eg 5.2.5. $\mathcal{A}$: Abel cat. having enough proj. (inj) objects.

$0 \to N \xrightarrow{f} L \xrightarrow{g} M \to 0$ be a short exact seq. in $\mathcal{A}$.

then $N \xrightarrow{f} L \xrightarrow{g} M \xrightarrow{h} N[1]$ is d.Δ. in $\mathcal{D}^b(\mathcal{A})$.

$h \in \mathrm{Hom}_{\mathcal{D}^b(\mathcal{A})}(M, N[1]) = \mathrm{Ext}^1_{\mathcal{A}}(M, N)$.

proof:

$$\begin{tikzcd}
N \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\wr"] & L \arrow[r, "\binom{0}{1}"] \arrow[d, equal] \arrow[dr, phantom, "\wr"] & \mathrm{Cone}(f) \arrow[r, "{(1,0)}"] \arrow[d, "{(0,g)}"] \arrow[dr, phantom, "?"] & N[1] \arrow[d, equal] \\
N \arrow[r, "f"] & L \arrow[r, "g"] & M \arrow[r, "h"] & N[1]
\end{tikzcd}$$

$0 \to N \xrightarrow{\alpha} I$

$0 \to N \xrightarrow{\alpha} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \to \cdots$

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "f"] \arrow[d, equal, "\mathrm{id}"'] \arrow[dr, phantom, "\wr"] & L \arrow[r, "g"] \arrow[d, dashed, "\exists \beta^0"] \arrow[dr, phantom, "\wr"] & M \arrow[r, "0"] \arrow[d, dashed, "\beta^1"] \arrow[dr, phantom, "\wr"] & 0 \arrow[r] \arrow[d, "0"] & \cdots \\
0 \arrow[r] & N \arrow[r] & I^0 \arrow[r, "d^0"] & I^1 \arrow[r, "d^1"] & I^2 \arrow[r] & \cdots
\end{tikzcd}$$

$d^1 \beta^1 g = d^1 d^0 \beta^0 = 0$

$d^1 \beta^1 = 0$.

$\beta^1 \in \ker \mathrm{Hom}_{\mathcal{A}}(M, d^1)$ Let $h = \overline{\beta^1} \in \mathrm{Ext}^1_{\mathcal{A}}(M, N) = \ker \mathrm{Hom}_{\mathcal{A}}(M, d^1) / \mathrm{Im}(\ )$.

$\Rightarrow \alpha[1] \backslash \tilde{h}$: $M \xrightarrow{\tilde{h}} I[1] \xLeftarrow{\alpha[1]} N[1]$ $\in \mathrm{Hom}_{\mathcal{D}^b(\mathcal{A})}(M, N[1])$.

$$\begin{tikzcd}
\mathrm{Cone}(f) \arrow[r, "\mathrm{id}\backslash(1,0)"] \arrow[d, "\mathrm{id}_M \backslash (0,g)"'] & N[1] \arrow[d, equal] \\
M \arrow[r, "\alpha[1]\backslash\tilde{h}"] & N[1]
\end{tikzcd}$$

$$\alpha[1]\backslash\tilde{h} \circ \mathrm{id}_M\backslash(0,g) = \alpha[1]\backslash(0, \tilde{h}g) \overset{?}{=} \mathrm{id}_{N[1]}\backslash(1,0) \quad \text{in } \mathcal{D}^b(\mathcal{A})$$

$$\begin{tikzcd}
 & N[1] \arrow[d, "{\alpha[1]}"] \arrow[dr, phantom, "\wr"] & \\
\mathrm{Cone}(f) \arrow[ur, "{(1,0)}"] \arrow[r, "{(\alpha[1],0)}"] \arrow[dr, "{(0,\tilde{h}g)}"'] \arrow[r, phantom, shift right=3ex, "\wr"] & I[1] & N[1] \arrow[ul, Rightarrow, "\mathrm{id}"'] \arrow[l, Rightarrow, "{\alpha[1]}"'] \arrow[dl, Rightarrow, "{\alpha[1]}"] \arrow[l, phantom, shift right=3ex, "\wr"] \\
 & I[1] \arrow[u, "\mathrm{id}"] &
\end{tikzcd}$$

prove $(\alpha[1], 0) \sim (0, \tilde{h}g)$ i.e. $(\alpha[1], -\tilde{h}g) \sim 0$.

