# 4.1

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$\mathcal{A}, \mathcal{B}$ : abelian cats.

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Prop: $(M_k)_{k\in\mathbb{N}}$ (R-mod), $N_n := \bigoplus_{k=1}^{n} M_k$, $N := \bigoplus_{k\in\mathbb{N}} M_k$. $N_n \xhookrightarrow{\varphi_{nm}} N_m$, $n \le m$

Then $\varinjlim_{\mathbb{N}} N_n = N$.

Pf:

$$\begin{tikzcd} N \arrow[d, shift right, "P_k"] & \\ M_k \arrow[u, shift right, "I_k"] \arrow[r, shift left, "i_k"] & N_n \arrow[l, shift left, "P_k"] \arrow[ul, dashed, "\theta_n"'] \end{tikzcd}, \quad k<n.$$

Suppose given $M$, $\{N_l \xrightarrow{s_l} M\}$ s.t. $s_l = s_j \circ \varphi_{lj}$.

$$\begin{tikzcd} N \arrow[rr, dashed, "\eta"] & & M \\ & N_l \arrow[ul, "\theta_l"'] \arrow[ur, "s_l"] \arrow[d, "\varphi_{lj}"] & \\ & N_j \arrow[uul, bend left, "\theta_j"] \arrow[uur, bend right, "s_j"'] & \end{tikzcd}$$

For any $M_l$, $\underline{M_l} \xrightarrow{i_l} N_l \xrightarrow{s_l} \underline{M}$, $\forall l$.

$\{M_l \xrightarrow{s_l \circ i_l} M\}$

By the universal property of $N$, $\exists ! \eta: N \to M$ s.t. $\underline{s_l \circ i_l = \eta \circ I_l}$.

$$\eta\circ\underline{\theta_l\circ i_l} = \eta\circ\theta_j\circ\underline{\varphi_{lj}\circ i_l} = \eta\circ\underline{\theta_j\circ i_l} = \eta\circ I_l$$

$$\eta\circ\theta_l\circ i_l = \eta\circ\theta_j\circ i_l$$

$$= s_l\circ i_l = s_j\circ\underline{\varphi_{lj}\circ i_l} = s_j\circ i_l$$

$\Rightarrow \eta\circ\theta_j\circ i_l = s_j\circ i_l$, $j \ge l$.

$\Rightarrow \eta\circ\theta_j\circ i_l\circ p_l = s_j\circ i_l\circ p_l$.

$\Rightarrow \sum_{l=1}^{j} \eta\circ\theta_j\circ i_l\circ p_l = s_j\circ\sum_{l=1}^{j} i_l p_l$,

$\Rightarrow \eta\circ\theta_j\circ\cancel{id_{N_j}} = s_j\circ\cancel{id_{N_j}}$

$\Rightarrow \eta\circ\theta_j = s_j$ $\forall j$. $\square$

$\varprojlim_{l\in\mathbb{N}} N_l = \prod_{l\in\mathbb{N}} M_l$.

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## §5.5. Derived cat of s.s. rings.

s.s. left Artin rings, or s.s. ring for short.

- Every s.e.s. is split.
- $\mathrm{Ext}^k(-,-)=0$, $k\ge 1$.

Claim 1: Every cpx $X$ in $D(R\text{-mod})$ is isom to $\bigoplus_{n\in\mathbb{Z}} H^n(X)[-n]$ (or $\prod_{n\in\mathbb{Z}} H^n(X)[-n]$)?

Pf: $X: \quad \cdots \to X^{-1} \to X^0 \to X^1 \to \cdots$.

Since $0 \to \ker d^0 \to X^0 \to \operatorname{Im} d^0 \to 0$ is split,

we have $X^0 \simeq \ker d^0 \oplus \operatorname{Im} d^0$

$0 \to \operatorname{Im} d^{-1} \to \ker d^0 \to H^0(X) \to 0$ is split.

