§ 4.5

Recall

By def: $\cdots 0 \to \underset{\substack{\uparrow \\ n}}{X} \to 0 \to 0 \cdots$ (stalk cpx of X) is hoproj $\Leftrightarrow$ X is proj

$\Leftarrow$ Prop 4.2.3 (1) $\mathrm{Hom}_{K(\mathcal{A})}(X, E) = 0$ for any acyclic cpx E

$\Rightarrow$ $\mathrm{Hom}_{K(A)}^{\bullet}(X, E)$ $\longrightarrow$ $\mathrm{Hom}^n(X,Y) = \prod_{p\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(X^p, Y^{p+n})$,

$f = (f^p)_{p\in\mathbb{Z}} \in \mathrm{Hom}^n(X,Y)$

$d^n(f) = ((d^n f)^p)_{p\in\mathbb{Z}} \in \mathrm{Hom}^{n+1}(X,Y)$

$(d^n f)^p = d_Y^{n+p} f^p + (-1)^{n+1} f^{p+1} d_X^p \in \mathrm{Hom}_{\mathcal{A}}(X^p, Y^{p+n+1})$

e.g. upper bounded proj cpx is a hoprojective cpx (prop 4.2.3 (1))

$$\mathrm{Hom}^n(X,Y) = \left(\begin{array}{l} 0 \\ \vdots \quad X^{-1} \longrightarrow Y^{-1+n} \\ f_0 : X \longrightarrow Y^n \\ 0 : X^1 \longrightarrow Y^{n+1} \\ \vdots \end{array}\right)$$

$$H^{n+1}(X,Y) = \left(\begin{array}{l} 0 \\ \vdots \\ f_0 \quad X \to Y^{n+1} \\ 0 \\ \vdots \end{array}\right)$$

$$H^{n+2}(X,Y) = \left(\begin{array}{l} 0 \\ \vdots \\ f_0 : X \to Y^{n+2} \end{array}\right)$$

Lem 4.4.3 $\mathcal{A}$ abelian cate. Then

① If $\{P_i^{\bullet}\}_{1\le i\le n}$ hoprojective cpx, $\Rightarrow \bigoplus_{i=1}^{n} P_i^{\bullet}$ is hoproj cpx

$$\begin{array}{rcl} \mathrm{Hom}^n(X^{\bullet}, Y^{\bullet}) & = & \mathrm{Hom}(X, Y^n) \\ & & \downarrow \\ \mathrm{Hom}^{n+1}(X^{\bullet}, Y^{\bullet}) & = & \mathrm{Hom}(X, Y^{n+1}) \\ & & \downarrow \\ H^{n+2}(X^{\bullet}, Y^{\bullet}) & = & \mathrm{Hom}(X, Y^{n+2}) \\ & & \uparrow \end{array}$$

acyclic complex $\Leftrightarrow$ X is proj obj

Recall:

- Thm 2.1 : $\forall X \in C^{-}(A)$ (resp $C^{-b}(A)$, pdim $<\infty$)

  $\exists P \xrightarrow{f} X$ quasi-iso , where $P \in K^{-}(P)$ (resp $K^{-b}(P)$, $K^{b}(P)$)

  projective c.p.x

- $K_{hproj}(A) = \{ P \in C(A) \mid \forall E^{\bullet} \in$ acyclic c.p.x . $\mathrm{Hom}^{\bullet}(P, E)$ acyclic c.p.x$\}$

- Cor 4.43. $K_{hproj}(A)$ are closed under:

  (1) finite sum

  (2) $\bigoplus_{i=0}^{\infty}$ (if $\exists \bigoplus_{i=0}^{\infty}$)

  ---

  (3). $X \to Y \to Z \to TX$

  two out of three $\in K_{hproj}(A)$, then so is the third.

  i.e. $K_{hproj}(A) = \mathrm{thick}(K(A))$

- $S: X \longrightarrow Y$ quasi-iso. , $P \in K_{hproj}(A)$

  $\mathrm{Hom}_{K(A)}(P, S) : \mathrm{Hom}_{K(A)}(P, X) \xrightarrow{\cong} \mathrm{Hom}_{K(A)}(P, Y)$

§. 4.5. Homotopy projective resolution for arbitrary complex.

- cocomplete abelian category $\mathcal{A}$:
  For $\mathbb{N} = I = \{0, 1, 2 \cdots \}$. If $M_i \in \mathrm{obj}\,\mathcal{A}$, Then $\bigoplus_{i \in I} M_i \in \mathrm{obj}\,\mathcal{A}$

- classical injection

$$\begin{array}{ccc} M_0 & \xrightarrow{m_0} & M_0 \\ \oplus & & \oplus \\ M_1 & \xrightarrow{m_1} & M_1 \\ \oplus & & \oplus \\ M_2 & \xrightarrow{m_2} & M_2 \\ \oplus & & \oplus \\ M_3 & \vdots & M_3 \\ \vdots & & \oplus \\ & & \vdots \end{array}$$

i.e $\exists$ infinite sequence

$$M_0 \xrightarrow{m_0} M_1 \xrightarrow{m_1} M_2 \longrightarrow M_3 \cdots \longrightarrow \cdots$$

let $\sigma_i : M_i \longrightarrow \bigoplus_{i=0}^{\infty} M_i$ be the classical injection

$$m_i \longmapsto \begin{pmatrix} 0 \\ \vdots \\ m_i \\ 0 \\ 0 \\ \vdots \end{pmatrix} i$$

