Chapter 6. Stable triangulated category

§6.1.

定义 6.1.1. (1) (D. Quillen [Q]) 设 $\mathcal{B}$ 是加法范畴, 并且 $\mathcal{B}$ 是某个 Abel 范畴 $\mathcal{A}$ 的扩张闭的全子范畴. 令 $\mathcal{S}$ 表示 $\mathcal{A}$ 中的那些短正合列作成的类, 这些短正合列中的每一项都属于 $\mathcal{B}$. 称二元组 $(\mathcal{B}, \mathcal{S})$ 为正合范畴.

Quillen 关于正合范畴 $\mathcal{B}$ 的上述定义依赖于一个外在的 Abel 范畴 $\mathcal{A}$.

$\mathcal{S} = \{0 \to X \to Y \to Z \to 0 \mid X, Z, Y \in \mathcal{B}.\}$

Def: An exact category is a pair $(\mathcal{A}, \mathcal{S})$, where $\mathcal{A}$ is an additive category, $\mathcal{S}$ is a class of short exact sequence $\boxed{\text{s.t.}}$

[a element of $\mathcal{S}$ is called conflation (admissible short exact sequence)

$$X \rightarrowtail Y \twoheadrightarrow Z \quad (\text{conflation})$$

inflation deflation]

(EX0): id_0 is deflation : ($0 \rightarrowtail 0 \twoheadrightarrow 0$ is a conflation)

(EX1): The composition of two deflations is a deflation

(EX2): $\forall g: Y \twoheadrightarrow Z$ deflation, $\forall b: Z' \to Z$, $\exists$ pull-back diagram

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "b'"'] \arrow[dr, phantom, "\circlearrowleft"] & Z' \arrow[d, "b"] \\ Y \arrow[r, two heads, "g"'] & Z \end{tikzcd}$$

s.t g': deflation

$(EX2)^{op}$: $f: X \rightarrowtail Y$ inflation, $\forall$ $a: X \to X'$, $\exists$ push-out

$$\begin{tikzcd} X \arrow[r, tail, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "a'"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

s.t f': inflation

Example: ① $\mathcal{A}$: abelian category

$\mathcal{S} = \{\text{all s.e.s}\}$

$(\mathcal{A}, \mathcal{S})$ is an exact category,

inflation (admiss. inj) = inj

deflation = surj

Abelian exact structure

② $\mathcal{A}$: additive category

$\mathcal{S} = \{\text{split short exact sequence}\}$

$(\mathcal{A}, \mathcal{S})$ ...

③ $\mathcal{B}$: abelian category

$\mathcal{A} \subseteq \mathcal{B}$ extension closed subcategory of $\mathcal{B}$

$\mathcal{S} = \{ 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0 \text{ s.e.s in } \mathcal{B} \mid X, Y, Z \in \mathcal{A} \}$

Conversely, if $(\mathcal{A}, \mathcal{E})$ is an exact category. ($\mathcal{A}$ is small category). Then $\exists\ \mathcal{B}$: abelian category s.t $\mathcal{A} \subseteq \mathcal{B}$

$\mathcal{S} = \{ \quad \}$

Thm: (Gabriel – Quillen embeding Thm)

Let $\mathcal{A}$ be a small - additive $\mathbb{k}$-category

$\Rightarrow$ all exact structure on $\mathcal{A}$ come in this way.

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(2) 设 $(\mathcal{B},\mathcal{S})$ 是正合范畴. $\mathcal{B}$ 中的对象 P 称为 $\mathcal{S}$- 投射对象, 如果对 $\mathcal{S}$ 中的任意短正合列 $0 \longrightarrow X \longrightarrow Y \xrightarrow{v} Z \longrightarrow 0$ 以及任意态射 $f: P \longrightarrow Z$, 均存在态射 $g: P \longrightarrow Y$ 使得 $f = vg$.