$\Rightarrow \ker d^0 \simeq \operatorname{Im} d^{-1} \oplus H^0(X) \Rightarrow X^0 \simeq H^0 \oplus \operatorname{Im} d^0 \oplus \operatorname{Im} d^{-1}$.

$$\Rightarrow X = \begin{array}{l} \cdots \to X^{-2} \to X^{-1} \to \operatorname{Im} d^{-1} \to 0 \to \cdots \\ \qquad\qquad\quad \| \\ \qquad H^{-1}(X)\oplus \operatorname{Im} d^{-2} \oplus \operatorname{Im} d^{-1} \quad \oplus \\ \cdots \to 0 \to H^0(X) \to 0 \to \cdots = H^0(X)[0] \\ \qquad\qquad\quad \oplus \\ \cdots \to \operatorname{Im} d^0 \to X^1 \to \cdots \end{array}$$

$$\begin{array}{l} \cdots \to X^{-2} \to \operatorname{Im} d^{-2} \to \cdots \\ \qquad\qquad \oplus \\ \cdots 0 \to H^{-1}(X) \to 0 \cdots = H^{-1}(X)[1] \\ \qquad\qquad \oplus \\ \cdots 0 \to \operatorname{Im} d^{-1} \xrightarrow{id} \operatorname{Im} d^{-1} \to 0 \cdots \end{array}$$

acyclic $= 0$ in $D(R\text{-mod})$.

$$= \begin{array}{l} \cdots \to X^{-2} \to \cancel{\operatorname{Im} d^{-2}} \to 0 \to \cdots \qquad R_2^l \\ \qquad \| \qquad\qquad \oplus \\ H^{-2}(X)\oplus \\ \cancel{\operatorname{Im} d^{-3}} \oplus \operatorname{Im} d^{-2} \quad H^{-1}(X)\oplus H^0(X) \oplus H^1(X) \\ \qquad\qquad\qquad \oplus \qquad\qquad\qquad R_2^r \\ \qquad 0 \to \operatorname{Im} d^1 \to X^2 \to \cdots \end{array}$$

$$\Rightarrow X = R_n^l \oplus \Big(\bigoplus_{|k|\le n} H^k(X)[-k]\Big) \oplus R_n^r, \quad \forall n. \quad (1).$$

View (1) as a eqn of direct sys over $(\mathbb{N}, \le)$.

$$X: \qquad \begin{array}{ccc} X & \xrightarrow{id_X} & X \\ n & \le & m \end{array}$$

$$\bigoplus_{|k|\le n} H^k(X)[-k], \quad \begin{array}{ccc} \bigoplus_{|k|\le n} H^k & \longleftrightarrow & \bigoplus_{|k|\le m} H^k \\ n & \le & m \end{array}$$

$$R_n^l: \quad \begin{array}{ll} n & \cdots \to X^{-m} \to X^{-m+1} \to \cdots \to X^{-n-1} \to X^{-n} \to \operatorname{Im} d^{-n} \\ \wedge\!| & \qquad \Downarrow \qquad\quad \Downarrow\Uparrow \qquad\qquad \Downarrow \qquad\quad \Downarrow \\ m & \cdots \to X^{-m} \to \operatorname{Im} d^{-m} \to 0 \to \cdots \to 0 \to 0 \to \cdots \end{array}$$

split.

Similar for $R_m^r$.

$$\varinjlim_{\mathbb{N}} ((1)) \Rightarrow X = \varinjlim \bigoplus_{|k|\le n} H^k(X)[-k] \oplus \varinjlim R_n^l \oplus \varinjlim R_n^r$$

$$= \bigoplus_{k\in\mathbb{Z}} H^k(X)[-k] \oplus 0 \oplus 0.$$

$$= \bigoplus_{k\in\mathbb{Z}} H^k(X)[-k]. \qquad \square$$

To show $\varinjlim R_n^r = 0$,

only need to show:

For given $M$, $R_n^r \xrightarrow{s_n} M$

s.t. $s_n = s_m \circ \varphi_{nm}$

$\Rightarrow s_n = 0, \ \forall n$.

$$\begin{tikzcd} 0 \arrow[rr] & & M \\ & R_n^r \arrow[ur, "s_n"] \arrow[d, "\varphi_{nm}"] & \\ & R_m^r \arrow[uur, "s_m"'] \arrow[uul] & \end{tikzcd}$$

Pf: $s_n^k = \underbrace{s_{k+1}^n}_{=0} \circ \varphi^n_{n,k+1} = 0, \ \forall n, k$

$\Rightarrow s_n = 0, \ \forall n$.