Consider $\sigma_{i+1} m_i : \; m_i \longrightarrow \bigoplus_{i=0}^{\infty} M_i \quad (i \geq 0)$

induced a unique morphism $m : \bigoplus_{i=0}^{\infty} M_i \overset{m}{\dashrightarrow} \bigoplus_{i=0}^{\infty} M_i$

$$\begin{tikzcd} \bigoplus_{i=0}^{\infty} M_i \arrow[r, dashed, "m"] & \bigoplus_{i=0}^{\infty} M_i \\ M_i \arrow[u, "\sigma_i"] \arrow[ur, "\sigma_{i+1} m_i"'] & \end{tikzcd}$$

$(i \leq j \leq k)$

$X_i \in \mathcal{C} \quad \forall i \in I$

$i \leq j$, $\exists f_{ji} : X_i \to X_j$

s.t $f_{ii} = \mathrm{id}_{X_i}$ $\quad f_{kj} f_{ji} = f_{ki}$

Recall $\underbrace{(X_i, f_{ji})}$

① $\begin{tikzcd} X_i \arrow[rr, "f_{ji}"] \arrow[dr, "f_i"'] & & X_j \arrow[dl, "f_j"] \\ & X & \end{tikzcd}$ $\quad (X, f_i)$

② $\begin{tikzcd} X_i \arrow[rr, "f_{ji}"] \arrow[dr, "g_i"'] & & X_j \arrow[dl, "g_j"] \\ & Y & \end{tikzcd}$ $\quad (Y, g_i)$

Lemma 4.5.1. $\mathcal{A}$: cocomplet Abelian category

$m : \bigoplus_{i=0}^{\infty} M_i \longrightarrow \bigoplus_{i=0}^{\infty} M_i$

$\mathrm{Coker}(1-m) = \varinjlim M_n$

Pf: let $\pi_i : \bigoplus_{i=0}^{\infty} M_i \longrightarrow \mathrm{Coker}(1-m)$

$\varphi_n = \pi\sigma_n : M_n \longrightarrow \mathrm{Coker}(1-m)$

By (4.2). $m\sigma_i = \sigma_{i+1} m_i \quad \forall i \geq 0$

$\Downarrow$

① $\boxed{\varphi_n = \varphi_{n+1} m_n}$

$$\begin{tikzcd}
M_n \arrow[rr, "m_n"] \arrow[dr, "\varphi_n"'] & & M_{n+1} \arrow[dl, "\varphi_{n+1}"] \\
& \mathrm{Coker}(1-m) &
\end{tikzcd}$$

② Let $f_n : M_n \longrightarrow X$ satify $f_n = f_{n+1} m_n$ ($\forall n \geq 0$)

$$\begin{tikzcd}
& M_n \arrow[rr, "m_n"] \arrow[dl, "\sigma_n"'] \arrow[dr, "f_n"] & & M_{n+1} \arrow[dl, "f_{n+1}"] \\
\bigoplus M_n \arrow[rr, dashed, "\exists! g"] & & X &
\end{tikzcd}$$

③ By univarsal property of direct sum

$\exists!\ g : \bigoplus_{i\in I} M_n \longrightarrow X$ s.t $g\sigma_n = f_n \ \forall n \geq 0$.

claim $\boxed{g(1-m) = 0}$

$$\begin{tikzcd}
\bigoplus_{i\in I} M_i \arrow[r, "1-m"] & \bigoplus_{i\in I} M_i \arrow[r, "\pi"] \arrow[dr, "g"'] & \mathrm{Coker}(1-m) \arrow[d, dashed, "\exists! \beta"] \\
& & X
\end{tikzcd}$$

s.t $g = \beta \circ \pi$

then $\boxed{f_n = g\sigma_n = \beta\pi\sigma_n = \beta\varphi_n}$ $\forall n \geq 0$ #

$\boxed{\varinjlim M_n = \mathrm{Coker}(1-m)}$

③ $\exists!\ g : X \to Y$ s.t

$g \circ f_i = g_i \quad \forall i \in I$

$\varinjlim X_i = X$

$$\begin{tikzcd}
\mathrm{Coker}(1-m) \arrow[rr, dashed, "\exists!\beta"] & & X \\
& M_n \arrow[ul, "\varphi_n"'] \arrow[ur, "f_n"] \arrow[d, "m_n"] & \\
& M_{n+1} \arrow[uul, "\varphi_{n+1}"] \arrow[uur, "f_{n+1}"'] &
\end{tikzcd}$$

$g(1-m) = 0 \iff g(1-m)\sigma_i = 0 \quad \forall i \geq 0$

$g\sigma_n - g\, m\sigma_n$

$= g\sigma_n - g\sigma_{n+1} m_n$

$= f_n - f_{n+1} m_n = 0$

Lemma 4.5.2

$\mathcal{A}$: Grothendieck category

then $1-m: \bigoplus_{i=0}^{\infty} M_i \longrightarrow \bigoplus_{i=0}^{\infty} M_i$ is a injective.

def: • P is called a generator, if $\mathrm{Hom}_{\mathcal{A}}(P,-)$ is a faithful functor

• $\mathcal{A}$ is a cocomplete Abelian category with with a generator and $\mathcal{A}$ has an exact $\varinjlim$, we call $\mathcal{A}$ is a Grothendieck category

Pf: By definition of direct sum: $\exists!\ \pi_i: \bigoplus_{i=0}^{\infty} M_i \longrightarrow M_i \quad \forall i \geq 0$

s.t $\pi_i \sigma_i = id_{M_i}$; $\pi_i \sigma_j = 0 \quad \forall i \neq j$

Let $p_i: \prod_{i=0}^{\infty} M_i \longrightarrow M_i$ be a canonical surjective.

