§4.9. Grothendieck ring of s.s. tensor categories.

· k is an algebraically closed field.

prop. 4.9.1 If $\mathcal{C}$ is a s.s. multitensor category, then $Gr(\mathcal{C})$ is a based ring. ...... s.s. tensor category, then $Gr(\mathcal{C})$ is a unital based ring.
If $\mathcal{C}$ is a (multi) fusion category, then $Gr(\mathcal{C})$ is a (multi) fusion ring.

proof: locally finite abelian cat. $\mathcal{C}$. $Gr(\mathcal{C})$ is a $\mathbb{Z}_+$-ring with unit $[1]$.
$\mathbb{Z}_+$-basis: simple obj. (iso. classes). I_0: class of simple subobjects of 1.
The involution $*$ is the duality map.
(by prop 4.8.1. ${}^*V \cong V^*$, $\Rightarrow V = V^{**}$)

· $\mathrm{Hom}(V_i, V_j) = \begin{cases} k & i=j \\ 0 & i\neq j \end{cases}$

$\mathrm{Hom}(V_i \otimes V_j, 1) \cong \mathrm{Hom}(V_j, V_i^*) = \mathrm{Hom}(V_j, V_{i^*}) = \begin{cases} k, & j=i^* \\ 0, & i\neq j \end{cases}$

· 1: simple.

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Rem. 4.9.2: The conclusion of prop 4.9.1 fails for non-s.s. tensor categories.
example 3.1.9 (ii) P51.

(3.2) $\tau(b_i b_j) = \begin{cases} 1 & \text{if } i=j^* \\ 0 & \text{if } i\neq j^* \end{cases}$ is fails.

i.e. $[X\otimes X^* : 1] > 1$ for a simple object X.

Consider X the 2-dim irreducible rep. of the group S_3 over field Char. 2.

$[X\otimes X^* : 1]$

$\mathrm{Hom}(X, X) = k$.

$$\frac{X\otimes X^*}{1 \quad -1} \quad \text{standard rep.} \quad -1 \quad 1$$

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Example 4.9.3
Let $\mathcal{C}$ be the category of finite dimensional rep. of the Lie alg $sl_2(\mathbb{C})$
Then simple objects of this category are irreducible rep. V_m of dim $m+1$
for $m = 0, 1, 2, \dots$; $V_0 = 1$.
The Grothendieck ring of $\mathcal{C}$ is determined by Clebsch–Gordan rule:

$$(4.1) \quad V_i \otimes V_j = \bigoplus_{l=0}^{\min(i,j)} V_{i+j-2l}.$$

duality map: identity. (unital based ring)

· Let $\mathcal{C}$ be a s.s. multitensor cat. with simple objects $\{X_i\}_{i\in I}$.
$1 = \bigoplus_{i\in I_0} X_i$, Let $H_{ij}^l := \mathrm{Hom}(X_l, X_i \otimes X_j)$

$H_{pj}^p = \mathrm{Hom}(X_p, X_p\otimes X_j) = k$
$= \mathrm{Hom}(X_p, X_p)$

$H_{ip}^p = \mathrm{Hom}(X_p, X_i\otimes X_p) = k$.

where $X_p \in \mathcal{C}_{ij}$, $p \in I$ and $i,j \in I_0$. $X_i \otimes \mathcal{C} \otimes X_j$

$X_p = X_i \otimes Y \otimes X_j$. $X_i \otimes X_i = X_i$, $\forall i \in I_0$

$X_i \otimes X_p \otimes X_j = X_p$.

· we have $X_i \otimes X_j = \bigoplus_{\ell} H_{ij}^{\ell} \otimes X_\ell$ Define tensor product.

$X_i \otimes X_j = \bigoplus_{\ell} [X_i \otimes X_j : X_\ell] X_\ell = \bigoplus_{\ell} \dim \mathrm{Hom}(X_\ell, X_i \otimes X_j) X_\ell$
$= \bigoplus_{\ell} n_\ell X_\ell \cong \bigoplus_{\ell} \bigoplus_{n_\ell} k \otimes X_\ell = \bigoplus_{\ell} H_{ij}^{\ell} \otimes X_\ell$.

$(X_{i_1} \otimes X_{i_2}) \otimes X_{i_3} \cong \bigoplus_{i_4} \bigoplus_{j} H_{i_1 i_2}^{j} \otimes H_{j i_3}^{i_4} \otimes X_{i_4}$

$X_{i_1} \otimes (X_{i_2} \otimes X_{i_3}) \cong \bigoplus_{i_4} \bigoplus_{\ell} H_{i_1 \ell}^{i_4} \otimes H_{i_2 i_3}^{\ell} \otimes X_{i_4}$.

