

Recall

§ 4.5

By def: $\dots \rightarrow X \rightarrow 0 \rightarrow 0 \dots$ (stalk cpx of X) is hoproj $\Leftrightarrow X$ is proj

\Leftarrow Prop 4.2.3 (i) $\text{Hom}_{\mathcal{R}(A)}(X, E) = 0$ for any acyclic cpx E

\Rightarrow $\text{Hom}_{\mathcal{R}(A)}(X, E) \rightarrow \text{Hom}^n(X, Y) = \prod_{p \in \mathbb{Z}} \text{Hom}_{\mathcal{R}(A)}(X^p, Y^{p+n})$
 $f = (f^p)_{p \in \mathbb{Z}} \in \text{Hom}^n(X, Y)$

$$d^n(f) = ((d^n f)^p)_{p \in \mathbb{Z}} \in \text{Hom}^{n+1}(X, Y)$$

$$(d^n f)^p = d_Y^{n+1} f^p + (-1)^{n+1} f^{p+1} d_X^n \in \text{Hom}_{\mathcal{R}(A)}(X^p, Y^{p+n+1})$$

e.g. upper bounded proj cpx is a hoprojective cpx (Prop 4.2.3 (ii))

$$\text{Hom}^n(X, Y) = \begin{pmatrix} 0 : X^{-1} \rightarrow Y^{-1+n} \\ \vdots : X \rightarrow Y^n \\ f_0 : X \rightarrow Y^n \\ \vdots : X^1 \rightarrow Y^{n+1} \end{pmatrix}$$

$$\text{Hom}^{n+1}(X, Y) = \begin{pmatrix} 0 \\ f_0^1 : X \rightarrow Y^{n+1} \\ \vdots \end{pmatrix}$$

Lem 4.4.3 A abelian cate. Then

① If $(P_i)_{1 \leq i \leq n}$ hoprojective cpx $\Rightarrow \bigoplus_{i=1}^n P_i$ is hoproj cpx

$$\text{Hom}^{n+2}(X, Y) = \begin{pmatrix} 0 \\ f_0^1 : X \rightarrow Y^{n+2} \\ \vdots \end{pmatrix}$$

$$\text{Hom}^n(X, Y) = \text{Hom}(X, Y^n)$$

$$\text{Hom}^{n+1}(X, Y) = \text{Hom}(X, Y^{n+1})$$

$$\text{Hom}^{n+2}(X, Y) = \text{Hom}(X, Y^{n+2})$$

} acyclic complex
 $\Leftrightarrow X$ is proj obj



Recall:

• Thm 2.1: $\forall X \in \bar{C}(A)$ (resp $C^b(A)$, $\text{pdim} < \infty$)

$\exists p \xrightarrow{f} X$ quasi-iso, where $p \in K^{-1}(\bar{P})$ (resp $K^b(P)$, $K^b(P)$)

\uparrow
projective c.p.x

• $K_{\text{hproj}}(A) = \{p \in C(A) \mid \forall E \leftarrow \text{acyclic c.p.x. } \text{Hom}^*(p, E) \text{ acyclic c.p.x}\}$

• Cor 4.43. $K_{\text{hproj}}(A)$ are closed under:

(1) finite sum

(2) $\bigoplus_{i=0}^{\infty}$ (if $\exists \bigoplus_{i=0}^{\infty}$)

(3) $X \rightarrow Y \rightarrow Z \rightarrow TX$

two out of three $\in K_{\text{hproj}}(A)$, then so is the third.

i.e. $K_{\text{hproj}}(A) = \text{thick}(K(A))$

• $S: X \rightarrow Y$ quasi-iso, $p \in K_{\text{hproj}}(A)$

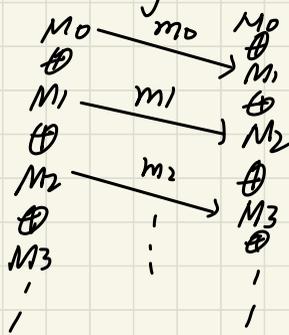
$\text{Hom}_{K(A)}(p, S) : \text{Hom}_{K(A)}(p, X) \xrightarrow{\cong} \text{Hom}_{K(A)}(p, Y)$

3.4.5. Homotopy projective resolution for arbitrary complex.

• Cocomplete abelian category \mathcal{A} :

For $I = \mathbb{N} = \{0, 1, 2, \dots\}$. If $M_i \in \text{Obj } \mathcal{A}$, Then $\bigoplus_{i \in I} M_i \in \text{Obj } \mathcal{A}$

• classical injection



i.e. \exists infinite sequence

$$M_0 \xrightarrow{m_0} M_1 \xrightarrow{m_1} M_2 \rightarrow M_3 \rightarrow \dots \rightarrow \dots$$

let $\sigma_i : M_i \longrightarrow \bigoplus_{i=0}^{\infty} M_i$ be the classical injection

$$m_i \longrightarrow \begin{pmatrix} 0 \\ \vdots \\ m_i \\ \vdots \\ 0 \end{pmatrix}_i$$

Consider $\sigma_{i+1} m_i : M_i \longrightarrow \bigoplus_{i=0}^{\infty} M_i \quad (i \geq 0) \quad (i \leq j \leq k)$

Induced a unique morphism

$$m : \bigoplus_{i=0}^{\infty} M_i \xrightarrow{m} \bigoplus_{i=0}^{\infty} M_i$$