By definition of $\prod_{i=0}^{\infty}$, $\exists!\ \sigma: \bigoplus_{i=0}^{\infty} M_i \longrightarrow \prod_{i=0}^{\infty} M_i$

s.t $p_i \sigma = \pi_i \quad i \geq 0$

$$\begin{tikzcd} X \arrow[r, "\beta"] & \bigoplus_{i=0}^{\infty} M_i \arrow[r, dashed, "\exists!\sigma"] \arrow[dr, "\pi_i"'] & \prod_{i=0}^{\infty} M_i \arrow[d, "p_i"] \\ & & M_i \end{tikzcd}$$

By a property of Grothendieck category : σ is injective

Let $\beta: X \longrightarrow \bigoplus_{i=0}^{\infty} M_i$ s.t $(1-m)\beta = 0$ aim : $\beta = 0$

claim $(1-m)\beta = 0 \implies \beta = 0$

Since σ is injective,

$$\sigma\beta = 0$$

$$\Updownarrow \quad \text{by defi of } \prod_{i=0}^{\infty}$$

$$p_i\,\sigma\beta = 0$$

$$\Updownarrow$$

$$\underline{\pi_i\beta = 0} \xLeftarrow{(1-m)\beta = 0} \pi_i m\beta = 0 \quad \forall i \geq 0$$

$$\Uparrow$$

$$\pi_i m\beta = m_{i-1}\,(\pi_{i-1} m\beta)$$

claim $\begin{cases} \pi_0 m = 0 \\ \pi_i m = m_{i-1} \circ \pi_{i-1} \end{cases}$

归纳法 $\pi_0 m\beta = 0$, $\pi_{i-1} m\beta = 0$ $\Rightarrow$ $\pi_i m\beta = m_{i-1}\left(\pi_{i-1} m\beta\right) = 0$ $\Rightarrow$ $\pi_i m\beta = 0$

只需 claim: $\pi_i m \sigma_j \overset{?}{=} m_{i-1} \circ \pi_{i-1} \sigma_j$

$$\begin{cases} \pi_i \sigma_{j+1} m_j = \delta_{i,j+1}\, m_j = m_{i-1}\, \pi_{i-1}\, \sigma_j \\ \pi_0 m \sigma_j = \pi_0 \sigma_{j+1} m_j = 0 \\ j \geq 0 \end{cases}$$

(4.1) $m\sigma_i = \sigma_{i+1} m_i$

$\pi_i \sigma_j = 0 \quad \forall i \neq j$

#

Rmk: (AB4) : (exact direct sum)

If $\exists \bigoplus_{i=0}^{\infty}$ in $\mathcal{A}$. $\forall$ $0 \to L_i \to M_i \to N_i \to 0$ $\forall i \in I$. I arbitrary index set. s.e.s in $\mathcal{A}$

$$\Downarrow$$

$$0 \to \bigoplus_{i=0}^{\infty} L_i \to \bigoplus_{i=0}^{\infty} M_i \to \bigoplus_{i=0}^{\infty} N_i \to 0 \quad \text{s.e.s in } \mathcal{A}$$

i.e $\bigoplus_{i=0}^{\infty}$ preserves injective

(preserves surjective $\Leftarrow$ trivial)

Cor: 4.5.3: Let $\mathcal{A}$ be a Grothendieck category:

(i) $\mathcal{A}$ has (Ab4)

(ii) Let $v_i : P_i \to X_i$ quasi-iso. in $\mathcal{C}(\mathcal{A})$ $\forall i \in I$

then $\bigoplus v_i : \bigoplus_{i \in I} P_i \longrightarrow \bigoplus_{i \in I} X_i$ is quasi-iso.

Pf: (ii) from (i)

$$\ker\left(\bigoplus_{i=0}^{\infty} d_i^n\right) = \bigoplus_{i \in I} \ker d_i^n$$

$$\mathrm{Im}\left(\bigoplus_{i=0}^{\infty} d_i^n\right) = \bigoplus_{i \in I} (\mathrm{Im}\, d_i^n)$$

$$\bigoplus v_i : \bigoplus_{i \in I} P_i \longrightarrow \bigoplus_{i \in I} X_i$$

$$H^n\left(\bigoplus_{i=0}^{\infty} v_i\right) = \bigoplus_{i=0}^{\infty} H^n(v_i) : \bigoplus_{i \in I} H^n(P_i) \longrightarrow \bigoplus_{i \in I} H^n(X_i)$$

$\boxed{\text{Thm 4.5.4}}$

$\mathcal{A}$: Grothendieck category with enough projective object

$\forall X \in \mathcal{C}(A)$ has a homotopy projective resolution $f: P \xrightarrow[①]{} X$, $P \in K_{hproj}(\mathcal{A})$

Moreover, P is projective complex. ② $(f\ P \twoheadrightarrow X)$ ③

if $\alpha: X \to Y$ is a homotopy equivalence.

$c_X: P \to X$
are homotopy proj. resolution
$c_Y: Q \to Y$

then $\exists$ $u: P \to Q$ s.t.
$$\begin{tikzcd} P \arrow[r, "c_X"] \arrow[d, "u"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "\alpha"] \\ Q \arrow[r, "c_Y"'] & Y \end{tikzcd}$$
④

Pf: ①: By Thm 4.2.1

$v_n: P_n \longrightarrow \tau_{\le n} X$ (左温和截断) and $\exists$ infinite sequence

$\tau_{\le 0} X \xrightarrow{u_0} \tau_{\le 1} X \xrightarrow{u_1} \tau_{\le 2} X \xrightarrow{u_2} \tau_{\le 3} X \xrightarrow{u_3} \cdots$