$\mathrm{Hom}(X_\ell, X_i \otimes X_j)$

$\mathrm{Hom}(X_{\ell'}, X_i \otimes X_j)$

$\mathrm{Hom}(f, X_i \otimes X_j)$

$H_{ij}^{\ell} \otimes X_\ell \longrightarrow H_{ij}^{\ell'} \otimes X_{\ell'}$

The associativity constraint $\Rightarrow$

(4.12) $\Phi_{i_1 i_2 i_3}^{i_4} : \bigoplus_{j} H_{i_1 i_2}^{j} \otimes H_{j i_3}^{i_4} \xrightarrow{\sim} \bigoplus_{\ell} H_{i_1 \ell}^{i_4} \otimes H_{i_2 i_3}^{\ell}$

The matrix blocks of these isom.

(4.13) $(\Phi_{i_1 i_2 i_3}^{i_4})_{j\ell} : H_{i_1 i_2}^{j} \otimes H_{j i_3}^{i_4} \longrightarrow H_{i_1 \ell}^{i_4} \otimes H_{i_2 i_3}^{\ell}$

are called 6j-symbols.

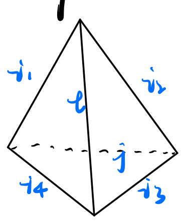
Groth ~~Categorfive~~ fusion cat.

D_8 Q_8 K_8

Example 4.9.4.

$\mathcal{C}$: the cat. of f.d. rep of $sl_2(\mathbb{C})$.
Then H_{ij}^{ℓ} are 0- or 1-dim by (4.11). $V_i \otimes V_j = \bigoplus_{\ell=0}^{\min(i,j)} V_{i+j-2\ell}$.

Fact: the map $(\Phi_{i_1 i_2 i_3}^{i_4})_{j\ell}$ is a map between nonzero (i.e. 1-dim) spaces iff:



+ every face's perimeter is even.

$V_i \otimes V_j = \bigoplus_{\ell=0}^{\min(i,j)} V_{i+j-2\ell}$. $H_{ij}^{\ell} = \mathrm{Hom}(X_\ell, \oplus X_{i+j-2k}) \neq 0$.

① even odd odd.

② even even even

X_ℓ appears $X_{i+j}, \cdots, X_{|i-j|}$

Racach coefficients or classical 6j-symbols.

Exercise 4.9.5.

Write down explicitly the relation on 6j-symbols coming from the pentagon identity. If $\mathcal{C} = \mathrm{Rep}(sl_2(\mathbb{C}))$, the relation is called the Elliott-Biedenharn relation.

proof: "The pentagon axiom"

$$\begin{tikzcd}
 & ((X_{i_1}\otimes X_{i_2})\otimes X_{i_3})\otimes X_{i_4} \arrow[dl] \arrow[dr, "a_{X_{i_1}\otimes X_{i_2}, X_{i_3}, X_{i_4}}"] & \\
(X_{i_1}\otimes (X_{i_2}\otimes X_{i_3}))\otimes X_{i_4} \arrow[d] \arrow[rr, phantom, "\circlearrowright"] & & (X_{i_1}\otimes X_{i_2})\otimes (X_{i_3}\otimes X_{i_4}) \arrow[d] \\
X_{i_1}\otimes ((X_{i_2}\otimes X_{i_3})\otimes X_{i_4}) \arrow[rr] & & X_{i_1}\otimes (X_{i_2}\otimes (X_{i_3}\otimes X_{i_4}))
\end{tikzcd}$$

