcal{B})$

by (TR4)

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[d, "gf"] & & \\
Y \arrow[r, "g"] \arrow[d] & Z \arrow[r] \arrow[d] & C_3 \arrow[r] \arrow[d, equal] & Y[1] \arrow[d] \\
C_1 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & C_3 \arrow[r] \arrow[d] & C_1[1] \\
X[1] \arrow[r, equal] & X[1] \arrow[r, "f{[1]}"] & Y[1] &
\end{tikzcd}$$

where $C_1, C_3 \in \mathcal{B}$.

$C_1 \to C_2 \to C_3 \to C_1[1]$ d.△.

$C_2 \in \mathcal{B}$.

$gf \in \varphi(\mathcal{B})$.

(FR2) If $X \xrightarrow{s} Y \xleftarrow{f} Z$, $s \in \varphi(\mathcal{B})$

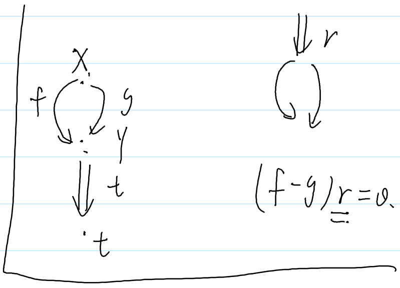
$$\begin{tikzcd}
 & Z \arrow[r, "hf"] \arrow[d, "f"] & k \arrow[r] \arrow[d, equal] & X'[1] \arrow[r, "t{[1]}"] \arrow[d] & Z[1] \arrow[d] \\
(X \arrow[r, "s"] & Y) \arrow[r, "h"] & k \arrow[r] & X[1] \arrow[r] & Y[1]
\end{tikzcd}$$

d.△. (first row) d.△. (second row) $\underline{k \in \mathcal{B}}$.

$$\begin{tikzcd}
X' \arrow[r, "t"] \arrow[d, "g"'] & Z \arrow[r] \arrow[d, "f"] & k \arrow[r] \arrow[d, equal] & X'[1] \arrow[d] \\
X \arrow[r, "s"] & Y \arrow[r] & k \arrow[r] & X[1]
\end{tikzcd}$$

$\underline{t \in \varphi(\mathcal{B})}$

(FR3) If $f, g : X \to Y$, $t : Y \to Z$ $t \in \varphi(\mathcal{B})$



$tf = tg$. $\underline{t(f-g) = 0.}$

$k \xrightarrow{u} Y \xrightarrow{t} Z \to k[1]$ $k[1] \in \mathcal{B}$, $\underline{k \in \mathcal{B}.}$

$\mathrm{Hom}_{\mathcal{K}}(X, \_)$ ↷.

$$\mathrm{Hom}_{\mathcal{K}}(X, k) \xrightarrow{\mathrm{Hom}_{\mathcal{K}}(X,u)} \mathrm{Hom}_{\mathcal{K}}(X, Y) \xrightarrow{\mathrm{Hom}_{\mathcal{K}}(X,t)} \mathrm{Hom}_{\mathcal{K}}(X, Z) \quad \text{exact}$$

$\mathrm{Hom}_{\mathcal{K}}(X,t)\ (f-g) = t(f-g) = 0$ $f - g \in \ker \mathrm{Hom}_{\mathcal{K}}(X,t) = \mathrm{Im}\, \mathrm{Hom}_{\mathcal{K}}(X, u)$.

$\exists s : X \to k$ $\mathrm{Hom}_{\mathcal{K}}(X, u)(s) = us = f - g$

[illegible]

$\mathrm{Hom}_k(X,t)(f-g) = t(f-g) = 0$ $\quad f-g \in \ker \mathrm{Hom}_k(X,t) = \mathrm{Im}\,\mathrm{Hom}_k(X,u)$.

$\exists s : X \longrightarrow K \quad \mathrm{Hom}_k(X,u)(s) = us = f-g$

$W \xrightarrow{s'} X \xrightarrow{s} K \to W[1]$ $\underline{\underline{\text{d.}\Delta}}$. $\quad K \in \mathcal{B}. \quad s' \in \varphi(\mathcal{B})$

$ss' = 0 \quad (f-g)s' = uss' = 0 \quad fs' = gs'$, $s' \in \varphi(\mathcal{B})$.