对偶地, $\mathcal{B}$ 中的对象 I 称为 $\mathcal{S}$- 内射对象, 如果对 $\mathcal{S}$ 中任意短正合列 $0 \longrightarrow X \xrightarrow{u} Y \longrightarrow Z \longrightarrow 0$ 以及任意态射 $f: X \longrightarrow I$, 均存在态射 $g: Y \longrightarrow I$ 使得 $f = gu$.

$$\begin{tikzcd} & & P \arrow[d, dashed, "\exists g"'] \arrow[dr, "f"] & & \\ 0 \arrow[r] & X \arrow[r] & Y \arrow[r, "v"] & Z \arrow[r] & 0 \end{tikzcd}$$

s.t $f = vg$ $\quad$ P : $\mathcal{S}$-proj

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "u"] \arrow[dr, "f"'] & Y \arrow[r] \arrow[d, dashed, "\exists g"] & Z \arrow[r] & 0 \\ & & I & & \end{tikzcd}$$

s.t $f = gu$

(3) 设 $(\mathcal{B},\mathcal{S})$ 是正合范畴. 称 $(\mathcal{B},\mathcal{S})$ 有足够多的 $\mathcal{S}$- 投射对象, 如果对每个 $X\in\mathcal{B}$, 存在 $\mathcal{S}$ 中的短正合列 $0\longrightarrow K\longrightarrow P\stackrel{\pi}{\longrightarrow}X\longrightarrow 0$ 使得 P 是 $\mathcal{S}$- 投射的.

对偶地, 称 $(\mathcal{B},\mathcal{S})$ 有足够多的 $\mathcal{S}$- 内射对象, 如果对每个 $X\in\mathcal{B}$, 存在 $\mathcal{S}$ 中的短正合列 $0\longrightarrow X\stackrel{\sigma}{\longrightarrow}I\longrightarrow C\longrightarrow 0$ 使得 I 是 $\mathcal{S}$- 内射的.

(4) 称正合范畴 $(\mathcal{B},\mathcal{S})$ 是 Frobenius 范畴, 如果 $(\mathcal{B},\mathcal{S})$ 有足够多的 $\mathcal{S}$- 投射对象和足够多的 $\mathcal{S}$- 内射对象, 并且一个对象是 $\mathcal{S}$- 投射的当且仅当它是 $\mathcal{S}$- 内射的.

(B,S) : exact category

$\Longrightarrow$ (B,S) is Frobenius category

S-proj $=$ S-injective

(5) 设 $(\mathcal{B},\mathcal{S})$ 是 Frobenius 范畴. $\mathcal{B}$ 的稳定范畴 $\underline{\mathcal{B}}$ 是如下定义的加法范畴:

- $\underline{\mathcal{B}}$ 中的对象就是 $\mathcal{B}$ 中的对象;
- 对 $X,Y\in\underline{\mathcal{B}}$, $\mathrm{Hom}_{\underline{\mathcal{B}}}(X,Y)$ 是商群 $\mathrm{Hom}_{\mathcal{B}}(X,Y)/I(X,Y)$, 其中 $I(X,Y)$ 是由可通过 $\mathcal{S}$- 内射对象分解的态射 f (即, 存在 $g:X\longrightarrow I$, $h:I\longrightarrow Y$ 使得 $f=hg$, 其中 I 是 $\mathcal{S}$- 内射的) 作成的集合. 易知 $I(X,Y)$ 是 $\mathrm{Hom}_{\mathcal{B}}(X,Y)$ 的子群.

对 $u\in\mathrm{Hom}_{\mathcal{B}}(X,Y)$, 用 $\underline{u}$ 表示 $\mathrm{Hom}_{\underline{\mathcal{B}}}(X,Y)$ 中的元 $u+I(X,Y)$.

$\boxed{\underline{B}}$: $\mathrm{obj}(\underline{B}) = \mathrm{obj}(B)$

$\mathrm{Hom}_{\underline{B}}(X,Y) = \mathrm{Hom}_B(X,Y)$

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \\ & P=I \arrow[ur] & \end{tikzcd} \in I(X,Y)$$

$u\in\mathrm{Hom}_B(X,Y)$

$\underline{u} = u + I(X,Y)$

设 $(\mathcal{B},\mathcal{S})$ 是 Frobenius 范畴．给定 $X\in\mathcal{B}$, 我们有 $\mathcal{S}$ 中的短正合列 $0\longrightarrow X\xrightarrow{m_X} I(X)\xrightarrow{p_X} X'\longrightarrow 0$, 其中 $I(X)$ 是 $\mathcal{S}$- 内射的．$\mathcal{S}$ 中任何两个这样的短正合列间有下述关系．