0 degree

$$\begin{tikzcd}
\mathrm{Cone}(f) \arrow[d, "{(\alpha[1], -\tilde{h}g)}"] & : & 0 \arrow[r] & N \oplus 0 \arrow[r, "\begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix}"] \arrow[d, "{(\alpha, 0)}"] \arrow[dr, phantom, "1"] & 0 \oplus L \arrow[r] \arrow[d, "{(0, -\beta^1 g)}"] \arrow[dl, dashed, "{(0,\beta^0)}"'] \arrow[dr, phantom, "2"] & 0 \arrow[r] \arrow[dl, dashed] & \cdots \\
I[1] & & 0 \arrow[r] & I^0 \arrow[r, "-d^0"] & I^1 \arrow[r] & I^2 \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & N & \xrightarrow{f} & L & \xrightarrow{g} & M & \xrightarrow{0} & 0 & \longrightarrow & \cdots \\
 & & \Big\| \mathrm{id} & \textcircled{0}\,\exists & \downarrow \exists\beta^0 & \exists\,\textcircled{1} & \downarrow \beta^1 & \exists & \downarrow 0 & & \\
0 & \longrightarrow & N & \xrightarrow{\alpha} & I^0 & \xrightarrow{d^0} & I^1 & \xrightarrow{d^1} & I^2 & \longrightarrow & \cdots
\end{array}$$

Tate alg_A. $\mathcal{D}^b(A\text{-mod})$ indecomp. obj

§5.8. Derived functor

· $\mathcal{A}, \mathcal{B}$: Abel cat. $F: \mathcal{A} \to \mathcal{B}$ additive functor $\longrightarrow F: K(\mathcal{A}) \xrightarrow{\Delta\text{-fun.}} K(\mathcal{B})$

$\rightsquigarrow F: \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{B})$. $F[1] \overset{\mathrm{id}}{\cong} [1]F$

If F map quasi-isom into quasi-isom $\Rightarrow F: \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ Δ fun.
$X^\cdot \longmapsto F(X^\cdot)$

$X^\cdot \overset{s}{\Leftarrow} \cdot \overset{a}{\Rightarrow} Y^\cdot \longmapsto F(X^\cdot) \overset{F(s)}{\Leftarrow} \cdot \overset{F(a)}{\Rightarrow} F(Y^\cdot)$

eg: M : non-proj obj in $\mathcal{A}$.

$\mathrm{Hom}_{K(\mathcal{A})}(M, -) : K(\mathcal{A}) \longrightarrow K(Ab)$ is a Δ-functor.

$X^\cdot$ acyclic complex. $\mathrm{Hom}_{K(\mathcal{A})}(M, X^\cdot)$ is not acyclic complex.

Q: How can we get functor $\mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ induced by F?

· Let $\mathcal{L}$ be a Δ-subcat of $K(\mathcal{A})$. $Q_{\mathcal{L}} := \mathcal{L} \cap Q$ is a saturated compatible multi system, $Q_{\mathcal{L}}^{-1}\mathcal{L} = \mathcal{L}_Q$.

Def 5.8.1 $\mathcal{A}$: Abel cat. $\mathcal{L}$: Δ subcat of $K(\mathcal{A})$.
If natural functor $\mathcal{L}_Q \to \mathcal{D}(\mathcal{A})$ is fully faithful, then say $\mathcal{L}$ is localizing subcat of $K(\mathcal{A})$.

Lem 5.1.7 (P125). S : multi sys. $D < \mathcal{C}$. $S \cap \mathcal{D}$

(i) if $s: X \to X'$, $s \in S$, $X \in D$, then $\exists f: X' \to X''$ s.t. $X'' \in \mathcal{D}$ and $fs \in S$

or

(ii) if $s: X' \to X$, $s \in S$, $X \in D$, then $\exists f: X'' \to X'$ s.t. $X'' \in D$ s.t. $sf \in S$.