(FR4) If $\underline{f \in \varphi(\mathcal{B})}$, (iff $f[1] \in \varphi(\mathcal{B})$)

$\exists$ d.Δ $\quad X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1] \qquad Z \in \mathcal{B}$.

$Y[1] \xrightarrow{f[1]} Y[1] \longrightarrow \underline{Z[1]} \longrightarrow X[2] \qquad Z[1] \in \mathcal{B}. \qquad f[1] \in \varphi(\mathcal{B})$

$f[-1] \in \varphi(\mathcal{B})$.

(FR5)
$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, dashed, "h"] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1] \end{tikzcd}$$
d.Δ

d.Δ $\quad f, g \in \varphi(\mathcal{B}) \quad \exists h \in \varphi(\mathcal{B})$ s.t. $\circlearrowright$

by (Lemma 4x4)
$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & X[1] \arrow[d] \\ X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & X'[1] \arrow[d] \\ X'' \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[d] & Z'' \arrow[r] \arrow[d] & X''[1] \arrow[d] \\ X[1] \arrow[r] & Y[1] \arrow[r] & Z[1] \arrow[r] & X[2] \end{tikzcd}$$

$\therefore \exists h$ s.t. $\circlearrowright$

$f, g \in \varphi(\mathcal{B}) \quad X'', Y'' \in \mathcal{B}$.

$\therefore Z'' \in \mathcal{B}. \quad h \in \varphi(\mathcal{B})$

$\therefore \varphi(\mathcal{B})$ is a compatible multiplication system

Denote $S := \varphi(\mathcal{B})$

$\forall Z \in \mathcal{B} \quad 0 \xrightarrow{0} Z \xrightarrow{id_Z} Z \longrightarrow 0$ d.Δ. $\quad 0 : 0 \to Z \in \varphi(\mathcal{B})$

by (Prop 3.2.7) $\therefore F(0) : F(0) \to F(Z)$ iso. $\underline{F(Z) = 0} \quad \mathcal{B} \subseteq \ker F$.

Next suppose $\mathcal{B}$ is thick, then we show $\varphi(\mathcal{B})$ saturated.

If $r : V \to X$, $s : X \to Y$, $t : Y \to Z$, s.t. $sr \in \varphi(\mathcal{B})$, $\underline{ts \in \varphi(\mathcal{B})}$. then we show $s \in \varphi(\mathcal{B})$

pf: by (TR4)
$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "s"'] & X \arrow[d, "ts"] & & \\ Y \arrow[r, "t"] \arrow[d, "g"'] & Z \arrow[r] \arrow[d] & C \arrow[r, "f"] \arrow[d, equal] & Y[1] \arrow[d, "g[1]"] \\ S \arrow[r, dashed, "p"] \arrow[d] & W \arrow[r, dashed] \arrow[d] & C \arrow[r, "g[1]f"] \arrow[d, "f"] & S[1] \\ X[1] \arrow[r, equal] & X[1] \arrow[r] & Y[1] & \end{tikzcd}$$

$\because ts \in \varphi(\mathcal{B})$

$\therefore W \in \mathcal{B}$.

consider d.Δ. $\quad V \xrightarrow{sr} Y \xrightarrow{h} K \longrightarrow V[1]$

$sr \in \varphi(\mathcal{B}) \qquad K \in \mathcal{B}$.

$-(sr)[1] \quad -h[1]$.

$\mathrm{Hom}_K(\_\_, S[1]) \curvearrowright \quad V[1] \longrightarrow Y[1] \longrightarrow K[1] \longrightarrow V[2]$

$$\mathrm{Hom}_K(K[1], S[1]) \xrightarrow{\mathrm{Hom}_K(-h[1], S[1])} \mathrm{Hom}_K(Y[1], S[1]) \xrightarrow{\mathrm{Hom}_K(-(sv)[1], S[1])} \mathrm{Hom}_K(V[1], S[1]) \quad \text{exact}$$

