4.11 Tensor subcat.

Def 4.11.1 A (multi)tensor subcat. $\mathcal{C}$ in a (multi)tensor cat. $\mathcal{D}$.
: a full subcat.
closed under taking $\begin{cases} \text{subquotients} \\ \otimes \\ * \quad (\text{contain } 1) \end{cases}$

Ex. 4.11.2 (1) G group $\mathcal{C} = Vec_G^w$ $w \in Z^3(G, \mathbb{k}^\times)$
Show: tensor subcat. of $\mathcal{C}$ are in bijective with subgroups of G.
the subcat. corresponding to a subgroup $H \subset G$ consists of v.s supported on H.

pf: $\mathcal{D}$; $V_h \neq 0$ $h \in H$ $V_g = 0$. $g \notin H$
$\mathcal{D}$: tensor subcat $\begin{cases} \text{subquotient} \quad \checkmark \\ \otimes \quad (V \otimes W)_h = \bigoplus_{x,y \in H, xy = h} V_x \otimes W_y \quad \checkmark \\ * \quad (V_h)^* = V_{h^{-1}} \quad \checkmark \end{cases}$

$Gr(Vec_G^w) = \mathbb{Z}G$ ($\mathbb{Z}_+$ ring)
$\mathbb{Z}_+$ subring ; $\mathbb{Z}H$, $H \leq G$. (tensor subcat)
by Prop 4.10.3 the cat. of $\mathbb{Z}H$ are Vec_H^w ☑.

(2) G. finite group. $\mathbb{k} = \overline{\mathbb{k}}$ charac. 0
Show: tensor subcat. of $Rep(G)$ are in bijective with $N \triangleleft G$
$\simeq Rep(G/N)$

pf: $Rep(G/N)$ tensor subcat.
by Example 3.6.10 $A = k_0(G)$ be the based ring of characters of a finite group G. any based subring of $B \subset A$ is of the form $B = k_0(G/N)$, $\underline{\underline{N \triangleleft G}}$.
($G \overset{\rho_i}{\curvearrowright} V_i$ $\to$ simple $\ker \rho_i$) $\underline{\underline{\bigcap \ker \rho_i = N}}$ ☑.

Let $\mathcal{C}$ fusion cat. $A = Gr(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C}$.
extend the involution $X \longmapsto X^*$ to A by anti-linearly.
Let $\iota: A \to \mathbb{C}$ linear map s.t. $\iota([X]) = \dim_{\mathbb{C}} Hom_{\mathcal{C}}(1, X)$ $\forall X \in \mathcal{C}$.
<u>Then $(A, *)$ is $*$-alg. $\iota: A \to \mathbb{C}$ positive trace</u> (Def 3.7.1)

pf: by Prop 4.9.1 $Gr(\mathcal{C})$ unital based ring.
· $(A, *)$ is $*$-alg.
$\forall [X], [Y] \in Gr(\mathcal{C})$ $([X][Y])^* = [Y]^*[X]^*$. $[X]^{**} = [X]$

$(\lambda[X])^* = \bar\lambda [X]^* \quad \forall \lambda \in \mathbb{C}.$

$\forall a, b \in A \quad (ab)^* = b^* a^* \quad a^{**} = a \quad (\lambda a)^* = \bar\lambda a^*. \Rightarrow (A, *)$ is $*$-alg.

- l is a positive trace

Let X_i, $i \in I$ be represent. of iso classes of simple obj in $\mathcal{C}$. $(|I| < \infty)$

show: $l(ab) = l(ba)$ $\quad l(a^*) = \overline{l(a)}$ $\quad l(aa^*) > 0$, $a \neq 0$.

pf: $l([X_i \otimes X_j]) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{C}}(1, X_i \otimes X_j) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{C}}(1 \otimes X_j^*, X_i) = \begin{cases} 0 & X_i \neq X_j^* \\ 1 & X_i = X_j^* \end{cases}$

$l([X_j \otimes X_i]) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{C}}(1 \otimes X_i^*, X_j) = \begin{cases} 0 & X_i^* \neq X_j \quad (X_i \neq X_j^*) \\ 1 & X_i^* = X_j \quad (X_i = X_j^*) \end{cases}$