Then $(S \cap D)^{-1} D \to S^{-1}C$ is fully faithful.

eg. 5.8.2. (1) If $* \in \{b, -, +, \text{empty}\}$, $K^*(\mathcal{A})$

(2) the intersection of localizing subcat is localizing subcat.

$X \in \mathcal{L}_1 \cap \mathcal{L}_2$. $\forall\ X \xrightarrow{s} X' \xrightarrow{\exists f} Y$ s.t. $f \circ s \in Q$. $Y \in L_1$.

$\Rightarrow X \cong Y$ in $\mathcal{D}(\mathcal{A})$. $\Rightarrow Y \in (L_1 \cap L_2)_Q \Rightarrow Y \in L_1 \cap L_2$.

(3) $\mathcal{A}'$: extension closed Abel subcat.

$K_{\mathcal{A}'}(\mathcal{A}) := \{ X \in K(\mathcal{A}) \mid H^n(X) \in \mathcal{A}', \forall n \}$ is a $\triangle$-subcat of $K(\mathcal{A})$

$H^n(X[1]) = H^{n+1}(X) \in \mathcal{A}'$

· $\underset{K_{\mathcal{A}'}(\mathcal{A})}{\overset{\cap\!\!\!\!\shortparallel}{X}} \to Y \to \underset{K_{\mathcal{A}'}(\mathcal{A})}{\overset{\cap\!\!\!\!\shortparallel}{Z}} \to X[1]$ d.$\triangle$. in $K(\mathcal{A})$.

$\rightsquigarrow \quad \to H^{n-1}(Z) \to \underset{\mathcal{A}'}{H^n(X)} \to \underset{\mathcal{A}'}{H^n(Y)} \to \underset{\mathcal{A}'}{H^n(Z)} \to \cdots$

Claim : $(K_{\mathcal{A}'}(\mathcal{A}) \cap Q)^{-1} K_{\mathcal{A}'}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{A})$ is fully faithful

Using lem. 5.1.7.

$\underset{K_{\mathcal{A}'}(\mathcal{A})}{X^\cdot} \xrightarrow{s} X_1^\cdot \xrightarrow{id} X_1^\cdot$

$K^*_{\mathcal{A}'}(\mathcal{A})$ $\quad * \in \{b, -, +\}$ are localizing subcat.

Def. 5.8.3. $\mathcal{A}, \mathcal{B}$: Abel cat, $\mathcal{L}$ localizing subcat of $K(\mathcal{A})$.
$F : \mathcal{L} \longrightarrow K(\mathcal{B})$ a covariant (resp. contravariant) $\triangle$- functor.

Let $Q : \mathcal{L} \longrightarrow \mathcal{L}_Q$ and $Q_{\mathcal{B}} : K(\mathcal{B}) \longrightarrow \mathcal{D}(\mathcal{B})$ localizing functor.
Define the right derived functor of F. ( )

$RF : \mathcal{L}_Q \longrightarrow \mathcal{D}(\mathcal{B})$. together with a natural transf. $\xi : Q_{\mathcal{B}} \circ F \longrightarrow RF \circ Q$.
s.t. (RF, ξ) has university property :

$$\begin{tikzcd}
Q_{\mathcal{B}} \circ F \arrow[rr, "\xi"] \arrow[dr, "\varepsilon"'] & & RF \circ Q \arrow[dl, dashed, "\exists \eta \circ id_Q"] \\
& G \circ Q &
\end{tikzcd}$$

$G : \mathcal{L} \to K(\mathcal{B})$

$G : \mathcal{L}_Q \longrightarrow \mathcal{D}(\mathcal{B})$ $\triangle$- functor, $\quad \exists!\ \eta : RF \to G$.

Rem 5.8.4.
(1) RF. exists $\Rightarrow$ unique.

(2) left derived functor LF : $\xi : LF \circ Q \longrightarrow Q_{\mathcal{B}} \circ F$.

(3) $F : \mathcal{A} \to \mathcal{B}$ left exact $\rightsquigarrow$ $\triangle$- functor $K(\mathcal{A}) \to K(\mathcal{B})$.
Q : How to get right derived functor $RF : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$?