$\sigma_i \uparrow$ $\sigma_{i+1} m_i \uparrow$
 M_i

$$\begin{array}{l}
 X_i \in \mathcal{C} \quad \forall i \in I \\
 i \leq j, \exists f_{ji} : X_i \rightarrow X_j \\
 \text{Set } f_{ii} = \text{id}_{X_i} \quad f_{kj} f_{ji} = f_{ki}
 \end{array}$$

Recall (X_i, f_{ij})

$$\textcircled{1} \quad \begin{array}{ccc}
 X_i & \xrightarrow{f_{ij}} & X_j \\
 f_i \searrow & \cup & \swarrow f_j \\
 & X &
 \end{array} \quad (X, f_i)$$

$$\textcircled{2} \quad \begin{array}{ccc}
 X_i & \xrightarrow{f_{ij}} & X_j \\
 g_i \searrow & \cup & \swarrow g_j \\
 & Y &
 \end{array} \quad (Y, g_i)$$

Lemma 4.5.1. \mathcal{A} : Cocomplete Abelian category

$$m : \bigoplus_{i=0}^{\infty} M_i \longrightarrow \bigoplus_{i=0}^{\infty} M_i$$

$$\text{Coker}(1-m) = \varinjlim M_n$$

1) pf: let $\pi_i : \bigoplus_{i=0}^{\infty} M_i \rightarrow \text{Coker}(I-m)$

$$\varphi_n = \pi_n \circ \sigma_n : M_n \rightarrow \text{Coker}(I-m)$$

By (4.2). $m\sigma_i = \sigma_{i+1} m_i \quad \forall i \geq 0$

①

$$\varphi_n = \varphi_{n+1} m_n$$

$$\begin{array}{ccc} M_n & \xrightarrow{m_n} & M_{n+1} \\ \varphi_n \searrow & & \swarrow \varphi_{n+1} \\ & \text{Coker}(I-m) & \end{array}$$

② Let $f_n : M_n \rightarrow X$ satisfy $f_n = f_{n+1} m_n \quad (\forall n \geq 0)$

$$\begin{array}{ccc} M_n & \xrightarrow{m_n} & M_{n+1} \\ \sigma_n \swarrow & f_n \searrow & \swarrow f_{n+1} \\ & X & \\ \oplus M_n & \xrightarrow{\exists! g} & \end{array}$$

③ By universal property of direct sum

$$\exists! g : \bigoplus_{i \in \mathbb{Z}} M_i \rightarrow X \quad \text{s.t.} \quad g \sigma_n = f_n \quad \forall n \geq 0$$

claim

$$g(I-m) = 0$$

$$\begin{array}{ccc} \bigoplus_{i \in \mathbb{Z}} M_i & \xrightarrow{I-m} & \bigoplus_{i \in \mathbb{Z}} M_i \xrightarrow{\pi} \text{Coker}(I-m) \\ & & \searrow g \\ & & X \end{array} \quad \begin{array}{l} \exists! \beta \\ \beta \end{array}$$

$$\text{s.t. } g = \beta \circ \pi$$

then

$$f_n = g \sigma_n = \beta \pi \sigma_n = \beta \varphi_n$$

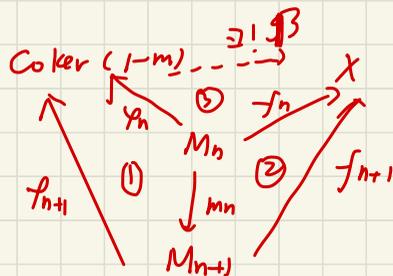
$\forall n \geq 0$

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$$\varinjlim M_n = \text{Coker}(I-m)$$

③ $\exists! g : X \rightarrow Y$ s.t.
 $g \circ f_i = g_i \quad \forall i \in I$

$$\varinjlim X_i = X$$



$$g(I-m) = 0 \iff g(I-m) \sigma_i = 0 \quad \forall i \geq 0$$

$$g \sigma_n - g m_n \sigma_n$$

$$= g \sigma_n - g \sigma_{n+1} m_n$$

$$= f_n - f_{n+1} m_n = 0$$

Lemma 4.5.2

\mathcal{A} : Grothendieck category

then $f_m: \bigoplus_{i=0}^{\infty} M_i \rightarrow \bigoplus_{i=0}^{\infty} M_i$ is a injective.

def: \mathcal{A} is called a generator, if $\text{Hom}_{\mathcal{A}}(p, -)$ is a faithful functor

- \mathcal{A} is a complete Abelian category with with a generator and \mathcal{A} has an exact \varinjlim , we call \mathcal{A} is a Grothendieck category

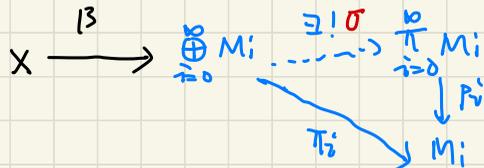
pf: By definition of direct sum: $\exists! \pi_i: \bigoplus_{i=0}^{\infty} M_i \rightarrow M_i \quad \forall i \geq 0$

s.t $\pi_i \circ \sigma_i = \text{id}_{M_i}$; $\pi_i \circ \sigma_j = 0 \quad \forall i \neq j$

Let $\rho_i: \prod_{i=0}^{\infty} M_i \rightarrow M_i$ be a canonical surjective.