$l([X_i \otimes X_j]) = l([X_j \otimes X_i])$

$\because \mathcal{C}$ is s.s. and l is linear $\quad \forall a, b \in A. \quad l(ab) = l(ba)$

$l((\lambda X_i)^*) = l(\bar\lambda X_i^*) = \bar\lambda\, l(X_i^*) = \bar\lambda\, l(X_i)$ $\qquad l(X_i^*) = \dim_{\mathbb{C}} \mathrm{Hom}(1, X_i^*)$

$= \overline{\lambda\, l(X_i)}$

$\forall a \in A \quad l(a^*) = \overline{l(a)}$

$l((\lambda X_i)(\lambda X_i)^*) = \lambda\bar\lambda\, l(X_i X_i^*) = \underbrace{\lambda\bar\lambda}_{>0} \underline{\dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{C}}(1, X_i \otimes X_i^*)}$

$\forall a \neq 0, a \in A \quad l(aa^*) > 0.$

unital based ring.
$\Rightarrow X_i \otimes X_i^*$ contain 1.

Lemma 4.11.3 $\mathcal{C}$ fusion cat. $\forall$ simple X in $\mathcal{C}$ $\exists N \in \mathbb{Z}_+$ s.t. $\mathrm{Hom}_{\mathcal{C}}(1, X^{\otimes N}) \neq 0$.

pf: • claim $[X] \in Gr(\mathcal{C}) \subset Gr(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C}$ is not nilpotent. i.e. $X^{\otimes n} \neq 0$

pf: $n=1$ $\quad X \neq 0$ ✓

suppose $n-1$ ✓ $\quad X^{\otimes n-1} \neq 0$

then $ev_{X^{\otimes n-1}}: (X^{\otimes n-1})^* \otimes X^{\otimes n-1} \longrightarrow 1 \quad \neq 0$

$\because 1$ simple $\quad \therefore ev_{X^{\otimes n-1}}$ epi.

$\otimes$ biexact $\quad (X^{\otimes n-1})^* \otimes X^{\otimes n-1} \otimes X \longrightarrow X \quad$ epi.

if $X^{\otimes n} = 0$ then contra.

$\therefore X^{\otimes n} \neq 0$.

$\therefore$ claim ✓.

length$(X \otimes Y) \geq$ length(X) length(Y)
length$(X^{\otimes N}) \geq$ length$(X)^N = 1$

by lemma 3.7.6 $\quad l(X^{\otimes N}) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{C}}(1, X^{\otimes N}) \neq 0$.

Cor 4.11.4 $\mathcal{D} \neq 0$ a full abelian subcat. of a fusion cat. $\mathcal{C}$. which closed under subquotient and $\otimes$. Then $\mathcal{D}$ contains 1 and is rigid. i.e. is a tensor subcat. of $\mathcal{C}$.

pf: Let $X \in \mathcal{D}$ simple. Then $X^{\otimes n} \in \mathcal{D}$ for $\forall n \geq 1$.

For N as in lemma 4.11.3 $\quad \underline{\mathrm{Hom}_{\mathcal{D}}(1, X^{\otimes N})} \neq 0$

$\therefore \exists f \neq 0 : 1 \longrightarrow X^{\otimes N}$. $\quad = \mathrm{Hom}_{\mathcal{D}}(1, X \otimes X^{\otimes N-1}) = \mathrm{Hom}(X^*, X^{\otimes N-1}) \neq 0.$

$\because 1$ simple $\quad \ker f = 0 \quad \therefore f$ mono. $\quad \therefore 1 \in \mathcal{D}$.

$\downarrow$

$g \neq 0 : X^* \longrightarrow X^{\otimes N-1}$ mono

$\therefore X^* \in \mathcal{D}$

$\therefore$ rigid

4.12. Chevalley prop. of tensor cat.

[*] «Lie alg. and alg. group» Patrice Tauvel. Rupert. W.T. Yu 2005.