Thm 5.8.5 (The existness of right derived functor).
$\mathcal{A}, \mathcal{B}$: Abel cat, $\mathcal{L}$: localizing subcat of $K(\mathcal{A})$.
(1) Assume $F : \mathcal{L} \longrightarrow K(\mathcal{B})$ is covariant $\triangle$- functor and there is a $\triangle$-subcat $\mathcal{L}$ of $\mathcal{L}$ sat.

(ii) $\forall X \in \mathcal{L}$, $\exists$ quasi-isom $X \to I(X)$, $I(X) \in L$

(iii) If $I \in L$ is acyclic complex, $F(I)$ is a acyclic complex.

Then F has right derived functor (RF, ξ), s.t. $\forall X \in \mathcal{L}$,

$$RFQ(X) \cong Q_{\mathcal{B}} F(I(X)).$$

and

$$\xi(I) : Q_{\mathcal{B}} F(I) \xrightarrow{\sim} RFQ(I) \quad \forall I \in L.$$

proof:

Step 1, There is Δ-functor $\bar{F} : L_Q \to D(\mathcal{B})$ where $L_Q = (Q \cap L)^{-1} L$.
s.t.

$$\begin{tikzcd} L \arrow[r, "F|_L"] \arrow[d, "Q"'] \arrow[dr, phantom, "\wr"] & K(\mathcal{B}) \arrow[d, "Q_{\mathcal{B}}"] \\ L_Q \arrow[r, "\bar{F}"] & D(\mathcal{B}) \end{tikzcd} \quad (*)$$

$F|_L$ maps quasi-isom into quasi-isom.

Indeed, let $s : I_1 \to I_2$ $s \in Q$, in L.

$\Rightarrow$ d.t. $I_1 \xrightarrow{s} I_2 \to J \to I_1[1]$

$J \in L$. acyclic

$\Rightarrow F(I_1) \xrightarrow{F(s)} F(I_2) \to F(J) \to F(I_1)[1]$ d.t. in $K(\mathcal{B})$.

$F(J)$ acyclic $\Rightarrow F(s) \in Q$.

Step 2: $\exists$ Δ-functor $T : L_Q \to \mathcal{L}_Q$ s.t.

$$\begin{tikzcd} L \arrow[r] \arrow[d, "Q"'] \arrow[dr, phantom, "\wr"] & \mathcal{L} \arrow[d, "Q"] \\ L_Q \arrow[r, "T"] & \mathcal{L}_Q \end{tikzcd} \quad (**)$$

T: Δ-eq. $\quad U : \mathcal{L}_Q \to L_Q$.

- T: fully faithful.

$s : X \to X'$, $s \in Q \cap \mathcal{L}$. $X \in L$, $X' \in \mathcal{L}$.
by (ii) $\exists I(X') \in L$, s.t. $X' \xrightarrow{f} I(X')$ is quasi-isom.

$\Rightarrow f \cdot s \in Q \cap \mathcal{L}$. $\xRightarrow{\text{Lem 5.1.7}}$ T ff.

- T: dense. by (ii).

Step 3. define Δ-functor $RF = \bar{F} U : \mathcal{L}_Q \to D(\mathcal{B})$

Since for $\forall X \in \mathcal{L}$, $X \to I(X)$ quasi-isom. $\Rightarrow Q(X) \cong Q(I(X))$. in $\mathcal{L}_Q$.

Hence: $RFQ(X) \cong RFQ(I(X)) = \bar{F} U Q(I(X)) \overset{(**)}{\cong} \bar{F} Q(I(X)) \overset{(*)}{\cong} Q_{\mathcal{B}} F(I(X))$.

Step 4. Construct $\xi : Q_\beta \circ F \longrightarrow RF \circ Q$.

$\forall X \in \mathcal{L}$, let $Y := UQ(X) \in LQ$.

$\alpha : Id_{LQ} \simeq UT$ and $\beta : Id_{\mathcal{L}Q} \simeq TU$.