By definition of $\prod_{i=0}^{\infty}$, $\exists! \sigma: \bigoplus_{i=0}^{\infty} M_i \rightarrow \prod_{i=0}^{\infty} M_i$

s.t $\rho_i \circ \sigma = \pi_i \quad i \geq 0$



By a property of Grothendieck category: σ is injective

Let $\beta: X \rightarrow \bigoplus_{i=0}^{\infty} M_i$ s.t $(1-m)\beta = 0$

aim: $\beta = 0$

claim $(1-m)\beta = 0 \Rightarrow \beta = 0$

Since σ is injective,

$$\sigma\beta = 0$$

\updownarrow by defn of $\prod_{i=0}^{\infty} \mathbb{Z}$

$$\beta_i \sigma\beta = 0$$

$$\underline{\pi_i \beta = 0} \xleftarrow{(1-m)\beta=0} \pi_i m\beta = 0 \quad \forall i \geq 0$$

claim $\begin{cases} \pi_0 m = 0 \\ \pi_i m = m_{i-1} \circ \pi_{i-1} \end{cases}$

$$\pi_i m\beta = m_{i-1} (\pi_{i-1} m\beta)$$

\exists 归纳法 $\pi_0 m\beta = 0$ \Rightarrow $\pi_i m\beta = m_{i-1} (\pi_{i-1} m\beta) = 0 \Rightarrow \pi_i m\beta = 0$

只需 claim: $\pi_i m \sigma_j = m_{i-1} \circ \pi_{i-1} \sigma_j$

(4.1) $m\sigma_i = \sigma_{i+1} m_i$
 $\pi_i \sigma_j = 0 \quad \forall i \neq j$

$$\left\{ \begin{aligned} \pi_i \sigma_{j+1} m_j &= \delta_{i,j+1} \cdot m_j = m_{i-1} \pi_{i-1} \sigma_j \\ \pi_0 m \sigma_j &= \pi_0 \sigma_{j+1} m_j = 0 \end{aligned} \right.$$

#

Rmk: (AB4) : (exact direct sum)

If $\exists \bigoplus_{i=0}^{\infty}$ in \mathcal{A} . \forall $0 \rightarrow L_i \rightarrow M_i \rightarrow N_i \rightarrow 0$ $\forall i \in I$. I arbitrary index set.
S.E.S in \mathcal{A}

$$0 \rightarrow \bigoplus_{i=0}^{\infty} L_i \rightarrow \bigoplus_{i=0}^{\infty} M_i \rightarrow \bigoplus_{i=0}^{\infty} N_i \rightarrow 0 \quad \text{S.E.S in } \mathcal{A}$$

i.e. $\bigoplus_{i=0}^{\infty}$ preserves injective

(preserves surjective \leftarrow trivial)

Cor: 4.5.3: Let \mathcal{A} be a Grothendieck category:

(i) \mathcal{A} has (AB4)

(ii) Let $v_i: P_i \rightarrow X_i$ quasi-iso. in \mathcal{A} $\forall i \in I$

Then $\bigoplus_{i \in I} v_i: \bigoplus_{i \in I} P_i \rightarrow \bigoplus_{i \in I} X_i$ is quasi-iso.

∴ pf: (ii) from (i)

$$\ker \left(\bigoplus_{i \in I} d_i^n \right) = \bigoplus_{i \in I} \ker d_i^n$$

$$\text{Im} \left(\bigoplus_{i \in I} d_i^n \right) = \bigoplus_{i \in I} (\text{Im } d_i^n)$$

$$\bigoplus_{i \in I} v_i: \bigoplus_{i \in I} P_i \rightarrow \bigoplus_{i \in I} X_i$$

$$H^n \left(\bigoplus_{i \in I} v_i \right) = \bigoplus_{i \in I} H^n(v_i): \bigoplus_{i \in I} H^n(P_i) \rightarrow \bigoplus_{i \in I} H^n(X_i)$$

Thm 4.5.4

\mathcal{A} : Grothendieck category with enough projective object

$\forall X \in \mathcal{L}(\mathcal{A})$ has a homotopy projective resolution $f: P \xrightarrow{\textcircled{1}} X$, $P \in K_{\text{proj}}(\mathcal{A})$

Moreover, P is projective complex. ($f: P \xrightarrow{\textcircled{2}} X$)

If $g: X \rightarrow Y$ is a homotopy equivalence. $\textcircled{3}$

$C_X: P \rightarrow X$

are homotopy proj. resolution

$C_Y: Q \rightarrow Y$

then $\exists u: P \rightarrow Q$ st

$$\begin{array}{ccc} P & \xrightarrow{C_X} & X \\ u \downarrow & & \downarrow d \\ Q & \xrightarrow{C_Y} & Y \end{array} \quad \textcircled{4}$$

$\textcircled{1}$ Pf: $\textcircled{1}$: By Thm 4.2.1

$v_n: P_n \rightarrow T_{\leq n} X$ (左无限截断) and \exists infinite sequence.

$$T_{\leq 0} X \xrightarrow{u_0} T_{\leq 1} X \xrightarrow{u_1} T_{\leq 2} X \xrightarrow{u_2} T_{\leq 3} X \xrightarrow{u_3} \dots$$