$= \bigoplus_{k\in\mathbb{Z}} H^k(X)[-k]$. $\square$

Pf: $s_n^k = \underline{s_{k+1}^n} \circ \varphi_{n,k+1}^n = 0$. $\forall n, k$.
$\Rightarrow s_n = 0$, $\forall n$.

$$\varprojlim_{\mathbb{N}} ((1)) \Rightarrow X = \varprojlim \bigoplus_{|k|\le n} H^k(X)[-k] \oplus \varprojlim R_n^l \oplus \varprojlim R_n^r$$
$$= \prod_{|k|\le n} H^k(X)[-k] \qquad \square$$

Claim: Indec obj in $D(R\text{-mod})$ has the form of $M[i]$ where $M$: in $R$-mod and vice versa.

Pf: Suppose $X$ indec, then $\exists i$ s.t. $\underline{H^i(X) \neq 0}$.

$$X = \begin{array}{c} \to X^{i-1} \to \mathrm{Ind}^{i-1} \to 0 \to \cdots \\ \oplus \\ H^i(X)[-i] \neq 0. \\ \hline \oplus \\ 0 \to \mathrm{Ind}^i \to \cdots \end{array}$$

Since $X$ indec, $H^i(X)[-i] \neq 0$, $\Rightarrow X = H^i(X)[-i]$. Hence $H^i(X) = M$, in. $\square$

Claim 3. $D(R\text{-mod}) \simeq \prod_{\mathbb{Z}} R\text{-Mod}$ $\quad (\cdots \to 0 \to \overset{n}{M} \to 0 \to 0 \to \cdots)$

Pf: By claim 2, obj $D(R\text{-mod}) = \underline{\prod_{n\in\mathbb{Z}}} \underline{R\text{-mod}[n]}$ $\quad \mathcal{C}\times\mathcal{D}$

$$\text{obj } D(R\text{-mod}) = \{ (\cdots, M_i, M_{i+1}, \cdots) \}$$
$$\text{Mor}(D(R\text{-mod})) = ( \quad \underset{i}{\uparrow}, \underset{j+1}{\uparrow} \quad )$$

Only need to show, if $i \neq j$, $\mathrm{Hom}_D(M[i], N[j]) = 0$.

In fact, $\mathrm{Hom}_D(M[i], N[j]) = \mathrm{Hom}_D(M, N[j-i])$
$$= \mathrm{Ext}_R^{j-i}(M, N) = 0. \quad \square$$

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§5.6. $D^-$ of hereditary rings.

left hereditary rings.

- submod of left proj mods are also proj.

Recall: $D^-(R\text{-mod}) \simeq D^-(\mathcal{P})$, $\mathcal{P}$: full subcat of all proj obj.

Claim: $\forall X \in D^-(R\text{-mod})$, $X = \prod_{n\in\mathbb{Z}} H^n(X)[-n]$.

Pf: Since $D^-(R\text{-mod}) \simeq D^-(\mathcal{P})$ W.l.o.g. $X \in D^-(\mathcal{P})$.

$$X: \cdots \to P^{-r} \xrightarrow{d^{-r}} \cdots \xrightarrow{d^{-2}} P^{-1} \xrightarrow{d^{-1}} P^0 \to 0.$$
$$P^{-1} \searrow \mathrm{Ind}^{-1} \nearrow P^0$$

$\Rightarrow \mathrm{Ind}^{-1}$ is proj, so $P^{-1} = \ker d^{-1} \oplus \mathrm{Ind}^{-1}$.

$$\Rightarrow X = \cdots \to P^{-2} \xrightarrow{d^{-2}} \ker d^{-1} \to 0$$
$$\oplus$$
$$0 \to \mathrm{Ind}^{-1} \to P^{0} \to 0 \;=\; H^0(X)[0].$$

$$\rightsquigarrow X = \underbrace{R_m^d}_{\text{inver sys}} \oplus \left(\bigoplus_{n=1}^{m} H^n(X)[-n]\right)$$

$$\overset{\varprojlim(-)}{\rightsquigarrow} X = \varprojlim R_m^d \oplus \varprojlim \bigoplus_{n\geq 1}^{m} H^n$$
$$= 0 \oplus \prod_{n\in\mathbb{Z}} H^n(X)[-n]. \quad \square.$$

Claim 2. Indec obj in $D^-(R\text{-mod})$ has the form of $M[i]$ where $M$ is indec in $R$-mod.