(Def) <u>alg. group</u>: group; alg. variety

multi. μ_G, inverse ι_G are morphisms of alg. varieties.

alg. variety affine $\Rightarrow$ affine alg. group.

e.g. $GL_n(\mathbb{k})$ is an alg. group. $\quad \underline{\delta}: M_n(\mathbb{k}) \longrightarrow \mathbb{k}$

$M \longmapsto \det(M)$

$D(\delta) = GL_n(\mathbb{k})$ is the set of maximal ideals

m which $\underline{\delta(m) \neq 0}$

Any closed subgroup of an alg. group is an alg. group

M subset of $\underline{\underline{\mathbb{k}^n}}$ The closure of M, with respect to the Zariski topology.

is $\underline{V}(I(\underline{\underline{M}}))$.

element: unipotent.

unipotent radical: the largest normal <u>unipotent subgroup</u> of G.

$R_u(G)$

f unipotent: $f - 1$ nilpotent

G reductive if $R_u(G) = \{e\}$.

we say that V is a rational G-mod if the map $\rho: G \to GL(V)$

is <u>a morphism of alg. groups.</u>

$\downarrow$

$\downarrow$

group hom ; a morphism of alg. varieties.

Thm 4.12.1 (Chevalley) Let $\mathbb{k}$ be filed char. 0. Then the tensor product of f.d. simple rep of any group or Lie alg. over $\mathbb{k}$ is s.s.

pf: Let V be f.d. v.s. over $\mathbb{k}$

$G \subset \underline{GL(V)}$ is a Zariski closed subgroup.

claim: Lemma 4.12.2 If V is a s.s. rep of G, then G is reductive.

pf: Let $\underline{\underline{V}}$ be a nonzero rational rep of an affine alg. group G

Let U be the unipotent radical of G.

Let $V^U = \{ v \mid hv = v, h \in U \} \subset V$.

$\because U$ is normal subgroup. $\therefore V^U$ is subrep.

($\forall g \in G$, $v \in V^U$. 要证 $gv \in V^U$, $u \in U$. $u_1 = g^{-1} u g \in U$

$g^{-1} u g(v) = u_1(v) = v$ $\quad u g(v) = g(v)$ $\therefore g(v) \in V^U$)

Since U is unipotent, then $V^U \neq 0$.

(by [*, Thm 22.3.6 (2)] If V is a non-zero rational G-mod, then $V^{\overset{\text{unipotent}}{G}} \neq 0$)

If V is irr. then $\underline{V^U} = V$. i.e. U acts trivially on V.

$\therefore$ if V is s.s. and U acts trivially on V,

$\therefore U = 1$. $\therefore G$ is reductive 证.

Now let G be any group. $\underline{V, W}$ f.d. irr. rep of G

Let G_V, G_W be the Zariski closure of the images of G in $GL(V)$, $GL(W)$

Then by lemma 4.12.2 G_V, G_W are reductive.

Let G_{VW} be the Zariski closure of the image of G in $G_V \times G_W \subseteq GL(V) \times GL(W)$

Let U be the unipotent radical of G_{VW}

$p_V : G_{VW} \to G_V$ $\quad p_W : G_{VW} \to G_W$ be the proj.

$\because p_V$ surj. $\therefore p_V(U)$ is a normal unipotent subgroup of $\underline{G_V} \to$ reductive

$\therefore p_V(U) = 1$; $p_W(U) = 1$.

$\therefore U = 1$. $\therefore G_{VW}$ is reductive

Let G'_{VW} be the closure of image of G in $GL(V \otimes W)$

Then $\underline{G'_{VW}}$ is quotient of G_{VW}, it is also reductive

([*, Prop 21.2.2 (2)] G reductive iff $\mathfrak{g}$ is $\underline{\text{reductive}}$ and all the elements of the centre of $\mathfrak{g}$ are s.s.) adj. rep. is s.s.

$\because$ char$(\mathbb{k}) \neq 0$. rep $V \otimes W$ is complete reducible as a rep of $\underline{G'_{V,W}}$.

hence of G.

$\because$ char$(\mathbb{k}) \neq 0$, rep $V \otimes W$ is complete reducible as a rep of $\mathfrak{g}_{V,W}$.
hence of G.
([*, Thm 27.3.3 (i) $\Leftrightarrow$ (ii)) ☒

Def 4.12.3. A tensor cat. is said to have the Chevalley prop if $\mathcal{C}_0$ of s.s. object of $\mathcal{C}$ is a tensor subcat.

claim: A tensor cat. has the Chevalley prop $\Leftrightarrow$ $\forall$ simple obj. X, Y. $X \otimes Y$ s.s.
" $\Rightarrow$ " ✓
" $\Leftarrow$ " ✓.