$\Rightarrow \beta_X : Q(X) \xrightarrow{\sim} TUQ(X) = TQ(Y) = Q(Y)$.

$\beta_X : X \xRightarrow{s} Z \xLeftarrow{t} Y \rightsquigarrow F(X) \xRightarrow{F(s)} F(Z) \xLeftarrow{F(t)} F(Y)$ in $K(\mathcal{B})$.

$Z \Leftarrow L$, $\quad Z'$

$\rightsquigarrow \xi_X : Q_\beta F(X) \longrightarrow Q_\beta(F(Y)) = \bar{F}Q(Y) = \bar{F}UQ(X) = RFQ(X)$.

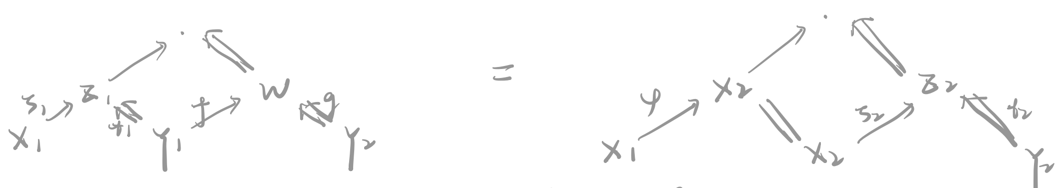
ξ_I is isom $\forall I \in L$.

ξ is a natural transf.

$\forall \varphi : X_1 \to X_2$ in $\mathcal{L}$, $Y_i = UQ(X_i) \in LQ$.

$$\begin{tikzcd} X_1 \arrow[r, "\beta_{X_1}"] \arrow[d, "Q(\varphi)"'] \arrow[dr, phantom, "\circlearrowright"] & T(Y_1) \arrow[d, "TU(Q(\varphi))"] \\ X_2 \arrow[r, "\beta_{X_2}"] & T(Y_2) \end{tikzcd}$$

$\Rightarrow TU(\varphi)\beta_{X_1} = \beta_{X_2}\varphi$. i.e.



where $TU(\varphi) = TUQ(\varphi) = Y_1 \xrightarrow{f} W \xLeftarrow{g} Y_2$, $W \in L$.

$\Rightarrow F(TU(\varphi) \cdot \beta_{X_1}) = F(\beta_{X_2} \cdot Q(\varphi))$.

i.e. $(FY_1 \xrightarrow{Ff} FW \xLeftarrow{Fg} F(Y_2)) \circ \xi_{X_1} = \xi_{X_2} \circ Q_\beta F(\varphi)$.

$\parallel ?$

$RFQ(\varphi)$.

$$\Rightarrow \begin{tikzcd} Q_\beta F(X_1) \arrow[r, "\xi_X"] \arrow[d, "Q_\beta F(\varphi)"'] \arrow[dr, phantom, "\circlearrowright"] & RFQ(X_1) \arrow[d, "RFQ(\varphi)"] \\ Q_\beta F(X_2) \arrow[r, "\xi_{X_2}"] & RFQ(X_2) \end{tikzcd}$$

Claim: if $\varphi : X_1 \to X_2$ in $\mathcal{L}$, $\underline{TUQ(\varphi) = Y_1 \xrightarrow{f} W \xLeftarrow{g} Y_2}$?, $W \in L$.

$\Rightarrow RFQ(\varphi) = FY_1 \xrightarrow{F(f)} FW \xLeftarrow{F(g)} FY_2$

· Suppose $UQ(\varphi) = Y_1 \xrightarrow{u} V \xLeftarrow{v} Y_2 \in LQ$. $V \in L$.