By prop 4.2.8(1) $\quad \rho_X \longrightarrow \rho_Y$ we have C.D. in $K(A)$
$$\begin{tikzcd} \rho_X \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \rho_Y \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

$$\begin{tikzcd} P_0 \arrow[r, "\rho_0"] \arrow[d, "v_0"] & P_1 \arrow[r, "\rho_1"] \arrow[d, "v_1"] & P_2 \arrow[r, "\rho_2"] \arrow[d, "v_2"] & \cdots \arrow[r] & P_n \arrow[r, "\rho_n"] \arrow[d, "v_n"] & \cdots \\ \tau_{\le 0} X \arrow[r, "u_0"] & \tau_{\le 1} X \arrow[r, "u_1"] & \tau_{\le 2} X \arrow[r, "u_2"] & \cdots \arrow[r] & \tau_{\le n} X \arrow[r, "u_n"] & {} \end{tikzcd}$$
$(*)$

By Cor 12.16.2 $\mathcal{C}(\mathcal{A})$ is Grothendieck category. By (4.2) $(m v_i = v_{i+1} m_i \ \ \forall i \ge 0)$

we have chain maps: $\quad u: \bigoplus_{i=0}^{\infty} \tau_{\le i} X \longrightarrow \bigoplus_{i=0}^{\infty} \tau_{\le i} X$

$\lfloor \rho : \bigoplus_{i=0}^{\infty} P_i \longrightarrow \bigoplus_{i=0}^{\infty} P_i$

s.t. $\boxed{\begin{array}{l} u\sigma_i^X = \sigma_{i+1}^X u_i \\ \rho\sigma_i^P = \sigma_{i+1}^P \rho_i \end{array}}$ $\forall i \geq 0$

where $\begin{cases} \sigma_i^P : P_i \longrightarrow \bigoplus_{i=0}^{\infty} P_i \\ \sigma_i^X : \tau_{\leq i} X \longrightarrow \bigoplus_{i=0}^{\infty} \tau_{\leq i} X \end{cases}$ $\forall i \geq 0$

By universal property of $\bigoplus_{i=0}^{\infty} P_i$

$\exists! \oplus v_i : \bigoplus_{i=0}^{\infty} P_i \longrightarrow \bigoplus_{i=0}^{\infty} \tau_{\leq i} X$ s.t. $\sigma_i^X v_i = (\oplus v_i)\sigma_i^P$ $\forall i \geq 0$

$$\begin{tikzcd}
P_i \arrow[r, "\sigma_i^P"] \arrow[d, "v_i"'] & \bigoplus_{i=0}^{\infty} P_i \arrow[ddl, dashed, "\exists! \oplus v_i"] \\
\tau_{\leq i} X \arrow[d, "\sigma_i^X"'] & \\
\oplus \tau_{\leq i} X &
\end{tikzcd}$$

(4.6) $u\sigma_i^X = \sigma_{i+1}^X u_i$ $\rho\sigma_i^P = \sigma_{i+1}^P \rho_i$ $i \geq 0$

(4.7) $\sigma_i^X v_i = (\oplus v_i)\sigma_i^P$ $\forall i \geq 0$

we have c.d. in $K(A)$:

$$\begin{tikzcd}
\bigoplus_{i=0}^{\infty} P_i \arrow[r, "1-\rho"] \arrow[d, "\oplus v_i"'] & \bigoplus_{i=0}^{\infty} P_i \arrow[d, "\oplus v_i"] \\
\bigoplus_{i=0}^{\infty} \tau_{\leq i} X \arrow[r, "1-u"'] & \bigoplus_{i=0}^{\infty} \tau_{\leq i} X
\end{tikzcd}$$

$\left(u(\bigoplus_{i=0}^{\infty} v_i)\sigma_i^P = (\bigoplus_{i=0}^{\infty} v_i)\rho\sigma_i^P \right.$ $\forall i \geq 0$ follow from (4.6) (4.7)

$u(\sigma_i^X v_i) \overset{(4.6)}{=} \sigma_{i+1}^X u_i v_i$ $\quad (\bigoplus_{i\in I} v_i)\sigma_{i+1}^P \rho_i \overset{(4.7)}{=} \sigma_{i+1}^X v_{i+1} \rho_i$

$\sigma_{i+1}^X u_i v_i \overset{(*)}{\Leftarrow} \sigma_{i+1}^X v_{i+1} \rho_i$ ✓

It follows that:

$$\begin{tikzcd}
\bigoplus_{i=0}^{\infty} P_i \arrow[r, "1-e"] \arrow[d, "\oplus v_i"'] & \bigoplus_{i=0}^{\infty} \arrow[r] \arrow[d, "\oplus v_i"] & \mathrm{Cone}(1-e) \arrow[r] \arrow[d, dashed, "\exists h"] & \left(\bigoplus_{i=0}^{\infty} P_i\right)[1] \arrow[d, "(\oplus v_i)[1]"] \\
\bigoplus_{i=0}^{\infty} \tau_{\le i} X \arrow[r, "1-u"'] & \bigoplus_{i=0}^{\infty} \tau_{\le i} X \arrow[r] & \mathrm{Cone}(1-u) \arrow[r] & \left(\bigoplus_{i=0}^{\infty} \tau_{\le i} X\right)[1]
\end{tikzcd}$$

Cor 4.5.2 $\Longrightarrow$ $\oplus v_i$ quasi-iso + Thm 2.4.1 + five lemma $\Rightarrow$ $h \to$ quasi-iso.

long exact sequence

By cor 4.5.2 $(1-u)$ is injective, $\Rightarrow$ s.e.c as following:

$$0 \to \bigoplus_{i\in I} P_i \xrightarrow{1-u} \bigoplus_{i\in I} P_i \to \mathrm{Coker}(1-u) \to 0$$