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§5.8.

Recall: (Right derived functor).

$\mathcal{A}, \mathcal{B}$, $\mathcal{L}$ is a localization subcat of $K(\mathcal{A})$ (e.g. $K^*(\mathcal{A})$).

(1). $F: \mathcal{L} \to K(\mathcal{B})$ tri functor, suppose $\exists$ tri subcat $L$ of $\mathcal{L}$ s.t.

(1). $\forall x \in \mathcal{L}$, $\exists$ $X \xrightarrow{s} I(X) \in Q$, $I(X) \in L$.

(2), $I \overset{\in L}{\text{is}}$ acyclic, $\Rightarrow F(I)$ is acyclic.

$\Rightarrow$ $\exists$ Right derived functor $RF$, s.t.

$$RF\,Q(X) \simeq Q_{\mathcal{B}} F(I(X)).$$

(2), ...

Cor 5.5.6. (1). $F: \mathcal{A} \to \mathcal{B}$ add functor, $F$ also denote the tri functor $F: K(\mathcal{A}) \to K(\mathcal{B})$. Suppose $\mathcal{A}$ has enough inj obj. Then Right derived functor $R^+F: D^+(\mathcal{A}) \to D(\mathcal{B})$ $\exists$, satisfying

$$R^+F \circ Q(X) \simeq Q_{\mathcal{B}} \circ F(I(X)). \quad \forall X \in K^+(\mathcal{A}).$$

where $I(X)$ inj resol of $X$. $\mathcal{I}$: full subcat of inj obj.

In addition, for any s.e.s. $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$. We have a s.e.s. cpx of $\mathcal{I}$ $\quad 0 \to I(X) \to I(Y) \to I(Z) \to 0$.

and a s.e.s. of cpx of $\mathcal{B}$

$$0 \to F(I(X)) \to F(I(Y)) \to F(I(Z)) \to 0$$

and a s.e.s. of cpx of $\mathcal{B}$

$$0 \to F(I(X)) \to F(I(Y)) \to F(I(Z)) \to 0$$

and a l.e.s. in $\mathcal{B}$:

$$0 \to R^0F(X) \to R^0F(Y) \to R^0F(Z) \to R^1F(X) \to \cdots$$

where $R^nF(X) := H^nR^+F(X) = H^nQ_{\mathcal{B}}F(I(X)) = \boxed{H^nF(I(X))}$

(2). By duality $F: \mathcal{A}^{op} \to \mathcal{B}$, ....

Pf: (1) By thm 5.8.5: let $\mathcal{L} = K^+(\mathcal{A})$, $L = K^+(\mathcal{I})$.

Given $0 \to X \to Y \to Z \to 0$ s.es in $\mathcal{A}$. By Horseshoe

Lem, we have $0 \to I(X) \to \underline{I(Y)} \to I(Z) \to 0$.

$I(Y)^n = I(X)^n \oplus I(Z)^n$.

$\xrightarrow{F \text{ add functor}}$ $0 \to F(I(X)) \to F(I(Y)) \to F(I(Z)) \to 0$ in $C(\mathcal{B})$

$\xrightarrow{\text{fundamental thm}}$ l.e.s. $\square$.

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§ 5.9. RHom & Ext.