$gs = 0 \quad gsv = 0 \quad \therefore g[1](-sv[1]) = 0.$

i.e. $\mathrm{Hom}_K(-sv[1], S[1])(g[1]) = 0.$

$\therefore \exists k : k[1] \to S[1],\ \underline{g[1] = -kh[1]}.$

$$\begin{tikzcd} C \arrow[r, "{g[1]f}"] \arrow[dr, "{-h[1]f}"'] & S[1] \\ & \underline{k[1] \in \mathcal{B}} \arrow[u, "k"'] \end{tikzcd}$$

in d.△. $\quad \underline{C \xrightarrow{g[1]f} S[1]} \longrightarrow W[1] \longrightarrow C[1],\ W[1] \in \mathcal{B}.$

$\because \mathcal{B}$ is thick. $\quad \therefore C,\ S[1] \in \mathcal{B}. \quad S \in \mathcal{B}. \quad s \in \varphi(\mathcal{B})$

$\therefore \varphi(\mathcal{B})$ saturated.

At last. suppose $\underline{Z \in \ker F}$, $\quad 0 \xrightarrow{0} Z \xrightarrow{id_Z} Z \longrightarrow 0.$ d.△.

$0 \xrightarrow{F(0)} F(Z) \longrightarrow \underset{\substack{\| \\ 0}}{\underline{F(Z)}} \longrightarrow 0 \qquad F(0)$ iso.

$\because S = \varphi(\mathcal{B})$ saturated. by Thm 3.4.2 (iv) $\quad 0 : 0 \to Z \in \varphi(\mathcal{B})$

$Z \in \mathcal{B}.$

$\therefore \ker F = \mathcal{B}.$

Coro 3.5.6 K is △. cat. $\mathcal{S}$ is the class of saturated comp. multi. sys. of K.
$\mathcal{N}$ is the class of thick subcat.

Let $\psi : \mathcal{S} \to \mathcal{N} \qquad S \longmapsto \psi(S) \qquad \forall S \in \mathcal{S}$

$\varphi : \mathcal{N} \longrightarrow \mathcal{S} \qquad \mathcal{B} \longmapsto \varphi(\mathcal{B}) \qquad \forall \mathcal{B} \in \mathcal{N}.$

Then ψ, φ <u>保序</u>互逆映射.

$\psi(S) := \{ Z \in K \mid \exists$ d.△. $X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X$, s.t. $f \in S \}$ ①

$\varphi(\mathcal{B}) := \{ f : X \to Y \mid \exists$ d.△, $X \xrightarrow{f} Y \longrightarrow \underline{\underline{Z}} \longrightarrow X$, s.t. $Z \in \underset{\sim}{\mathcal{B}} \}$ ②

pf: show $\underline{\varphi\psi(S)} = \underline{S}$, $\underline{\psi\varphi(\mathcal{B})} = \underline{\mathcal{B}}$. $\quad \forall S \in \mathcal{S}.\ \forall \mathcal{B} \in \mathcal{N}.$

$\forall f : X \to Y \in S.$ ① $\exists$ d.△. $X \xrightarrow{f} Y \to Z \to X[1],\ \underline{Z} \in \psi(S)$

$f \in \varphi\psi(S)$ ②

$\forall f \in \varphi\psi(S),$ ② $\exists$ d.△. $\underline{X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1]}$, $Z \in \psi(S).$

① $\exists$ d.△, $X' \xrightarrow{s} Y' \longrightarrow Z \longrightarrow X'[1] \quad s \in S.$

by Prop 3.2.7. in $S^{-1}K \quad F(s)$ iso.

$F(X') \xrightarrow{F(s)} F(Y') \longrightarrow F(Z) \longrightarrow F(X'[1]) \qquad F(Z) = 0.$

$F(X) \xrightarrow{F(f)} F(Y) \longrightarrow \underline{F(Z)} \longrightarrow F(X[1]) \qquad F(f)$ iso.

by Thm 3.4.2, $\quad f \in S.$

$\therefore \varphi\psi(S) = S.$

$\forall Z \in \Psi\varphi(\mathcal{B})$. ① $\exists$ d.$\triangle$, $X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1]$ $f \in \varphi(\mathcal{B})$

② $\exists$ d.$\triangle$, $X \xrightarrow{f} Y \to Z' \to X[1]$. $Z' \in \mathcal{B}$. $Z \overset{u}{\cong} Z'$. $Z \in \mathcal{B}$.