Thus, Chevalley's thm. implies the cat. f. d. rep. of any group or Lie alg. over a field of char. 0 has the Chevalley Prop.

Prop 4.12.4. A tensor cat. in which every simple obj. is invertible has the Chevalley Prop

pf: $\forall$ X, Y simple obj. X, Y invertible.
by Prop 2.11.3(2) $X \otimes Y$ invertible.
by Exercise 4.3.11(5) $\therefore X \otimes Y$ is simple $\therefore$ Chevalley Prop.

Rm 4.12.5. pointed (Def 5.11.1.)

X in a locally finite abelian cat. $\mathcal{C}$.
$Lw(X)$ the Loewy length of X.
$0 = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n = X$. X_{i+1}/X_i is max s.s. obj. of X/X_i.
$Lw(X) = n$.
$\mathcal{C}_i$: full subcat. of obj. of $\mathcal{C}$ of Loewy length $\leq i+1$
$\underset{\leq 1}{\mathcal{C}_0} \subset \underset{\leq 2}{\mathcal{C}_1} \subset \underset{\leq 3}{\mathcal{C}_2} \subset \cdots$ coradical filtration of $\mathcal{C}$.

(Prop 4.12.6) In a tensor cat. with the Chevalley prop.
(4.16) $Lw(X \otimes Y) \leq Lw(X) + Lw(Y) - 1$
Thus $\mathcal{C}_i \otimes \mathcal{C}_j \subset \mathcal{C}_{i+j}$

pf: Let $X(i)$ $0 \leq i \leq m$ $Y(j)$ $0 \leq j \leq n$ be the successive quotients of the socle filtration
$0 = X_0 \subset X_1 \subset \cdots \subset X_{m+1} = X$ $X_{i+1}/X_i = X(i)$ $0 \leq i \leq m$.
$0 = Y_0 \subset Y_1 \subset \cdots \subset Y_{n+1} = Y$ $Y_{j+1}/Y_j = Y(j)$ $0 \leq j \leq n$

$X(i)$ $Y(j)$ s.s.

$Z = X\otimes Y$. has a filtration with successive quotient $Z(r) = \bigoplus_{i+j=r} X(i)\otimes Y(j)$

$0\le r\le m+n$

$0 = Z_0$ $\quad Z_1/Z_0 = Z(0)$ $\quad Z_2/Z_1 = Z(1)$ $\cdots$ $Z_{m+n+1}/Z_{m+n} = Z(m+n)$

$\underline{0=Z_0} \subset Z_1 \subset \cdots \subset Z_{m+n+1} = Z$. is a filtration of $Z = \underline{X\otimes Y}$.

$\therefore Lw(X\otimes Y) \le Lw(X) + Lw(Y) - 1$. (4.16)

$\therefore \mathcal{C}_i \otimes \mathcal{C}_j \subset \mathcal{C}_{i+j}$. 証.

Rm 4.12.7 $\quad$ (4.16) $\Longrightarrow$ Chevalley prop

X, Y simple $\quad Lw(X\otimes Y) \le Lw(X) + Lw(Y) - 1 = 1$.

$\therefore X\otimes Y \in Ob(\mathcal{C}_0)$ $\quad$ s.s.

Exercise 4.12.8. Let $\mathrm{char}\,\Bbbk = p$, G a finite group.

Show that $\mathrm{Rep}_{\Bbbk}(G)$ has Chevalley prop iff G has a normal p-Sylow subgroup

$p^n \mid |G|$, $p^{n+1} \nmid |G|$.

$V_1 \cdots V_n$

$\rho_i : G \hookrightarrow GL(V_i)$ $\qquad G/\ker\rho_i$

4.13. Groupoids.

Def: small cat. $\quad$ morphism : iso. $\quad (X, G)$ $\quad$ obj. $\quad$ morphism

$\underline{s, t : G \longrightarrow X}$ $\qquad \forall a\in G$. $\quad \underset{s(a)}{\cdot} \xrightarrow{a} \underset{t(a)}{\cdot}$

$\forall a, b\in G$. if $ta = sb$ $\qquad \underset{s(a)}{\cdot} \xrightarrow{a} \underset{t(a)=s(b)}{\cdot} \xrightarrow{b} \underset{t(b)}{\cdot}$

composite ab exist $\qquad \mu : G\times_X G \longrightarrow G$

$\qquad\qquad (a, b) \longmapsto ab$.