By prop 2.3.2 $\exists$ quasi iso g:

$$\begin{tikzcd}
\mathrm{Cone}(1-u) \arrow[r, "e"] & \mathrm{Coker}(1-u) \\
\mathrm{Cone}(1-e) \arrow[u, "h"] \arrow[ur, "f=gh"'] &
\end{tikzcd}$$

Thus, we have quasi-iso $f = gh : \mathrm{Cone}(1-e) \longrightarrow \mathrm{Coker}(1-u) \overset{\text{Cor 4.5.1}}{=} \varinjlim \tau_{\le n} X \overset{\text{lem 2.6.2(ii)}}{=} X$

①

$$f : \mathrm{Cone}(1-e) \longrightarrow X$$

Since every P_i is proj. complex $\Longrightarrow$ $\oplus P_i \in K_{hproj}(A)$

($P \subseteq K_{hproj}$ + K_{hproj} closed under $\bigoplus_{i=0}^{\infty}$)

d.s.: $\bigoplus_{i=0}^{\infty} P_i \xrightarrow{1-e} \bigoplus_{i=0}^{\infty} P_i \longrightarrow \mathrm{Cone}(1-e) \to \left(\bigoplus_{i=0}^{\infty} P_i\right)[1]$

$\Longrightarrow$ $\mathrm{Cone}(1-e) \in K_{hproj}(A)$

Moreover, proj. c.p. X.

②

$\boxed{\text{For surjective}}$ ③

$(\_) \longrightarrow X$;

$K_{hproj}(A)$

add zero object in $K(A)$

$\cdots 0 \to P \to P \to 0 \cdots$ ③

(contractible complex)

$\boxed{\text{For unique up to homotopy equi.}}$ ④

Lemma 4.4.4 (iii) $\forall$ quasi-iso $s: X \longrightarrow Y$, $P \in K_{hproj}(A)$

$$\mathrm{Hom}_{K(A)}(P, s): \mathrm{Hom}_{K(A)}(P, X) \xrightarrow{\simeq} \mathrm{Hom}_{K(A)}(P, Y) \quad : \text{iso as abelian group.}$$

in particular $\mathrm{Hom}_{K(A)}(P, s): \mathrm{Hom}_{K(A)}(P, Q) \xrightarrow{\simeq} \mathrm{Hom}_{K(A)}(P, Y)$

$$u \longmapsto c_Y \circ u$$

i.e. $\exists! u$ s.t.

$$\begin{tikzcd} P \arrow[r, "u"] \arrow[d, "c_X"'] & Q \arrow[d, "c_Y"] \\ X \arrow[r, "s"'] \arrow[ur, phantom, "s"] & Y \end{tikzcd}$$

and u quasi-iso.

By Cor 4.4.5. $u: Q \to P$ quasi-iso. $P, Q \in K_{hproj}(A) \Rightarrow u$ homotopy equi.v #.

Cor 4.5.5. $\mathcal{A}$: Grothendieck category with enough proj. $Q \in K_{hproj}(\mathcal{A})$

$\exists$ P proj. complex s.t $P \simeq Q$ (homotopy equiv.)

i.e $K^-(\mathcal{P}) \subseteq K_{hproj}(A) \subseteq K(\mathcal{P})$

$\mathcal{P}$: full subcategory consisting of all proj. obj.

Pf: By Thm 4.5.4. $\exists$ quasi-iso $f: P \longrightarrow Q$

P proj. comple.

By Cor 4.4.5. $f: P \xrightarrow{\text{homotopy equivalence}} Q$

Rmk: ① homotopy proj complex $\xrightarrow{\text{homotopy equivelence}}$ projective complex

② a homotopy proj complex is not projective complex in general

③ homotopy proj comple $\cap$ proj complex $=:$ dg-proj complex

轴度形(齐次分支为投射模)

prop 4.5.6 (Similar as prop 4.2.8))

$\mathcal{A}$: Grothendieck category with enough proj.

(1) $i: K_{hproj}(\mathcal{A}) \rightleftarrows K(A): \rho$ $\quad i \dashv \rho$ (ρ righ adjont to i)

$\forall X \in K(A)$, $c_X: \rho X \longrightarrow X$

$$\begin{array}{c} X \\ f\downarrow \\ Y \end{array} \text{ in } K\mathcal{A} \longrightarrow \begin{tikzcd} \rho_X \arrow[r, "\rho_f"] \arrow[d, "c_X"'] \arrow[dr, phantom, "\wr"] & \rho_Y \arrow[d, "c_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

(2). ρ is a triangulated functor

Similar as prop 4.2.8

4.2.8 (Cor 4.2.4 —— 4.5.6 lemma 4.4.4 (iii)
Cor 4.2.6 —— + Cor 4.4.5))

<u>prop 4.5.7</u>: $\mathcal{A}$: Grothendieck category with enough proj.