$\forall Z \in \mathcal{B}$. ② d.$\triangle$, $0 \xrightarrow{0} Z \xrightarrow{id_Z} Z \longrightarrow 0$ $0: 0 \to Z \in \varphi(\mathcal{B})$.

$\exists$ d.$\triangle$, $0 \xrightarrow{0} Z \longrightarrow Z' \longrightarrow 0$ $Z' \in \Psi\varphi(\mathcal{B})$ $Z' \overset{u}{\cong} Z$, $Z \in \Psi\varphi(\mathcal{B})$

$\mathcal{B}$. $\triangle$ subcat. $S := \varphi(\mathcal{B})$

$\mathcal{K}/\mathcal{B} := S^{-1}\mathcal{K}$. : $\mathcal{K}$关于$\mathcal{B}$的 Verdier Quotient

$F: \mathcal{K} \to \mathcal{K}/\mathcal{B}$. : Verdier functor.

(Coro 3.5.7) Let $\mathcal{B}$ be the $\triangle$ subcat. then

(1) $F: \mathcal{K} \longrightarrow \mathcal{K}/\mathcal{B}$. $\triangle$ functor (Thm 3.4.2 (ii))

$\mathcal{B} \subseteq \ker F$ (Lemma 3.3.5)

$F(s)$ is iso., $\forall s \in \varphi(\mathcal{B})$ (Thm 3.4.2 (ii))

(2) If $H: \mathcal{K} \longrightarrow \mathcal{T}$ is $\triangle$ functor, $\forall$ object $X \in \mathcal{B}$. $H(X) = 0$ in $\mathcal{T}$.

then $\exists !$ $\triangle$ functor $G: \mathcal{K}/\mathcal{B} \longrightarrow \mathcal{T}$ s.t.

$$\begin{tikzcd}
\mathcal{K} \arrow[r, "H"] \arrow[dr, "F"'] & \mathcal{T} \\
& \mathcal{K}/\mathcal{B} \arrow[u, "G"']
\end{tikzcd}$$

(Thm 3.4.2 (iii))

(Lemma. 3.5.5)

(3) If $\mathcal{B}$ is thick, then $S := \varphi(\mathcal{B})$ saturated comp. multi sys. $\mathcal{B} = \ker F$.

$F(f)$ iso iff $f \in S := \varphi(\mathcal{B})$. a/s in $\mathcal{K}/\mathcal{B}$) is iso $\Longleftrightarrow$ $a \in S$. (coro 3.3.4)

3.6.

(Prop 3.6.1) $\mathcal{B}$ is a $\triangle$ subcat. , If $\forall$ index set I and a family of object $\underline{X_i,}$ $i \in I$, in $\mathcal{B}$. $\underline{\bigoplus_i X_i \text{ exists}}$, $\bigoplus_i X_i \in \mathcal{B}$. then $\mathcal{B}$ is thick.

pf: If $\underline{X \oplus Y \in \mathcal{B}.}$ we show $X \in \mathcal{B}$, $Y \in \mathcal{B}$.

$L := X \oplus Y \oplus X \oplus \cdots$

By Eilenberg's swindle $L = (X \oplus Y) \oplus (X \oplus Y) \oplus \cdots\cdots$ $\in \mathcal{B}$.

$L = X \oplus (Y \oplus X) \oplus (Y \oplus X) \oplus \cdots\cdots$ $Y \oplus X \overset{u}{\cong} X \oplus Y$

$L \overset{u}{\cong} X \oplus L$. $\in \mathcal{B}$.

d.$\triangle$, $X \xrightarrow{\binom{1}{0}} X \oplus L \xrightarrow{(0,1)} L \xrightarrow{0} X[1]$

$L \in \mathcal{B}$, $X \oplus L \in \mathcal{B}$. $\therefore X \in \mathcal{B}$.

$L' := Y \oplus X \oplus Y \oplus X \oplus \cdots$ $\Longrightarrow Y \in \mathcal{B}$. by Prop 3.5.2(2) $\mathcal{B}$ is thick