Recall: $X, Y \in K(\mathcal{A})$. $\mathrm{Hom}^\bullet(X, Y) :=$

$$\mathrm{Hom}^n(X, Y) = \prod_{p \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(X^p, Y^{p+n}),$$

$$d^n := (d^n \underset{\substack{\uparrow \\ \mathrm{Hom}^n(X,Y)}}{f})^p = \partial_Y^{n+p} f^p + (-1)^{n+1} f^{p+1} \partial_X^p.$$

$\longrightarrow$ we have tri functor

$$\mathrm{Hom}^\bullet(X, -): K(\mathcal{A}) \longrightarrow K(\mathrm{Ab}).$$

$$\mathrm{Hom}^\bullet(-, Y): K(\mathcal{A}) \longrightarrow K(\mathrm{Ab}).$$

$\longrightarrow$ tri bifunctor.

$$\mathrm{Hom}^\bullet(-,-): K^{op}(\mathcal{A}) \times K(\mathcal{A}) \to K(\mathrm{Ab}),$$
$$(X, Y) \mapsto \mathrm{Hom}^\bullet(X, Y).$$

Lem 5.9.1 (1) Suppose $\mathcal{A}$ has enough inj obj. $X \in K(\mathcal{A})$, $Y \in K^+(\mathcal{A})$.

$\underline{Y \in C(\mathcal{I})}$. If $Y$ is exact or $X$ is exact, $\Rightarrow \mathrm{Hom}^\bullet(X, Y)$ is exact.

$Y \in C(I)$. If $Y$ is exact or $X$ is exact, $\Rightarrow \mathrm{Hom}^\cdot(X, Y)$ is exact.

(2) ...

Pf: (1). By Key formula,

$$H^n \mathrm{Hom}^\cdot(X, Y) = \mathrm{Hom}_{K(\mathcal{A})}(X, Y[n]).$$

If $X$ is exact. By prop 4.3.3. (If $C$ acyclic, $I \in C^+(I) \Rightarrow \mathrm{Hom}_{K(\mathcal{A})}(C, I) = 0$).

$\Rightarrow H^n \mathrm{Hom}^\cdot(X, Y) = 0$.

If $Y$ is exact. By Cor 4.3.3. $Y \to 0 \in Q$. $\Rightarrow \mathrm{Id}_Y \sim 0. \Rightarrow Y$ is contractible

$\Rightarrow Y = 0$ in $K^+(\mathcal{A})$. $\Rightarrow H^n \mathrm{Hom}^\cdot(X, Y) = \mathrm{Hom}_{K(\mathcal{A})}(X, Y[n]) = 0$. $\square$.

Suppose $\mathcal{A}$ has enough inj obj, tri functor

$$\mathrm{Hom}^\cdot(X, -) : K^+(\mathcal{A}) \to K(\mathcal{Ab}).$$

By thm 5.8.5, $\exists$ right derived functor

$$R\mathrm{Hom}^\cdot(X, -) : D^+(\mathcal{A}) \to D(\mathcal{Ab}).$$

Suppose $\mathcal{A}$ has enough proj obj,

$$\mathrm{Hom}^\cdot(-, Y) : K^-(\mathcal{A}) \to K(\mathcal{Ab}).$$

$\longrightarrow \exists$ right derived functor

$$R\mathrm{Hom}^\cdot(-, Y) : D^-(\mathcal{A}) \to D(\mathcal{Ab}).$$

Prop 5.9.2. $\mathcal{A}$ has enough inj & proj obj. Then we have

$$R\mathrm{Hom}^\cdot(-, -) : D^-(\mathcal{A})^{op} \times D^+(\mathcal{A}) \to D(\mathcal{Ab}).$$

i.e. $\forall X \in K^-(\mathcal{A})$, $Y \in K^+(\mathcal{A})$, we have nat isom

$$R\mathrm{Hom}^\cdot(X, -)(Y) \simeq R\mathrm{Hom}^\cdot(-, Y)(X).$$

Denote by $R\mathrm{Hom}^\cdot(X, Y)$.

Pf: $\left.\begin{array}{ll} Y \xrightarrow{s} I & \text{inj resol} \\ P \xrightarrow{t} X & \text{proj resol} \end{array}\right\} \in Q$.