$\forall X \in K(\mathcal{A})$, $\exists$ split s.e.s

$$0 \longrightarrow X \longrightarrow \underset{\substack{\uparrow \\ \text{acyclic}}}{E} \longrightarrow \underset{\substack{\uparrow \\ \text{dg-projective}}}{P} \longrightarrow 0$$

$\longrightarrow \xrightarrow{} P \xrightarrow{f} X \longleftarrow$ 投射维数

pf: let $f: P \to X$ quasi-iso. and $P \in K_{hproj}(\mathcal{A})$

Then $\exists$ d.t.

$$\underset{\substack{\uparrow \\ \text{quasi-iso}}}{P \xrightarrow{f}} X \longrightarrow \underset{\substack{\uparrow \\ \text{acyclic cpx}}}{\mathrm{Cone}(f)} \longrightarrow P[1]$$

By thm 4.5.4

then $0 \longrightarrow X \longrightarrow \mathrm{Cone}(f) \longrightarrow P[1] \to 0$ #

split s.e.s

Rmk 4.5.8: $\mathcal{T}$: triangulated category with $\bigoplus_{i=0}^{\infty}$ for arbitary infinite morphism sequence.

$$U_0 \xrightarrow{u_0} U_1 \xrightarrow{u_1} U_2 \to U_3 \to \cdots$$

Let $u: \bigoplus_{i=0}^{\infty} U_i \longrightarrow \bigoplus_{i=0}^{\infty} U_i$ s.t. $u\sigma_i = \sigma_{i+1} u_i \quad (\forall i \geq 0)$

define: $\mathrm{Cone}(1-u)$ is called homotopy limit of infinite sequence.

denote by $\underrightarrow{\mathrm{holim}}\, u_i$ (or $\underrightarrow{\mathrm{holim}}\, U_i$)

then $\triangle$ in $\mathcal{T}$ can be writen as:

$$\bigoplus_{i=0}^{\infty} U_i \xrightarrow{1-u} \bigoplus_{i=0}^{\infty} U_i \longrightarrow \underrightarrow{\mathrm{holim}}\, U_i \longrightarrow \Big(\bigoplus_{i=0}^{\infty} U_i\Big)[1]$$

unique up to iso.

re introduced Thm 4.5.4 by $P = \underrightarrow{\mathrm{holim}}\, P_i$ (dg-proj complex)

$$\begin{tikzcd}
P_0 \arrow[r, "\rho_0"] \arrow[d, "v_0"'] & P_1 \arrow[r, "\rho_1"] \arrow[d, "v_1"] & P_2 \arrow[r, "\rho_2"] \arrow[d, "v_2"] & \cdots \arrow[r] & P_n \arrow[r, "\rho_n"] \arrow[d] & \cdots \\
\tau_{\leq 0}X \arrow[r] & \tau_{\leq 1}X \arrow[r] & \tau_{\leq 2}X \arrow[r] & \cdots \arrow[r] & \tau_{\leq n}X \arrow[r] & \cdots
\end{tikzcd}$$

$\cdots$

prop 4.5.9. $\mathcal{A}$: Grothendieck category with enough proj. $\mathcal{P}$: full subcategory consisting of projective obj.

Let $\mathrm{Tri}(\mathcal{P}) \supseteq \mathcal{P}$ the smallest triangulated category closed under $\bigoplus_{i=0}^{\infty}$ as a full subcategory of $K(\mathcal{A})$

Then $\mathrm{Tri}(\mathcal{P}) = K_{hproj}(\mathcal{A})$

Pf: ① $\mathrm{Tri}(\mathcal{P}) \subseteq K_{hproj}(\mathcal{A})$ ✓

② $K_{hproj}(\mathcal{A}) \subseteq \mathrm{Tri}(\mathcal{P})$

$\boxed{\text{claim: } K^-(\mathcal{P}) \subseteq \mathrm{Tri}(\mathcal{P})}$ ✓

then $\forall P \in K_{hproj}(\mathcal{A})$

By the pf of Thm 4.5.4 + claim

$\exists$ quasi-iso $Q \longrightarrow P$ ⟶ $\bigoplus_{i=0}^{\infty} P_i$

Q: dg proj. complex, by 4.4.5

$P \xrightarrow{\simeq} Q$: homotopy equivalence

$\Rightarrow$ By TR1 $\Rightarrow$ $P \in \mathrm{Tri}(\mathcal{P})$

For claim: Let $P \in K^-(\mathcal{P})$

Consider

$$P_{\geq 0} \xrightarrow{m_0} P_{\geq 1} \xrightarrow{m_1} P_{\geq -2} \xrightarrow{m_{-2}} P_{\geq -3} \to \cdots$$

m_i: injective $P_{\geq i} \to$ proj. complex

i.e. $P_{\geq i} \in \mathrm{Tri}(\mathcal{P})$

By Cor 2.6.2 (iv)

$$\boxed{\varinjlim_{i \leq 0} P_{\geq i} = P}$$

$\cong \mathrm{Coker}(1-m)$

$$\mathrm{Cone}\left(\bigoplus_{i=0}^{-\infty} P_{\geq i} \xrightarrow{1-m} \bigoplus_{i=0}^{-\infty} P_{\geq i}\right) = \varinjlim_{i \leq 0} P_{\geq i} = P$$

By Cor 4.5.2 $\Rightarrow$ $1-m$ injective

By prop 2.3.2

$\mathrm{Cone}(1-m) \to \mathrm{Coker}(1-m) = P$ quasi-iso

proj comple ↑ proj compl

$\Rightarrow P \simeq \mathrm{Cone}(1-m) \in \mathrm{Tri}(\mathcal{P})$ (closed under $\bigoplus_{i=0}^{-\infty}$)

§ 4.6. Homotopy injective resolution for arbitrary complex #

§5. Derived category

§5.1: Recall in chapter 3.