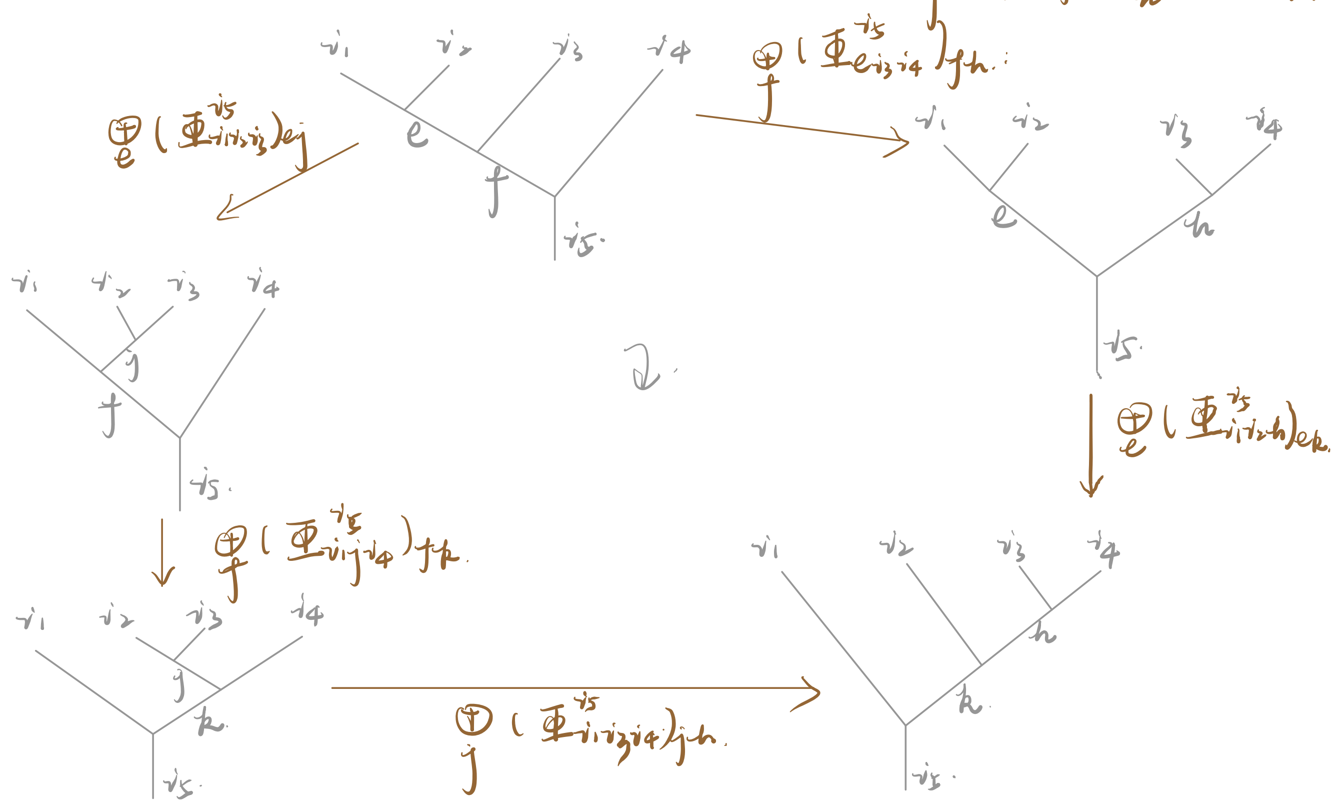
Note that.

$$((X_{i_1}\otimes X_{i_2})\otimes X_{i_3})\otimes X_{i_4} = \bigoplus_{e,f,i_5} H^e_{i_1i_2}\otimes \underline{H^f_{ei_3}\otimes H^{i_5}_{fi_4}}\otimes X_{i_5}$$

$$\bigoplus_{e,f} H^e_{i_1i_2}\otimes H^f_{ei_3}\otimes H^{i_5}_{fi_4} \longrightarrow \bigoplus_{e,h} H^e_{i_1i_2}\otimes H^{i_5}_{eh}\otimes H^h_{i_3i_4}$$

$$(X_{i_1}\otimes X_{i_2})\otimes (X_{i_3}\otimes X_{i_4}) = \bigoplus_{e,h,i_5} H^e_{i_1i_2}\otimes \underline{H^{i_5}_{eh}\otimes H^h_{i_3i_4}}\otimes X_{i_5}$$

$$\bigoplus_f H^f_{ei_3}\otimes H^{i_5}_{fi_4} \to \bigoplus_h H^{i_5}_{eh}\otimes H^h_{i_3i_4}$$



$$\Rightarrow \bigoplus_e (\Phi^{i_5}_{i_1 i_2 h})_{ek} \circ \bigoplus_f (\Phi^{i_5}_{e i_3 i_4})_{fh} = \bigoplus_j (\Phi^{i_5}_{i_2 i_3 i_4})_{jh} \circ \bigoplus_f (\Phi^{i_5}_{i_1 j i_4})_{fk} \circ \bigoplus_e (\Phi^{i_5}_{i_1 i_2 i_3})_{ej}$$

$$\Rightarrow (\Phi^{i_5}_{i_1 i_2 h})_{ek} (\Phi^{i_5}_{e i_3 i_4})_{fh} = \bigoplus_j (\Phi^{i_5}_{i_2 i_3 i_4})_{jh} \circ (\Phi^{i_5}_{i_1 j i_4})_{fk} \circ (\Phi^{i_5}_{i_1 i_2 i_3})_{ej}$$

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Example 4.9.6. $\mathcal{C} = Rep(U_q(sl_2))$ q is a root of unity.
$(\Phi^{i_4}_{i_1 i_2 i_3})_{je}$ are called q-Racah Coefficients or quantum 6j-sym.