$s(ab) = s(a)$ $\quad t(ab) = t(b)$ $\qquad \mu$: associ.

obj. set.

$\forall x\in X$, $\underline{u_x: \text{acts as identity.}}$ $\qquad x \circlearrowleft u_x$. $\qquad X \xrightarrow{u} G \xrightarrow{s;t} X$

$\underline{u : X \longrightarrow G. \quad x \longmapsto u_x.}$

$s(u_x) = t(u_x) = x$ $\qquad su = tu = id$.

$i : G \longrightarrow G$ $\qquad a \longmapsto a^{-1}$ $\qquad \underset{s(a)}{\cdot} \xrightarrow{a} \underset{t(a)}{\cdot}$, a^{-1}

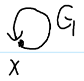
$s(a^{-1}) = t(a)$ $\quad t(a^{-1}) = s(a)$

$aa^{-1} = u(s(a))$ $\qquad a^{-1}a = u(t(a))$

Example. (1) Any group G is a groupoid $\mathcal{G}$ with a single obj.

whose set of morphism to itself is G.

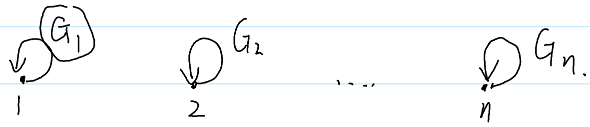
object x morphism G.



$\forall a \in G$. $s(a) = t(a) = x$ $i: a \longmapsto a^{-1}$ $u: x \longrightarrow \underline{1_x} = 1_G$.

$\mu(a, b) = m(a, b)$ asso.

A disjoint union $G = \coprod_\lambda G_\lambda$ $\lambda \in I$. is a groupoid, where G_λ group.



obj I

morphism G_λ $\lambda \in I$.

$\forall a_\lambda \in G_\lambda$ $s(a_\lambda) = t(a_\lambda) = \lambda$ $\mu: \underline{(a_\lambda, b_\lambda)} = a_\lambda b_\lambda$ ass.

$i: a_\lambda \longmapsto a_\lambda^{-1}$

$u: \lambda \longmapsto 1_\lambda$ (unit element of G_λ)

(2) Let X be a set $G = X \times X$. $(X\ G)$ groupoid.

object X

morphism $G = X \times X$

$s: X \times X \longrightarrow X$ $t: X \times X \longrightarrow X$

$(x, y) \longmapsto x$ $(x, y) \longmapsto y$.

$u: X \longrightarrow X \times X$ $i: X \times X \longrightarrow X \times X$

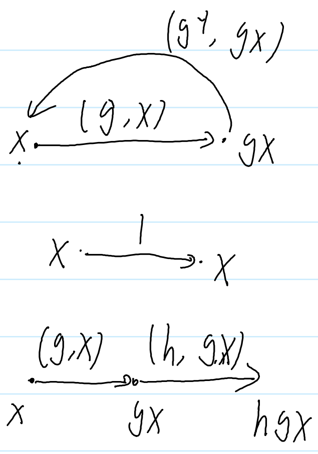
$x \longmapsto (\underline{x, x})$ $(\underline{x}, y) \longmapsto (y, x)$

$\mu: (X \times X) \times (X \times X) \longrightarrow X \times X$ associative

$((x, y), (y, z)) \longmapsto (x, z)$

$\forall x, y \in X$, $\exists$ unique morphism $(x, y): x \longrightarrow y$.

(3) transformation groupoid $T(G, X)$



object: X morphism: $G \times X$

$s(g, x) = x$ $t(g, x) = gx$

$u(x) = (\underline{\underline{1}}, \underline{\underline{x}})$

$i(g, x) = (g^{-1}, gx)$

$\mu: (G \times X) \times (G \times X) \longrightarrow G \times X$

$((g, x), (h, gx)) \longmapsto (gh, x)$