引理 6.1.3. 设 $(\mathcal{B},\mathcal{S})$ 是 Frobenius 范畴， $X\in\mathcal{B}$. 若

$$0\longrightarrow X\xrightarrow{m_X} I(X)\xrightarrow{p_X} X'\longrightarrow 0,\quad 和\quad 0\longrightarrow X\xrightarrow{m} I\xrightarrow{p} X''\longrightarrow 0$$

都是 $\mathcal{S}$ 中短正合列且 $I(X)$ 和 I 都是 $\mathcal{S}$- 内射对象．则在 $\mathcal{B}$ 中 $X'\cong X''$.

TX

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "m_X"] \arrow[d, equal] & I(X) \arrow[r, "p_X"] \arrow[d, dashed, "\alpha"] & X' \arrow[r] \arrow[d, dashed, "\beta"] & 0 \\
0 \arrow[r] & X \arrow[r, "m"] \arrow[d, equal] & I \arrow[r, "p"] \arrow[d, dashed, "\alpha'"] & X'' \arrow[r] \arrow[d, dashed, "\beta'"] & 0 \\
0 \arrow[r] & X \arrow[r, "m_X"] & I(X) \arrow[r, "p_X"] & X' \arrow[r] & 0
\end{tikzcd}$$

证: $\alpha'\alpha\circ m_X = m_X$

$(\alpha'\alpha - id_{I(x)})\, m_X = 0$

By universal property of coker

$(\alpha'\alpha - id_{I(x)}) = a p_X$

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "m_X"] & I(x) \arrow[r, "p(x)"] \arrow[dr, "\alpha'\alpha - id_{I(x)}"'] & X' \arrow[r] \arrow[d, dashed, "a"] & 0 \\
 & & & I(x) &
\end{tikzcd}$$

then $p(x)\, a\, p(x) = p(x)(\alpha'\alpha - id_{I(x)})$

$= (\beta'\beta - id_{I(x)})\, p_X$

$\Longrightarrow p(x)a = \beta'\beta - Id_{I(x)}$

$P_X a: X' \to I(x) \xrightarrow{p(x)} X'$

$\beta'\beta = id_{I(x)}$

$\beta\beta' = id_I$

引理 6.1.4. 设 $(\mathcal{B},\mathcal{S})$ 是 Frobenius 范畴．设 $u: X\longrightarrow Y$ 是 $\mathcal{B}$ 中的态射，且

$$0\longrightarrow X\xrightarrow{m_X} I(X)\xrightarrow{p_X} X'\longrightarrow 0,\quad 和\quad 0\longrightarrow Y\xrightarrow{m_Y} I(Y)\xrightarrow{p_Y} Y'\longrightarrow 0$$

是 $\mathcal{S}$ 中的正合列，其中 $I(X)$ 和 $I(Y)$ 均是 $\mathcal{S}$- 内射对象．则

(1) 存在 $\mathcal{B}$ 中态射 $Tu: X'\longrightarrow Y'$ 使得有下述交换图

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "m_X"] \arrow[d, "u"] & I(X) \arrow[r, "p_X"] \arrow[d, "I(u)"] & X' \arrow[r] \arrow[d, "Tu"] & 0 \\
0 \arrow[r] & Y \arrow[r, "m_Y"] & I(Y) \arrow[r, "p_Y"] & Y' \arrow[r] & 0
\end{tikzcd}$$

而且 Tu 不依赖于 $I(u)$ 的选择. 即, 若有另一个交换图

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "m_X"] \arrow[d, "u"] & I(X) \arrow[r, "p_X"] \arrow[d, "I'(u)"] & X' \arrow[r] \arrow[d, "u'"] & 0 \\ 0 \arrow[r] & Y \arrow[r, "m_Y"] & I(Y) \arrow[r, "p_Y"] & Y' \arrow[r] & 0 \end{tikzcd}$$

Tx　T(Y)

则 $\underline{Tu}=\underline{u'}$. 特别地, $\underline{T\mathrm{Id}_X}=\underline{\mathrm{Id}_{X'}}$, $\underline{T0}=\underline{0}$.