· Further, evaluation and coevaluation maps define elements.
(4.14) $\alpha_{ij} \in (H^j_{ii^*})^*$ and $\beta_{ij} \in H^j_{ii^*}$, $j \in I_0$

· coeval: $\mathbb{1} \longrightarrow X_i \otimes (X_i)^* = X_i \otimes X_{i^*}$

$$\bigoplus_{i \in I_0} X_i \longrightarrow \bigoplus_j H^j_{ii^*} \otimes X_j, \quad j \in I_0$$

defines $\beta_{ij} \in H^j_{ii^*}$

evaluation. $(X_i)^* \otimes X_i \longrightarrow \mathbb{1}$.

$$\bigoplus_j H^j_{i^* i} \otimes X_j \longrightarrow \bigoplus_{i \in I_0} X_i, \quad j \in I_0$$

defines $\alpha_{ij} \in H^j_{i^* i} = (H^j_{ii^*})^*$, $j^* = j$. $(ii^*)^* = ii^*$ & s.s.

§4.10. Categorification of based rings.

Def 4.10.1. Given a based ring R, its categorification over k is a s.s. multitensor cat over k, together with an isom of based rings

$$R \simeq Gr(\mathcal{C}).$$

· Let $\mathbb{Z}G$ be the unital based group ring of a group G. with basis $\{g \mid g \in G\}$ and $g^* = g^{-1}$.

prop 4.10.3. The categorifications of $\mathbb{Z}G$ are Vec^ω_G and they are parametrized by $H^3(G, k^\times)$.

proof: P38 $G \longrightarrow$ monoidal cat Vec^ω_G (s.s. tensor cat.).

ω is a 3-cocyle. invertible 降ω.只能为 Vec^ω_G

#

Rem 4.10.4. The tensor equivalence classes of categories Vec_G^ω are para... by $H^3(G, k^\times)/\mathrm{Out}(G)$.

rd

· Example 4.10.5 (Tambara – Yamagami fusion rings).

Let G : finite group, $TY_G := \mathbb{Z}G \oplus \mathbb{Z}X$

where X is a new basis vector with $gX = Xg = X$, $X^2 = \sum_{g \in G} g$

$X^* = X$. $\Rightarrow$ fusion ring.

· $\mathrm{FPdim}(g) = 1$, $\mathrm{FPdim}(X) = |G|^{\frac{1}{2}}$

$\mathrm{FPdim}(gg^{-1}) = \underline{\mathrm{FPdim}(g)}\,\mathrm{FPdim}(g^{-1}) = 1$

These rings are categorifiable iff G is abelian.

· Example 4.10.6. (Verlinde rings for sl_2)

Let $k \in \mathbb{Z}_+$. Define a unital $\mathbb{Z}_+$-ring $\mathrm{Ver}_k = \mathrm{Ver}_k(sl_2)$ with basis ϑ_i
$i = 0, \cdots, k$, $(\vartheta_0 = 1)$, duality : id.
truncated Clebsch – Gordan rule :

$$(4.15) \quad \vartheta_i \vartheta_j = \sum_{l = \max(i+j-k, 0)}^{\min(i,j)} \vartheta_{i+j-2l}.$$

$k = 12$, $i = 8$, $j = 9$.

$\vartheta_i\vartheta_j = \vartheta_{17} + \vartheta_{15} + \vartheta_{13} + \vartheta_{11} + \vartheta_9 + \vartheta_7 + \cdots$

$\vartheta_i\vartheta_j = \vartheta_7 + \vartheta_5 + \vartheta_3$

· $\mathrm{Ver}_0 = \mathbb{Z}$, $\mathrm{Ver}_1 = \mathbb{Z}[\mathbb{Z}_2]$, $\mathrm{Ver}_2 = TY_{\mathbb{Z}_2}$

· Exercise 4.10.7. Show that $\mathrm{FPdim}(\vartheta_j) = [j+1]_q := \dfrac{q^{j+1} - q^{-j-1}}{q - q^{-1}}$, where $q = e^{\frac{\pi i}{k+2}}$
$(q \neq 1)$

proof: ① $[j+1]_q$ is non-negative real number.