$\Rightarrow TUQ(\varphi) = UQ(\varphi).$

Suppose $UQ(\varphi) = Y_1 \xrightarrow{f} W \xleftarrow{g} Y_2$

$\Rightarrow RFQ(\varphi) = FUQ(\varphi) = F(Y_1 \xrightarrow{f} W \xleftarrow{g} Y_2).$

$= F(\qquad) F(\qquad).$

$= (F\,Q(g))^{-1} F(Q(f)) = (Q_\beta F(g))^{-1}(Q_\beta F(f))$

$= FY_1 \xrightarrow{Ff} FW \xleftarrow{Fg} FY_2$

step 5. University property: If $G: \mathcal{L}_Q \to \mathcal{D}(\beta)$ Δ-functor and $\varepsilon: Q_\beta F \longrightarrow GQ$ nat. transf, prove $\exists!\ \eta: RF \longrightarrow G$ s.t.

$$\begin{tikzcd} Q_\beta \circ F \arrow[rr, "\xi"] \arrow[dr, "\varepsilon"'] & {} \arrow[d, phantom, "\supseteq"] & RFQ \arrow[dl, dashed, "\exists!\ \eta \circ id_Q"] \\ & GQ & \end{tikzcd}$$

$\cdot$ ① $\forall X \in \mathcal{L}_Q$. Construct $\eta_X: RF(X) \longrightarrow G(X).$

$\varepsilon_X: Q_\beta F(X) \longrightarrow GQ(X),\qquad \beta_X = X \xrightarrow{s} Z \xleftarrow{t} Y$, $Z \leftarrow L.$

By nat. of ε.

$$\begin{tikzcd} Q_\beta F(X) \arrow[r, "\varepsilon_X"] \arrow[d] & GQ(X) \arrow[d] \\ Q_\beta F(Z) \arrow[r, "\varepsilon_Z"] & GQ(Z) \\ Q_\beta F(Y) \arrow[u, "\cong\ Q_\beta F(t)"] \arrow[r, "\varepsilon_Y"] & GQ(Y) \arrow[u, "GQ(t)\ \cong"'] \end{tikzcd}$$

(□) in $\mathcal{D}(\beta)$.

$Q_\beta F(Y) = RFQ(X)$

$$\begin{tikzcd} Q_\beta F(X) \arrow[r, "\varepsilon_X"] \arrow[d, "\xi_X"'] & GQ(X) \arrow[d, "G(\beta_X)"] \\ RFQ(X) \arrow[r, "\varepsilon_Y"] & GQ(Y) \end{tikzcd}$$

define $\eta_X := G(\beta_X)^{-1}\varepsilon_Y: RFQ(X) \longrightarrow GQ(X).$

② naturality.

$\forall \varphi: X_1 \to X_2$ in $\mathcal{L}_Q$. $\quad X_1 \to X = X_2 \quad$ assume $\varphi: X_1 \to X_2$ in $\mathcal{L}$.

$$\begin{tikzcd}
RFQ(X_1) \arrow[r, "\varepsilon_{Y_1}"] \arrow[d, "RFQ(\varphi)"] \arrow[dr, phantom, "(*)"] & GQ(Y_1) \arrow[r, "G(\beta_{X_1})^{-1}"] \arrow[d] \arrow[dr, phantom, "(**).\ \beta\ \text{nat.}"] & GQ(X_1) \arrow[d, "GQ(\varphi)"] \\
RFQ(X_2) \arrow[r, "\varepsilon_{Y_2}"] & GQ(Y_2) \arrow[r, "G(\beta_{X_2})^{-1}"] & GQ(X_2)
\end{tikzcd}$$

$(*)$ comm. $\iff \varepsilon_{Y_2}(FY_1 \xrightarrow{Ff} FW \xLeftarrow{Fg} FY_2) = GT(Y_1 \longrightarrow W \Leftarrow Y_2)\varepsilon_{Y_1}$.

$$\begin{aligned}
LHS = \varepsilon_{Y_2}(Q_\beta F(g))^{-1} Q_\beta F(f) &\overset{(0)}{=} (GQ(g))^{-1}\varepsilon_W Q_\beta F(f) \\
&= (GQ(g))^{-1} GQ(f)\,\varepsilon_{Y_1} \\
&\overset{\text{step2}}{=} G(TQ(g))^{-1}(TQ(f))\,\varepsilon_{Y_1} \\
&= GT(Y_1 \xrightarrow{f} W \xLeftarrow{g} Y_2)\,\varepsilon_{Y_1} \\
&= RHS.
\end{aligned}$$

which completes the proof. #