(2) 若 $\underline{u}=\underline{v}$, 则 $\underline{Tu}=\underline{Tv}$.

pf: if $\underline{u}=\underline{v}$, $\exists$ C.D.

$$\Rightarrow \begin{tikzcd} 0 \arrow[r] & X \arrow[r, "m_X"] \arrow[d, "{v,u}"'] & I(X) \arrow[r, "p_X"] \arrow[d, "{I(v),I(u)}"] \arrow[dl, "a"'] & X' \arrow[r] \arrow[d, "{T(u),T(v)}"] & 0 \\ 0 \arrow[r] & Y \arrow[r, "m_Y"] & I(Y) \arrow[r, "p_Y"] & Y' \arrow[r] & 0 \end{tikzcd}$$

$\because \underline{u}=\underline{v}$

$\therefore u-v\in \mathcal{I}(X,Y)$: $X \xrightarrow{\iota} I \xrightarrow{p} Y$, $m_X: X\to I(X)$, $f: I(X)\to I$

s.t. $\underline{u-v = a m_X}$

$(X \xrightarrow{m_X} I(X) \xrightarrow{pf} Y)$, $a = pf$

$m_Y(a m_X)$

$= m_Y(u-v)$

$= (I(u)-I(v))m_X$

$\Rightarrow (I(u)-I(v)-m_Y a)\circ m_X = 0$

$$\begin{tikzcd} X \arrow[r, "m_X"] & I(X) \arrow[r, "p_X"] \arrow[dr, "I(u)-I(v)-m_Y a"'] & X' \arrow[d, dashed, "\exists b"] \\ & & I(Y) \end{tikzcd}$$

$\Rightarrow I(u)-I(v)-m_Y a = bp_X$

$$\begin{aligned} \Rightarrow (Tu-Tv)p_X &= p_Y(I(u)-I(v)) = p_Y(bp_X+m_Y a) \\ &= p_Y b p_X + p_Y m_Y a \\ &= p_Y b p_X \end{aligned}$$

$\Rightarrow \underline{Tu-Tv = p_Y b}$: $X' \to I(Y) \xrightarrow{p_Y} Y' \in \underline{\mathcal{I}(X',Y')}$

$\Rightarrow \underline{Tu}=\underline{Tv}$. #

(T)

Remark: ① in Lemma 6.4:

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "m_X"] \arrow[d, "u"', "v"] & I(X) \arrow[r] & X' \quad X'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & Y \arrow[r] & I(Y) \arrow[r] & Y' \arrow[r] & 0 \end{tikzcd}$$

I

$\underline{Tu}=\underline{Tv}$

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d, "u"] & I \arrow[r] \arrow[d] & X'' \arrow[r] \arrow[d, dashed, "u'"] & 0 \\ 0 \arrow[r] & Y \arrow[r] & I(Y) \arrow[r] & Y' \arrow[r] & 0 \end{tikzcd}$$

$\underline{u'}\neq \underline{Tu}$　　$\underline{Tu}=\underline{u'\beta}$

$\beta\in \mathrm{Aut}(B)$

$T : \underline{\mathcal{B}} \longrightarrow \underline{\mathcal{B}}$

fix $0 \to X \xrightarrow{m_X} I(X) \xrightarrow{p_X} TX \to 0$

fix

$(\mathcal{B}, \mathcal{E})$: Frobenius category

fix: $S = \{ 0 \longrightarrow X \xrightarrow{m_X} I(X) \xrightarrow{p_X} TX \longrightarrow 0 \}$

By Lemma 6.13 + Lemma 6.4

$X \longmapsto TX$

$$\begin{tikzcd} X \arrow[d, "u"] & & TX \arrow[d, "Tu"] \\ Y & \longmapsto & TY \end{tikzcd} \qquad T\underline{u} := T\underline{u}$$

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d, "u"] \arrow[dr, phantom, "\circlearrowleft"] & I(X) \arrow[r, "p_X"] \arrow[d, "I(u)"] \arrow[dr, phantom, "\circlearrowleft"] & TX \arrow[r] \arrow[d, "T(u)"] & 0 \\ 0 \arrow[r] & Y \arrow[r] & I(Y) \arrow[r] & T(Y) \arrow[r] & 0 \end{tikzcd}$$