By prop 4.2.8(1) $P_X \rightarrow P_Y$ we have c.d. in $K(\mathcal{A})$

$$\begin{array}{ccccccc} & & \downarrow & \cong & \downarrow & & \\ & & X & \xrightarrow{\cong} & Y & & \\ P_0 & \xrightarrow{P_0} & P_1 & \xrightarrow{P_1} & P_2 & \xrightarrow{P_2} & \dots \rightarrow P_n \xrightarrow{P_n} \dots \\ \downarrow u_0 & & \downarrow u_1 & & \downarrow u_2 & & \downarrow u_n \\ T_{\leq 0} X & \xrightarrow{u_0} & T_{\leq 1} X & \xrightarrow{u_1} & T_{\leq 2} X & \xrightarrow{u_2} & \dots \rightarrow T_{\leq n} X \xrightarrow{u_n} \dots \end{array} \quad (*)$$

By Cor 12.16.2 $\mathcal{L}(\mathcal{A})$ is Grothendieck category. By (4.2) ($m_{i+1} = \overline{u_{i+1} m_i} \quad \forall i \geq 0$)

We have chain maps: $f: u: \bigoplus_{i=0}^{\infty} T_{\leq i} X \rightarrow \bigoplus_{i=0}^{\infty} T_{\leq i} X$

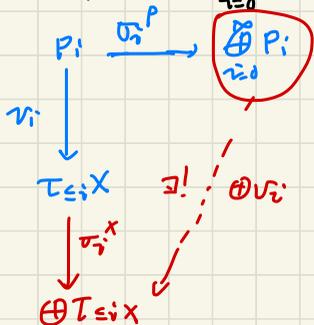
$$p: \bigoplus_{i=0}^{\infty} P_i \longrightarrow \bigoplus_{i=0}^{\infty} P_i$$

$$\text{s.t. } \begin{cases} u \sigma_i^X = \sigma_{i+1}^X u_i \\ p \sigma_i^P = \sigma_{i+1}^P p_i \end{cases} \quad \forall i \geq 0$$

$$\text{where } \begin{cases} \sigma_i^P: P_i \longrightarrow \bigoplus_{i=0}^{\infty} P_i \\ \sigma_i^X: T_{i+1} X \longrightarrow \bigoplus_{i=0}^{\infty} T_{i+1} X \end{cases} \quad \forall i \geq 0$$

By universal property of $\bigoplus_{i=0}^{\infty} P_i$

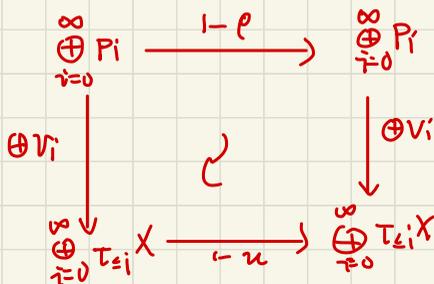
$$\exists! \bigoplus v_i: \bigoplus_{i=0}^{\infty} P_i \longrightarrow \bigoplus_{i=0}^{\infty} T_{i+1} X \quad \text{s.t.} \quad v_i^X u_i = (\bigoplus v_i) \sigma_i^P \quad \forall i \geq 0$$



$$(4.6) \quad u \sigma_i^X = \sigma_{i+1}^X u_i \quad p \sigma_i^P = \sigma_{i+1}^P p_i \quad i \geq 0$$

$$(4.7) \quad \sigma_i^X v_i = (\bigoplus v_i) \sigma_i^P \quad \forall i \geq 0$$

we have c.d. in $k(A)$:



$$\underbrace{u(\sigma_i^X v_i)}_{(4.6)} \stackrel{(4.7)}{=} \underbrace{v_{i+1}^X u_i v_i}_{(4.6)} \stackrel{(4.6)}{=} \underbrace{(\bigoplus_{i=0}^{\infty} v_i) \sigma_{i+1}^P}_{(4.6)} \stackrel{(4.7)}{=} \underbrace{(\bigoplus_{i=0}^{\infty} v_i) \sigma_i^P}_{(4.7)} \stackrel{(4.7)}{=} \underbrace{v_{i+1}^X v_{i+1} p_i}_{(4.7)} \quad \forall i \geq 0 \quad \text{follow from (4.6) (4.7)}$$

$$\sigma_{i+1}^X u_i v_i \stackrel{(4.6)}{=} \sigma_{i+1}^X u_{i+1} p_i \quad \checkmark$$

It follows that:

$$\begin{array}{ccccccc}
 \bigoplus_{i=0}^{\infty} P_i & \xrightarrow{1-p} & \bigoplus_{i=0}^{\infty} P_i & \longrightarrow & \text{Cone}(1-p) & \longrightarrow & \left(\bigoplus_{i=0}^{\infty} P_i \right) [1] \\
 \downarrow \oplus \nu_i & & \downarrow \oplus \nu_i & & \downarrow \cong h & & \downarrow (\oplus \nu_i)[1] \\
 \bigoplus_{i=0}^{\infty} \tau_{\leq i} X & \xrightarrow{1-u} & \bigoplus_{i=0}^{\infty} \tau_{\leq i} X & \longrightarrow & \text{Cone}(1-u) & \longrightarrow & \left(\bigoplus_{i=0}^{\infty} \tau_{\leq i} X \right) [1]
 \end{array}$$