② ring homomorphism.

Assume $i \leq j$. $\vartheta_i \otimes \vartheta_j$.
Case 1: $i+j \leq k$. $\mathrm{FPdim}(\vartheta_i\vartheta_j) = \dfrac{q^{i+j+2} - q^{j-i} + q^{-i-j-2} - q^{i-j}}{(q-q^{-1})^2} = \mathrm{FPdim}(\vartheta_i)\mathrm{FPdim}(\vartheta_j)$

case 2: $i+j>k$. $\quad \vartheta_i \otimes \vartheta_j = \vartheta_{i+j-2(i+j-k)} + \vartheta \cdots\cdots + \vartheta_{i+j-2i}$.

$$\mathrm{FPdim}(\vartheta_i\vartheta_j) = \frac{q^{-i-j+2k+2} - q^{j-i} + q^{i+j-2k-2} - q^{i-j}}{(q-q^{-1})^2}$$

$$q = e^{\frac{\pi i}{k+2}}, \quad q^{-i-j+2k+2} = q^{-i-j-2}.$$

$$q^{i+j-2k-2} = q^{i+j+2}.$$

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. Subring Ver_p^{0} spanned by ϑ_j, j even.

$k=3$. $\Rightarrow$ Yang-Lee fusion. basis 1, X, $X^2 = X+1$. $X^* = X$.

$\mathrm{FPdim}(X) = \frac{1+\sqrt5}{2}$ $\qquad \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

generalized Yang-Lee fusion ring YL_n, $X^2 = 1+nX$

Rem: Q: any $\mathbb{Z}_+$-ring R. $\longrightarrow$ Categorification.

skeletal s.s. cat. $\mathcal{C} = \mathcal{C}_R$, $\mathrm{Gr}(\mathcal{C}) = R$ basis: $\{X_i\}_{i\in I}$.

$\mathrm{obj}(\mathcal{C}_R) = \{\delta_X \mid X\in R\}$. $\qquad \mathrm{Hom}_{\mathcal{C}_R}(\delta_{X_i}, \delta_{X_j}) = \begin{cases} k, & X_i = X_j \\ 0, & X_i \neq X_j \end{cases}$

$\delta_{X_i}\otimes\delta_{X_j} = \bigoplus_{l\in I} N_{ij}^{l}\delta_{X_l}$, where $X_i\cdot X_j = \sum_{l\in I} N_{ij}^{l} X_l$.

id: $(\delta_X\otimes\delta_Y)\otimes\delta_Z = \delta_X\otimes(\delta_Y\otimes\delta_Z)$.

"associator" is not natural.

$(\mathrm{id}_{\delta_X}\otimes g)\otimes \mathrm{id}_{\delta_Z} \neq \mathrm{id}_{\delta_X}\otimes(g\otimes\mathrm{id}_{\delta_Z}) \qquad g: \delta_Y \to \delta_Y$.

$\delta_X = \delta_{X_i}$. $\quad \delta_Y = m\,\delta_{X_j}$. $\quad \delta_Z = \delta_{X_l}$.

$\mathrm{End}(\underline{(\delta_X\otimes\delta_Y)\otimes\delta_Z}) = \mathrm{End}(\underline{\delta_X\otimes(\delta_Y\otimes\delta_Z)}) = \bigoplus_s \mathrm{Mat}_{mn_s}(k)$

$n_s = \underline{\sum_p N_{ij}^{p} N_{pl}^{s}} = \underline{\sum_q N_{iq}^{s} N_{jl}^{q}}$

$(\delta_X\otimes\delta_Y)\otimes\delta_Z = (\delta_{X_i}\otimes m\,\delta_{X_j})\otimes\delta_{X_l} = \bigoplus_{p\in I} m\,N_{ij}^{p}\,\delta_{X_p}\otimes\delta_{X_l}$

$= \bigoplus_{s,p\in I} m\,N_{ij}^{p} N_{pl}^{s}\,\delta_{X_l}$.

$(\mathrm{id}_X\otimes g)\otimes\mathrm{id}_Z = \bigoplus_s \bigoplus_p \mathrm{id}_{N_{ij}^{p}}\otimes g\otimes \mathrm{id}_{N_{pl}^{s}}$

$$\mathrm{id}_X \otimes (g \otimes \mathrm{id}_Z) = \bigoplus_{s} \bigoplus_{q} \mathrm{id}_{\Lambda^s_{X_q}} \otimes g \otimes \mathrm{id}_{\Lambda^q_{Z_0}} \qquad \#$$