$T \in \mathrm{End}(\underline{\mathcal{B}})$

$T \in \mathrm{Aut}(\underline{\mathcal{B}})$

$\forall \; u : X \longrightarrow Y$

$$\begin{tikzcd} X \arrow[r, "m_X"] \arrow[d, "u"] \arrow[dr, phantom, "\circlearrowleft"] & I(X) \arrow[d, "i_u"] \\ Y \arrow[r, "v"] & C_u \end{tikzcd} \qquad \text{push-out } (u, m_X)$$

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "m_X"] \arrow[d, "u"] & I(X) \arrow[r, "p_X"] \arrow[d, "i_u"] & T(X) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y \arrow[r, "m_Y"] & e_u \arrow[r] & T(X) \arrow[r] & 0 \end{tikzcd} \qquad e_u \in \mathcal{E}.$$

$$X \xrightarrow{\binom{u}{m_X}} I(X) \oplus Y \xrightarrow{(v, -i_u)} C_u \longrightarrow 0$$

$$\underline{\Omega} = \{ X \xrightarrow{\underline{u}} Y \xrightarrow{\underline{v}} C_u \xrightarrow{\underline{w}} TX \}$$

$\underline{\mathcal{E}}$

(Lemma 1.4.4)

$0 \to X \to I(X) \to TX \to 0$

定理 6.2.1. ([Hap1], [Hap2]) 设 $(\mathcal{B}, \mathcal{S})$ 是 Frobenius 范畴. 则 $(\underline{\mathcal{B}}, T, \mathcal{E})$ 是三角范畴.

引理 6.2.2. 设 $(\mathcal{B}, \mathcal{S})$ 是 Frobenius 范畴，$v: Y \longrightarrow Z$ 是 $\mathcal{B}$ 中的态射．设有 $\mathcal{S}$ 中的短正合列 $0 \longrightarrow Y \xrightarrow{m_Y} I(Y) \xrightarrow{p_Y} TY \longrightarrow 0$ 和 $0 \longrightarrow Y \xrightarrow{m} I \xrightarrow{p} M \longrightarrow 0$, 其中 $I(Y)$ 和 I 是 $\mathcal{S}$- 内射对象．假设有如下行正合交换图

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "m_Y"] \arrow[d, "v"] & I(Y) \arrow[r, "p_Y"] \arrow[d, "i_v"] & TY \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Z \arrow[r, "j"] & X' = C_v \arrow[r, "j'"] & TY \arrow[r] & 0
\end{tikzcd}$$

和

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "m"] \arrow[d, "v"] & I \arrow[r, "p"] \arrow[d, "i'_v"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Z \arrow[r, "\widetilde{j}"] & \widetilde{X'} = C'_v \arrow[r, "\widetilde{j'}"] & M \arrow[r] & 0
\end{tikzcd}$$

则

(1) 存在 $\alpha, \alpha', \beta, \beta'$ 使得有如下的交换图

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "m_Y"] \arrow[d, equal] & I(Y) \arrow[r, "p_Y"] \arrow[d, "\alpha"] & TY \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & Y \arrow[r, "m"] \arrow[d, equal] & I \arrow[r, "p"] \arrow[d, "\alpha'"] & M \arrow[r] \arrow[d, "\beta'"] & 0 \\
0 \arrow[r] & Y \arrow[r, "m_Y"] & I(Y) \arrow[r, "p_Y"] & TY \arrow[r] & 0
\end{tikzcd} \tag{6.4}$$

且对使得图 (6.4) 交换的任意态射 $\alpha, \alpha', \beta, \beta'$, 必存在 $r: X' \longrightarrow \widetilde{X'}$ 和 $r': \widetilde{X'} \longrightarrow X'$ 使得

$$\begin{aligned}
\underline{r'}\,\underline{r} &= \mathrm{Id}_{X'}, & \underline{r}\,\underline{r'} &= \mathrm{Id}_{\widetilde{X'}} \\
r i_v &= i'_v \alpha, & r j &= \widetilde{j}, \\
r' i'_v &= i_v \alpha', & r' \widetilde{j} &= j, \\
\widetilde{j'} r &= \beta j', & j' r' &= \beta' \widetilde{j'};
\end{aligned}$$