Cor 4.5.2 $\implies \oplus \nu_i$ quasi-iso + Thm 2.4.1 + five lemma $\implies h \rightarrow$ quasi-iso
 \implies long exact sequence

By cor 4.5.2 $(1-u)$ is injective, \implies s.e.c as following:

$$0 \rightarrow \bigoplus_{i \in \mathbb{Z}} P_i \xrightarrow{1-u} \bigoplus_{i \in \mathbb{Z}} P_i \rightarrow \text{Coker}(1-u) \rightarrow 0$$

By prop 2.3.2 \exists quasi-iso $g: \text{Cone}(1-u) \xrightarrow{p} \text{Coker}(1-u)$
 $(\text{Cone}(1-p) \xrightarrow{\uparrow h} \text{Cone}(1-u) \xrightarrow{f=gh} \text{Coker}(1-u))$

Thus, we have quasi-iso $f = gh: \text{Cone}(1-p) \rightarrow \text{Coker}(1-u) \xrightarrow{\text{Cor 9.5.1}} \varinjlim \tau_{\leq n} X = X$
lem 2.6.2 (i)

$$f: \text{Cone}(1-p) \rightarrow X$$

Since every P_i is proj. complex $\implies \bigoplus P_i \in \text{Khpj}(A)$

$(\mathcal{P} \subseteq \text{Khpj} + \text{closed under } \bigoplus)$

d.s.: $\bigoplus_{i=0}^{\infty} P_i \xrightarrow{1-p} \bigoplus_{i=0}^{\infty} P_i \rightarrow \text{Cone}(1-p) \rightarrow \left(\bigoplus_{i=0}^{\infty} P_i \right) [1]$

$\implies \text{Cone}(1-p) \in \text{Khpj}(A)$
 Moreover, proj. cplx. (2)

For surjective

③

$$\begin{array}{c} \boxed{\quad} \longrightarrow X \\ \uparrow \\ \text{Khprij}(A) \end{array}$$

add zero object in $\mathcal{K}(A)$

$$\cdots \rightarrow 0 \rightarrow P \rightarrow P \rightarrow 0 \rightarrow \cdots \quad \textcircled{3}$$

(contractible complex)

For unique up to homotopy equi.

④

Lemma 4.4.4 (iii) \forall quasi-iso $s: X \rightarrow Y$, $P \in \text{Khprij}(A)$

$$\text{Hom}_{\mathcal{K}(A)}(P, s) = \text{Hom}_{\mathcal{K}(A)}(P, X) \xrightarrow{\cong} \text{Hom}_{\mathcal{K}(A)}(P, Y) \quad \text{iso as abelian group}$$

in particular $\text{Hom}_{\mathcal{K}(A)}(P, s) = \text{Hom}_{\mathcal{K}(A)}(P, Q) \xrightarrow{\cong} \text{Hom}_{\mathcal{K}(A)}(P, Y)$

ie $\exists! u$ s.t.

$$\begin{array}{ccc} u & \longrightarrow & \text{cylinder} \\ P & \xrightarrow{u} & Q \\ \text{ex} \downarrow & & \downarrow \text{cylinder} \\ X & \xrightarrow{\alpha} & Y \end{array}$$

and u quasi-iso.

By Lwr 4.4.5.

$u: Q \rightarrow P$ quasi-iso, $P, Q \in \text{Khprij}(A) \Rightarrow u$ homotopy equ. $v \#$.

Cor 4.5.5. \mathcal{A} : Grothendieck category with enough proj. $\mathcal{K} \text{proj}(\mathcal{A})$

$\exists P$ proj. complex s.t. $P \simeq Q$ (homotopy equiv.)

i.e. $K(P) \subseteq K \text{proj}(\mathcal{A}) \stackrel{\checkmark}{\subseteq} K(\mathcal{P})$

\mathcal{P} : full subcategory consisting of all proj. obj.

1.2.1: By Thm 4.5.4. \exists quasi-iso $f: P \rightarrow Q$

P proj. complex.

By Cor 4.4.5. $f: P \rightarrow Q$ homotopy equivalence

Rmk: \mathcal{O} homotopy proj complex $\xrightarrow{\text{homotopy equivalence}}$ projective complex

② a homotopy proj complex is not projective complex in general

③ homotopy proj complex \cap proj complex =: dg-proj complex

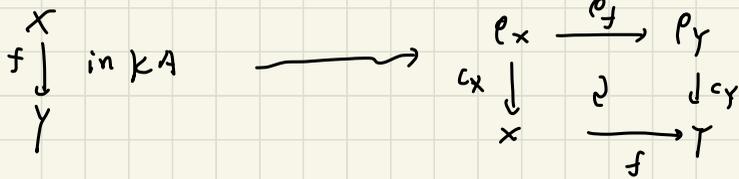
轴复形 (齐次复形为投射复形)

prop 4.5.6 (Similar as prop 4.2.8)

\mathcal{A} : Grothendieck category with enough proj.