$\mathcal{S} \to$ compatible multiplicative system of $(\mathcal{K}, [1])$

Then $(\mathcal{S}^{-1}\mathcal{K}, [1])$ is a triangulated category.

d. △. in $\mathcal{S}^{-1}\mathcal{K}$: iso. to the standard △

$F: \mathcal{K} \to \mathcal{S}^{-1}\mathcal{K}$, $F(\triangle)$

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

F }

$$X \xrightarrow{f/1} Y \xrightarrow{g/1} Z \xrightarrow{h/1} X[1]$$

- $F(f) = f/\mathrm{id}_X$ is iso iff $f \in \mathcal{S}$
- $X \in \mathcal{S}^{-1}\mathcal{K}$ iff $X \in$ thick corresponding to S
  $X \cong 0$

Fix $\mathcal{S} = \{$the class of quasi-iso.$\}$

then $\mathcal{S}^{-1}\mathcal{K}(A) = D(A)$

$S_*^{-1}\mathcal{K}^*(A) = D^*(A)$ $\quad * \in \{$空$, b, -, +\}$

prop 5.1.1. $\mathcal{A}$: Abelian category

$K^+(\mathcal{A})$: homotopy category of $\mathcal{A}$.

let Q_* be the classes of quasi-iso

Then Q_* is saturated compatible multiplicative system.

(1) $\{$ $fh \in S,\ hf \in S \Rightarrow f \in S$

(2) think $(K^+(\mathcal{A})) = \{$ acyclic c.p.x $\}$

Pf (FR1) $f \in S$ $g \in S \Rightarrow g \circ f \in S$

dual+ (FR2)

$$\begin{tikzcd} & X \arrow[d, Rightarrow, "s"] \\ E \arrow[r, "f"'] & Y \end{tikzcd} \Rightarrow \begin{tikzcd} D \arrow[r, "h"] \arrow[d, Rightarrow, "t"'] & X \arrow[d, Rightarrow, "s"] \\ E \arrow[r, "f"'] & Y \end{tikzcd}$$

dual+ (FR3) :

$$\begin{tikzcd} X \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \\ Y \arrow[d, Rightarrow, "t"] \\ Z \end{tikzcd} \Rightarrow \begin{tikzcd} K \arrow[d, Rightarrow, "s"] \\ X \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \\ Y \end{tikzcd}$$

(multiplicative system)

compatible (FR4) : $s \in S \Rightarrow s[1] \in S$

(FR5)

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & TX \arrow[d, "Tf"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & TX' \end{tikzcd}$$

$f . g \in S \Rightarrow h \in S$

FR1: $H^i(g \circ f) = H^i(g) \circ H^i(f)$ ✓

FR2:

$$\begin{tikzcd} & E \arrow[d, "f"] \\ X \arrow[r, "s"] & Y \end{tikzcd} \quad s \in Q_*$$

$$\Downarrow$$

$$\begin{tikzcd} D \arrow[r, "t"] \arrow[d, "h"'] & E \arrow[r, "\chi f"] \arrow[d, "f"] & S \arrow[r] \arrow[d, equal] & D[1] \arrow[d, "h[1]"] \\ X \arrow[r, "s"] & Y \arrow[r, "\chi"] & S \arrow[r] & X[1] \end{tikzcd}$$

$t \in Q_*$ acycle

FR3: let $h = f - g$ then $t h = 0$

$Y \xrightarrow{t} Z \hookrightarrow \Delta \quad S \xrightarrow{z} Y \xrightarrow{t} Z \longrightarrow S[1]$

Consider

$$\cdots \operatorname{Hom}_{K(A)}(X, S) \xrightarrow{(X, z)} \operatorname{Hom}(X, Y) \xrightarrow{(X, t)} \operatorname{Hom}(X, Z) \longrightarrow \cdots$$

$h \longmapsto t \circ h = 0$

$k \longmapsto h$

$\exists k \in \operatorname{Hom}(X, S)$ s.t. $h = z \circ k$

$X \xrightarrow{k} S \rightsquigarrow \Delta \quad K \xrightarrow{s} X \xrightarrow{k} S \to K[1]$

Since $t \in Q_*$ $\Rightarrow S$ acyclic

then $s \in Q_*$. $h \circ s = 0$

(FR4) ✓ (FR5) ✓ (Thm 2.4.1 + TR3 + five lemma)

For Q_* is saturated: $fh \in Q_*$, $kf \in Q_*$ $\Longrightarrow$ $H^*(f)\cdot H^*(h)$, $H^*(k)\cdot H^*(f)$ are iso of abelian group

$$\begin{tikzcd} K \arrow[r, "H^*(h)"] & A \arrow[r, "H^*(f)"] & B \arrow[r, "H^*(k)"] \arrow[ll, dashed, bend left, "\exists\ \text{inverse } l"] & C \arrow[ll, dashed, bend right, "\text{inverse } m"'] \end{tikzcd}$$

the $l \circ (H^*(f)) = id_A$

$H^*(f) \circ (m \circ H^*(k)) = id_B$

$\Longrightarrow$ $H^*(f)$ is an iso of abelian group

$\Longrightarrow$ $f \in Q_*$

Finally: $\mathcal{Y}(Q_*) = \{ Z \in K(\mathcal{A}) \mid \exists\ X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1]$ s.t $f \in Q_* \}$

$= \{$ classes of acyclic complex $\}$

Defi: Derived category of $\mathcal{A}$ defined by $D(\mathcal{A}) := Q^{-1} K(\mathcal{A})$

obj $D(\mathcal{A}) \longleftrightarrow$ obj $\mathcal{C}(\mathcal{A})$

$\mathrm{Hom}_{D(\mathcal{A})}(X,Y)$: $X \xLeftarrow{s \in Q} \cdot \xrightarrow{b} Y$

右分式 (左分式)