(2) 有如下三角的同构:

$$\begin{tikzcd}
Y \arrow[r, "\underline{v}"] \arrow[d, equal] & Z \arrow[r, "\underline{j}"] \arrow[d, equal] & X' = C_v \arrow[r, "\underline{j'}"] \arrow[d, dashed, "\underline{r}"] & TY \arrow[d, equal] \\
Y \arrow[r, "\underline{v}"] & Z \arrow[r, "\underline{\widetilde{j}}"] & \widetilde{X'} = C'_v \arrow[r, "\underline{\beta'}\,\underline{\widetilde{j'}}"] & TY
\end{tikzcd}$$

$\underline{r'r} = \underline{id}$

$\underline{rr'} = \underline{id}$

Pf: Due to 6.13 $\exists\ \alpha, \alpha', \beta, \beta'$ s.t (6.4)

and
$$\begin{cases} \alpha'\alpha - id_{I(Y)} = a p_Y & (4) \\ \beta'\beta - id_{T(Y)} = p_Y a & (5) \\ \beta'\beta = id\, T_Y & (6) \\ \beta\beta' = id\, M & (7) \end{cases}$$

Consider the pushout - diagram (8) (9)

$$\begin{tikzcd} Y \arrow[r, "m_Y"] \arrow[d, "v"'] & I(Y) \arrow[d, "i_v"] \arrow[ddr, bend left, "i_v\alpha"] & \\ Z \arrow[r, "j"'] \arrow[drr, bend right, "\tilde{j}"'] & X' \arrow[dr, "r"] & \\ & & \tilde{X}' \end{tikzcd}$$

$[\tilde{j}v = \tilde{i}_v m = \tilde{i}_v \alpha m_Y]$

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "m_Y"] \arrow[d, "v"] & I(Y) \arrow[r, "P_Y"] \arrow[d, "i_v"] & TY \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Z \arrow[r, "j"] & X' = e_v \arrow[r, "j'"] & TY \arrow[r] & 0 \end{tikzcd} \quad (1)$$

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "m"] \arrow[d, "v"] & I \arrow[r, "p"] \arrow[d, "\tilde{i}_v"] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Z \arrow[r, "\tilde{j}"] & \tilde{X}' \arrow[r, "\tilde{j}'"] & M \arrow[r] & 0 \end{tikzcd} \quad (2)$$

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "m_Y"] \arrow[d, equal] & I(Y) \arrow[r, "p_Y"] \arrow[d, "\alpha"] & TY \arrow[r] \arrow[d, "\beta"] & 0 \\ 0 \arrow[r] & Y \arrow[r, "m"] \arrow[d, equal] & I \arrow[r, "p"] \arrow[d, "\alpha'"] & M \arrow[r] \arrow[d, "\beta'"] & 0 \\ 0 \arrow[r] & Y \arrow[r, "m_Y"] & I(Y) \arrow[r, "p_Y"] & TY \arrow[r] & 0 \end{tikzcd} \quad (6.4)\ (3)$$

(10)

then $\exists\ r: X' \to \tilde{X}'$ s.t $r \circ j = \tilde{j}$ (11)

$r \circ i_v = \tilde{i}_v \alpha$ (12)

$$\begin{tikzcd} Y \arrow[r, "m_Y"] \arrow[d, "v"'] & I \arrow[d, "\tilde{i}_v"] \arrow[ddr, bend left, "i_v \circ \alpha'"] & \\ Z \arrow[r, "\tilde{j}"'] \arrow[drr, bend right, "j"'] & \tilde{X}' \arrow[dr, "r'"] & \\ & & X' \end{tikzcd}$$

s.t $r'\tilde{j} = j$ (13)

$r'\tilde{i}_v = i_v \circ \alpha'$ (14)

$[jv = \tilde{i}_v \cdot m_Y = i_v \circ \alpha' m_Y]$

$\Rightarrow\ r'rj \overset{(11)}{=} r'\tilde{j} \overset{(13)}{=} j \Rightarrow (r'r - id_{X'})\, j = 0$

$$\begin{tikzcd} Z \arrow[r, "j"] & X' \arrow[r, "j'"] \arrow[dr, "r'r - id_{X'}"'] & TY \arrow[d, dashed, "\exists b"] \\ & & X' \end{tikzcd}$$

s.t $r'r - id_{X'} = bj'$ (15)

$\Rightarrow (r'r - id_{X'})\, i_v = (bj')\, i_v = b P_Y$

on the other hand.