(1) $i: K \text{proj}(\mathcal{A}) \rightleftarrows K(\mathcal{A}): p \quad i \dashv p \quad (p \text{ right adjoint to } i)$

$\forall X \in K(\mathcal{A}), \quad \textcircled{C} X: \mathcal{P} \rightarrow X$

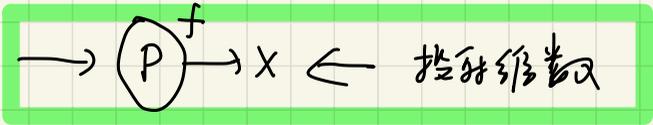


(2). ρ is a triangulated functor

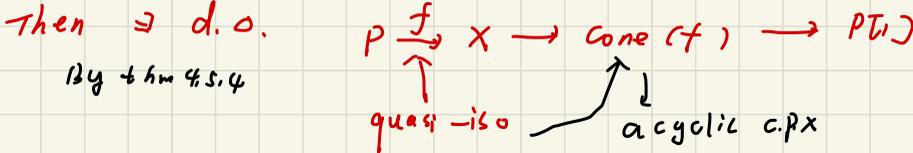
Similar as prop 4.2.8 $\left(\begin{array}{l} \text{Cor 4.2.4} \\ \text{Cor 4.2.6} \end{array} \right) \xrightarrow{\text{4.2.8}} \left(\begin{array}{l} \text{lemma 4.4.4 (iii)} \\ \text{+} \\ \text{Cor 4.4.5} \end{array} \right) \xrightarrow{\text{4.5.6}}$

prop 4.5.7: \mathcal{A} : Grothendieck category with enough proj.

$\forall x \in \mathcal{K}(\mathcal{A}), \exists$ split s.e.c



pf: let $f: P \rightarrow X$ quasi-iso and $P \in \mathcal{K}_{\text{hproj}}(\mathcal{A})$



By lem 4.5.4



Rmk 4.5.8: \mathcal{T} : triangulated category with $\bigoplus_{i=0}^{\infty}$ for arbitrary infinite morphism sequence.

$$U_0 \xrightarrow{u_0} U_1 \xrightarrow{u_1} U_2 \rightarrow U_3 \rightarrow \dots$$

$$\text{let } u: \bigoplus_{i=0}^{\infty} U_i \longrightarrow \bigoplus_{i=0}^{\infty} U_i \quad \text{s.t.} \quad u\sigma_i = \sigma_{i+1} d_i \quad (\forall i \geq 0)$$

define: $\text{Conel}(u)$ is called homotopy limit of infinite sequence.
denote by $\underline{\text{holim}} u$ (or $\underline{\text{holim}} U_i$)

then Δ in \mathcal{T} can be written as:

$$\bigoplus_{i=0}^{\infty} U_i \xrightarrow{1-u} \bigoplus_{i=0}^{\infty} U_i \longrightarrow \underline{\text{holim}} U_i \longrightarrow \left(\bigoplus_{i=0}^{\infty} U_i \right)[1]$$

↑

unique up to iso.

re introduced Thm 4.5.4 by $P = \underline{\text{holim}} P_i$ (dg-proj complex)

$$\begin{array}{ccccccc} P_0 & \xrightarrow{P_0} & P_1 & \xrightarrow{P_1} & P_2 & \xrightarrow{P_2} & \dots \rightarrow P_n \xrightarrow{P_n} \dots \\ r_0 \downarrow & & \downarrow r_1 & & \downarrow r_2 & & \downarrow \\ T_{s_0} X & \rightarrow & T_{s_1} X & \rightarrow & T_{s_2} X & \rightarrow & \dots \rightarrow T_{s_n} X \rightarrow \dots \\ \dots & & & & & & \end{array}$$

prop 4.5.9. \mathcal{A} : Grothendieck category with enough proj. \mathcal{P} : full subcategory consisting of projective obj.

Let $\text{Tri}(\mathcal{P}) \supseteq \mathcal{P}$ the smallest triangulated category closed under $\bigoplus_{i=0}^{\infty}$ as a full subcategory of $\mathcal{K}(\mathcal{A})$

Then $\text{Tri}(\mathcal{P}) \supseteq \mathcal{K}_{\text{hproj}}(\mathcal{A})$

1) $\text{Tri}(\mathcal{P}) \subseteq \mathcal{K}_{\text{hproj}}(\mathcal{A})$ ✓

2) $\mathcal{K}_{\text{hproj}}(\mathcal{A}) \subseteq \text{Tri}(\mathcal{P})$

claim: $\mathcal{K}(\mathcal{P}) \subseteq \text{Tri}(\mathcal{P})$

then $\forall P \in \mathcal{K}_{\text{hproj}}(\mathcal{A})$

By the pf of Thm 4.5.4 + claim

\exists quasi-iso $\mathcal{Q} \rightarrow \mathcal{P}$

\mathcal{Q} : dg proj. complex, by 4.4.5

$\mathcal{P} \xrightarrow{\simeq} \mathcal{Q}$: homotopy equivalence

\Rightarrow By TR1 $\Rightarrow P \in \text{Tri}(\mathcal{P})$

For claim: Let $P \in \mathcal{K}(\mathcal{P})$

Consider

$$P_{\geq 0} \xrightarrow{m_0} P_{\geq 1} \xrightarrow{m_1} P_{\geq 2} \xrightarrow{m_2} \dots$$

m_i : injective

$P_{\geq i} \rightarrow$ proj. complex

i.e. $P_{\geq i} \in \text{Tri}(\mathcal{P})$

By Cor 2.6.2 (iv)

$$\varinjlim_{i=0} P_{\geq i} = P$$

$$\text{Cone} \left(\varinjlim_{i=0} P_{\geq i} \xrightarrow{1-m} \varinjlim_{i=0} P_{\geq i} \right) \cong \text{Coker}(1-m)$$