Then $R\mathrm{Hom}^\cdot(X, -)(Y) \simeq \mathrm{Hom}^\cdot(X, I)$. nat for $Y$, for $X$

$R\mathrm{Hom}^\cdot(-, Y)(X) \simeq \mathrm{Hom}^\cdot(P, Y)$. nat for $X$. for $Y$.

$$\begin{tikzcd} R\mathrm{Hom}^\cdot(X, -)(Y) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Hom}^\cdot(X, -)(Y) \arrow[d] \\ R\mathrm{Hom}^\cdot(X', -)(Y) \arrow[r, "\simeq"] & \mathrm{Hom}^\cdot(X', -)(Y) \end{tikzcd}$$

Consider:

$$\mathrm{Hom}^\cdot(t, I) : \mathrm{Hom}^\cdot(X, I) \to \mathrm{Hom}^\cdot(P, I).$$

Consider:

$$\mathrm{Hom}^\bullet(f, I) : \mathrm{Hom}^\bullet(X, I) \to \mathrm{Hom}^\bullet(P, I).$$

$$\mathrm{Hom}^\bullet(P, s) : \mathrm{Hom}^\bullet(P, Y) \to \mathrm{Hom}^\bullet(P, I)$$

By key formula, $H^n \mathrm{Hom}^\bullet(f, I) = \mathrm{Hom}_{K(\mathcal{A})}(f, \underline{I[n]})$.

By Cor 4.3.4. $H^n \mathrm{Hom}^\bullet(f, I)$ is an isom of Abelian grp.

$\Rightarrow$ $\mathrm{Hom}^\bullet(f, I)$ is a Quasi-isom, Similar, so is $\mathrm{Hom}^\bullet(P, s)$.

Then $\underline{\mathrm{Hom}^\bullet(X, I) \simeq \mathrm{Hom}^\bullet(P, Y)}$ in $D(Ab)$. $\square$

not by the functorial of Hom & prop 4.2.5. & prop 4.7.5.

$X, Y \in D(\mathcal{A})$. Def

$$\mathrm{Ext}^i(X, Y) := \mathrm{Hom}_{D(\mathcal{A})}(X, Y[i]).$$

If $X, Y \in \mathcal{A}$ and $\mathcal{A}$ has enough proj obj or inj obj, then the above def coincide with the usual def of Ext.

**Prop 5.5.1** $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ s.e.s. in $C(\mathcal{A})$, $W \in C(\mathcal{A})$, then we have l.e.s of abelian grp

$$\cdots \to \mathrm{Ext}^i(W, X) \to \mathrm{Ext}^i(W, Y) \to \mathrm{Ext}^i(W, Z) \to \mathrm{Ext}^{i+1}(W, X)$$

&

$$\cdots \to \mathrm{Ext}^i(Z, W) \to \mathrm{Ext}^i(Y, W) \to \mathrm{Ext}^i(X, W) \to \mathrm{Ext}^{i+1}(Z, W)$$

Pf: By prop 5.1.2, $\exists\, h : Z \to X[1]$ in $D(\mathcal{A})$ s.t.

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1] \quad \text{D.T. in } D(\mathcal{A}).$$

Since $\mathrm{Hom}_{D(\mathcal{A})}(-, W)$ is cohom functor, we obtain the second exact seq. Similar the first one. $\square$

**Thm 5.9.4.** Suppose $\mathcal{A}$ has enough inj obj. $\forall X \in D(\mathcal{A})$, $Y \in D^+(\mathcal{A})$,

$$H^i R\mathrm{Hom}^\bullet(X, Y) \simeq \mathrm{Ext}^i(X, Y).$$

Pf: Let $I$ be the inj resol of $Y$. Then

$$\begin{aligned} H^i R\mathrm{Hom}^\bullet(X, Y) &\simeq H^i \mathrm{Hom}^\bullet(X, I) \\ &\simeq \mathrm{Hom}_{K(\mathcal{A})}(X, I[i]) \text{ (the key formula)} \\ &\simeq \mathrm{Hom}_{D(\mathcal{A})}(X, I[i]). \text{ (Lem 5.1.10)}. \end{aligned}$$

$$\simeq \mathrm{Hom}_{D(A)}(X, Y[i])$$

$$\simeq \mathrm{Ext}^i(X, Y). \quad \square$$