Composition: $b/s \circ a/r := bc/rt$

$$\begin{tikzcd}
& & A \arrow[dl, Rightarrow, "t"'] \arrow[dr, "c"] & & \\
& B \arrow[dl, Rightarrow, "r"'] \arrow[dr, "a"] & & C \arrow[dl, Rightarrow, "s"'] \arrow[dr, "b"] & \\
X & & Y & & Z
\end{tikzcd}$$

identity: $id_X / id_X = s/s$

" + ": $a/r + b/s = a'+b'/t$

$$\begin{tikzcd}
\cdot \arrow[r, "s'"] \arrow[d, Rightarrow, "r'"'] & \cdot \arrow[d, Rightarrow, "r"] \\
\cdot \arrow[r, Rightarrow, "s"'] & \cdot
\end{tikzcd}$$

$$\begin{cases} r\circ s' = s\circ r' = t \\ a' = as' \\ b' = br' \end{cases}$$

$\mathrm{Hom}_{D(A)}(X,Y) \leftarrow$ abelian group

zero obj: $0/t = 0/s$

- $\alpha = a/s$, $s \in Q$, α zero morphism iff $\exists t$. s.t $at = 0$, $st \in S$

· $X \in D(\mathcal{A})$ zero object $\Longleftrightarrow$ $\exists$ $0 : Z \longrightarrow X$ 为拟同构

acyclic

Similarly : $D^*(\mathcal{A})$

$F^* : K^*(\mathcal{A}) \longrightarrow D^*(\mathcal{A})$ (局部化函子)

$f \longmapsto F^*(f) = f/\mathrm{id}_X$

1 $F^*(f)$ is isomophism in $D^*(\mathcal{A})$ iff $\exists$ quasi-iso $t : X' \longrightarrow X$ st

$X' \xrightarrow{t} X \xrightarrow{f} Y$

$f \circ t \sim 0$ (同伦于 0)
(右分式)

iff: $\exists$ quasi-iso $s : Y \longrightarrow Y'$

$X \xrightarrow{f} Y \xrightarrow{s} Y'$

s.t $s \circ f \sim 0$ (左分式)
FR3

2 X is an zero obj $\Longleftrightarrow$ $X \in e^*(\mathcal{A})$ is acyclic ✓

Rmk: $F^*(f)$ is zero morphism in $D^*(\mathcal{A})$

But $f \not\sim 0$

Example $X' = \cdots 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0 \cdots$ acyclic complex.

Let $f = \mathrm{id}_X$. $f \sim 0$

But $f: X^\cdot \to 0$ is quasi-iso.

$F(f)$ is zero morphism in $D^*(\mathcal{A})$

$$id_{X^\cdot} = d^{n-1}s^n + s^{n+1}d^n$$

$$\begin{tikzcd}
\cdots & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "2"] \arrow[d] \arrow[dl, "0"'] & \mathbb{Z} \arrow[r, "\pi"] \arrow[d] \arrow[dl, "s_n"'] & \mathbb{Z}_2 \arrow[r] \arrow[d] \arrow[dl, "s_{n+1}"'] & 0 \arrow[r] \arrow[d] \arrow[dl, "0"'] & 0 \cdots \arrow[d] \\
& 0 \arrow[r, "0"'] & \mathbb{Z} \arrow[r, "2"'] & \mathbb{Z} \arrow[r, "\pi"'] & \mathbb{Z}_2 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

$$\boxed{2s_n = id}$$

$s_n = \frac{1}{2} \notin Hom(\mathbb{Z}, \mathbb{Z})$

Prop 5.1.2

$\mathcal{A}$: abelian category:

$0 \to X^\cdot \xrightarrow{f} Y^\cdot \xrightarrow{g} Z^\cdot \to 0$ in $\mathcal{C}(\mathcal{A})$

$\exists\, h \in Hom_{D^*(\mathcal{A})}(Z^\cdot, X[1])$ (右分式，非复形态射)

s.t. $X^\cdot \xrightarrow{f} Y^\cdot \to Z^\cdot \xrightarrow{h} X[1]$ is distinguished Δ in $D^*(\mathcal{A})$

Furthermore, all d. Δ in $D^*(\mathcal{A})$ are iso to this form Δ, $* \in \{\text{空}, +, -, b\}$

Pf: By prop 2.3.2 $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ (0.n): Cone(u) → Coker(u) = Z quasi-iso

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\begin{pmatrix}0\\0\\1\end{pmatrix}"] \arrow[d, equal] & \mathrm{Cyl}(u) \arrow[r, "\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}"] \arrow[d, "(0\,.\,1\,.\,u)=:s"] & \mathrm{Cone}(u) \arrow[r] \arrow[d, "(0\,,\,v)=t"] & 0 \\
0 \arrow[r] & X \arrow[r, "u"] & Y \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

$(0.1.u)$ <u>homotopy equiv.</u> $\Longrightarrow$ $(0.1.u)$ <u>quasi-iso</u>

wherer $s.$ $t.$ quasi-iso. $\Longrightarrow$ $D(A^b)$

$$\begin{tikzcd}
X \arrow[r] \arrow[d, equal] & \mathrm{Cyl}(f) \arrow[r] \arrow[d, "s"] & \mathrm{Cone}(f) \arrow[r, "w"] \arrow[d, "t"] & X[1] \arrow[d, equal] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \arrow[r, "h"] & X[1]
\end{tikzcd} \longrightarrow K(A).\ \text{d.o}$$

By lemma 2.3.1

$h = w \circ t$

s, t are iso in $D^b(A)$ $\Longrightarrow$ $X \xrightarrow[f]{} Y \xrightarrow[g]{} Z \to X[1]$ is d.o in $D^b(A)$

Furthermore: $\forall$ d.o in $D^b(A)$ is iso. to $\underbrace{X \to \mathrm{Cyl}(f) \to \mathrm{Cone}(f)}_{\text{chain split s.e.c}} \xrightarrow{w} X[1]$

$\Downarrow$

It is s.e.c ✓ #