By Cor 4.5.2 $\Rightarrow 1-m$ injective

By prop 3.2

$$\text{Cone}(1-m) \rightarrow \text{Coker}(1-m) = P \text{ quasi-iso}$$

proj complex \uparrow proj complex

$\Rightarrow P \simeq \text{Cone}(1-m) \in \text{Tri}(\mathcal{P})$ (closed under $\bigoplus_{i=0}^{\infty}$)

prop 5.1.1. A : Abelian category

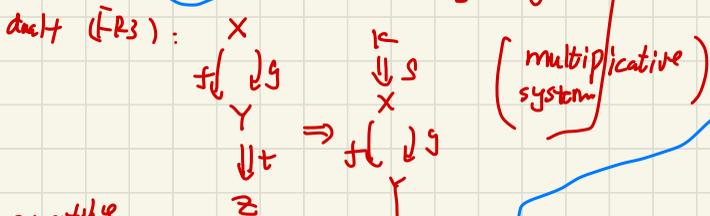
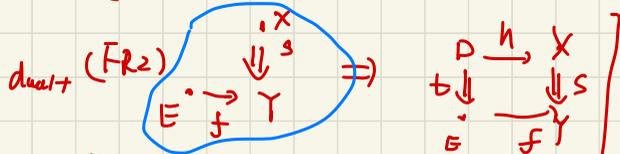
$K^{\mathcal{A}}$: homotopy category of A .

let $\mathcal{Q}_{\mathcal{A}}$ be the classes of quasi-isom.

Then $\mathcal{Q}_{\mathcal{A}}$ is saturated compatible multiplicative system.

- ① $\{ f \in S, h \in S \Rightarrow f \circ h \in S \}$
 ② think $(K^{\mathcal{A}})_{\mathcal{Q}_{\mathcal{A}}} = \{ \text{acyclic c.p.x} \}$

1st (FR1) $f \in S, g \in S \Rightarrow g \circ f \in S$

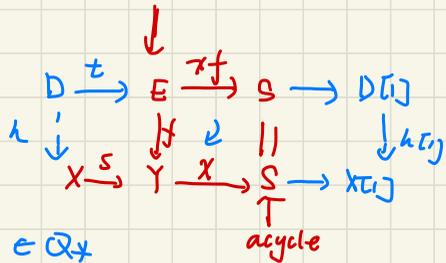
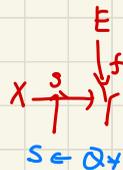


compatibility (FR4) $S \circ S \Rightarrow S \circ S$

(FR5) $X \rightarrow Y \rightarrow Z \rightarrow TX$
 $f \downarrow g \downarrow h \downarrow \tau f$
 $X' \rightarrow Y' \rightarrow Z' \rightarrow TX'$
 $f, g \in S \Rightarrow h \in S$

FR1: $H^i(g \circ f) = H^i(g) \circ H^i(f)$ ✓

FR2:



FR3: let $h = f \circ g$ then $t \circ h = 0$



$$\dots \text{Hom}_{K(\mathcal{A})}(X, S) \xrightarrow{(X, z)} \text{Hom}(X, Y) \xrightarrow{(X, t)} \text{Hom}(X, Z) \rightarrow \dots$$

$h \mapsto t \circ h = 0$

$$k \longmapsto h$$

$\exists k \in \text{Hom}(X, S)$ st $h = z \circ k$

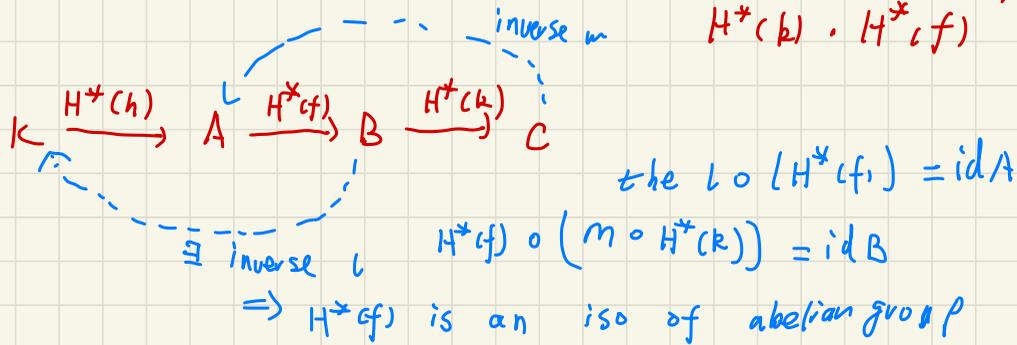
$$X \xrightarrow{k} S \hookrightarrow \Delta \xrightarrow{s} X \xrightarrow{p} S \rightarrow K[\tau]$$

then $S \in \mathcal{Q}_{\mathcal{A}}$. $h \circ s = 0$

Since $t \in \mathcal{Q}_{\mathcal{A}} \Rightarrow S$ acyclic

(FR4) ✓ (FR5) ✓ (Thm 2.4) + TR3 + five lemma

For \mathcal{Q}_* is saturated: $fh \in \mathcal{Q}_*$, $kf \in \mathcal{Q}_* \Rightarrow H^*(f) \cdot H^*(h) = H^*(k) \cdot H^*(f)$ are iso of abelian group



$\Rightarrow f \in \mathcal{Q}_*$

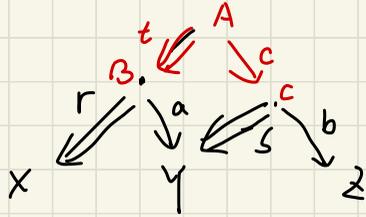
Finally: $\mathcal{Y}(\mathcal{Q}_*) = \{ z \in K(A) \mid \exists X \xrightarrow{f} Y \rightarrow Z \rightarrow X[1] \text{ s.t. } f \in \mathcal{Q}_* \}$
 $= \{ \text{classes of acyclic complex} \}$

Defi: Derived category of A defined by $D(A) := \mathcal{Q}^{-1} K(A)$

$\text{obj } D(A) \leftrightarrow \text{obj } K(A)$

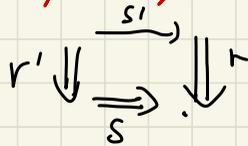
$\text{Hom}_{D(A)}(X, Y) : X \xleftarrow{s \in \mathcal{Q}} \cdot \xrightarrow{b} Y$
 右分式 (左分式)

Composition: $b/s \circ a/r := bc/rt$



identity: $id_X/id_X = s/s$

" + " : $a/r + b/s = a'+b'/t$



$$\begin{cases} r \circ s' = s \circ r' = t \\ a' = a s' \\ b' = b r' \end{cases}$$

$\text{Hom}_{\mathcal{D}(A)}(X, Y) \leftarrow$ abelian group

zero obj: $0/t = \delta/s$

$\cdot \mathcal{Q} = \alpha/s$, $s \in \mathcal{Q}$, \mathcal{Q} zero morphism iff $\exists t$. s.t. $at=0$
 $s+t \in \mathcal{S}$

$X \in \mathcal{D}(A)$ zero object $(\Leftrightarrow) \exists 0: Z \rightarrow X$ 为拟同构
← acyclic

Similarly: $\mathcal{D}^{\vee}(A)$

$F^{\vee}: k^{\vee}(A) \rightarrow \mathcal{D}^{\vee}(A)$ (局部化函子)

$f \mapsto F^{\vee}(f) = f / \text{id}_X$

① $F^{\vee}(f)$ is isomorphism in $\mathcal{D}^{\vee}(A)$ iff \exists quasi-iso $t: X' \rightarrow X$ st
 $f \circ t \sim 0$ (同构于 0)
 (右'式)

$$X' \xrightarrow{t} X \xrightarrow{f} Y$$

iff: \exists quasi-iso $s: Y \rightarrow Y'$

$$X \xrightarrow{f} Y \xrightarrow{s} Y'$$

st $s \circ f \sim 0$ (右'式)
 FR3

② X is an zero obj $\Leftrightarrow X \in e^{\vee}(A)$
 is acyclic ✓

Remark: $F^{\vee}(f)$ is zero morphism in $\mathcal{D}^{\vee}(A)$

But $f \neq 0$

Example $\mathcal{X}' = \dots \rightarrow 0 \rightarrow Z \xrightarrow{2} Z \rightarrow Z \rightarrow 0 \dots$

acyclic complex.

let $f = \text{id}_X$. $f \sim 0$

But $f \cdot X \rightarrow 0$ is quasi-iso

$F(f)$ is zero morphism in $D^*(A)$

$$\text{id}_X = d^{n+1} s^n + s^{n+1} d^n$$

$$\begin{array}{ccccccccccc}
 0 & \longrightarrow & Z & \xrightarrow{2} & Z & \xrightarrow{\pi} & Z_2 & \longrightarrow & 0 & \longrightarrow & 0 & \dots \\
 \downarrow & \swarrow 0 & \downarrow & \swarrow s_n & \downarrow & \swarrow s_{n+1} & \downarrow & \swarrow 0 & \downarrow & & & \\
 0 & \longrightarrow & Z & \xrightarrow{2} & Z & \xrightarrow{\pi} & Z_2 & \longrightarrow & 0 & \longrightarrow & 0 & \\
 & & \downarrow 0 & & \downarrow 2 & & \downarrow \pi & & \downarrow 0 & & & \\
 & & 0 & & Z & & Z_2 & & 0 & & &
 \end{array}$$

$$\boxed{2s_n = \text{id}} \quad \}$$

$$s_n = \frac{1}{2} \notin \text{Hom}(Z, Z)$$

Prop 5.1.2

\mathcal{A} : abelian category:

$$0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0 \text{ in } \mathcal{A}$$

$\exists h \in \text{Hom}_{D^*(\mathcal{A})}(Z, X[1])$ (右逆, 非正则态射)

\Rightarrow $X \xrightarrow{f} Y \rightarrow Z \xrightarrow{h} X[1]$ is distinguished Δ in $D^*(\mathcal{A})$

Furthermore, all $d. \Delta$ in $D^*(\mathcal{A})$ are iso to this form Δ , $\ast \in \{ \mathbb{Z}, +, -, 0 \}$

pf: By prop 2.3.2

$$0 \rightarrow X \xrightarrow{u} Y \xrightarrow{v} Z \rightarrow 0$$

(0.n): $\text{Cone}(u) \rightarrow \text{Coker}(u) = Z$ quasi-iso