$(r'r - id_{X'})\, i_v = r'r\, i_v - i_v$
$\overset{(12)}{=} \tilde{i}_v \alpha' \alpha - i_v = i_v(\alpha'\alpha - id_{I(Y)})$
$\overset{(4)}{=} i_v a P_Y$

$b P_Y = i_v a P_Y \Rightarrow b = \boxed{i_v \circ a}: I(Y) \xrightarrow{a} I(Y) \xrightarrow{i_v} X'$

$\Rightarrow b = 0$ in $\underline{B}$

$\Rightarrow \underline{r'r} = \underline{id_X}$ similarly $\underline{r}\,\underline{r'} = id_{\widetilde{X'}}$

$\left[\ \underline{\widetilde{j'}r = \beta j'}\ ,\quad j'r' = \beta'\widetilde{j}\ \right]$ ✓

$\underline{(\widetilde{j'}r - \beta j')\begin{pmatrix} j \\ -iv \end{pmatrix} = 0}$

满

$\left[\right.$ pushout diagram

$$\begin{tikzcd} Y \arrow[r, "m_Y"] \arrow[d, "v"] & I(Y) \arrow[d, "iv"] \\ Z \arrow[r, "j"] & X' \end{tikzcd}$$

$Y \to Z \oplus I(Y) \xrightarrow{(j\ -iv)} X' \to 0$ $\left.\right]$

$$\begin{cases} (\widetilde{j'}r - \beta j')j = \widetilde{j'}\underline{rj} = \underline{\widetilde{j'}\widetilde{j}} = 0 \\ (\widetilde{j'}r - \beta j')iv = 0 \end{cases}$$

lemma 1.4.41

Lemma 6.2.3. $(\underline{B}, T, \mathcal{E})$ satisfy (tr1) and (tr2)

Pf: $\Big[$ TR(1) $\begin{cases} \text{(i)} \ \forall\, \underline{u}: X \to Y,\ \exists\ X \xrightarrow{\underline{u}} Y \xrightarrow{\underline{v}} Z \xrightarrow{\underline{w}} TX \in \Omega \\ \text{(ii)} \ X \xrightarrow{id} X \to 0 \to TX \in \mathcal{E} \end{cases}$

TR(2). if $X \xrightarrow{\underline{u}} Y \xrightarrow{\underline{v}} Z \xrightarrow{\underline{w}} TX \in \Omega$

$\Rightarrow Y \xrightarrow{\underline{v}} Z \xrightarrow{\underline{w}} TX \xrightarrow{-\underline{Tu}} T(Y) \in \mathcal{E}\ \Big]$

TR1. (i): ✓

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d, "u"] & T(X) \arrow[r] \arrow[d] & TX \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & Y \arrow[r] & Z \arrow[r] & TY \arrow[r] & 0 \end{tikzcd} \rightsquigarrow X \xrightarrow{\underline{u}} Y \xrightarrow{\underline{v}} Z \xrightarrow{\underline{w}} TX \in \Omega$$

(ii) ✓

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "m_X"] \arrow[d, "id_X"] & I(X) \arrow[r, "p_X"] \arrow[d, dashed] & TX \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & X \arrow[r] & I(X) \arrow[r] & TX \arrow[r] & 0 \end{tikzcd} \rightsquigarrow \begin{tikzcd} X \arrow[r] & X \arrow[r] & I(X) \arrow[r] \arrow[d, dashed] & TX \\ X \arrow[r, "id_X"] & X \arrow[r] & 0 \arrow[r] & TX \end{tikzcd}$$

TR2: $X \xrightarrow{\underline{u}} Y \xrightarrow{\underline{v}} Z \xrightarrow{\underline{w}} TX \in \Omega$

$Y \to Z \to TX \to TY \quad \in \Omega \subset \mathcal{E}$

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ 0 \arrow[r] & Y \arrow[r, "m_Y"] \arrow[d, "v"] \arrow[dr, phantom, "②"] & I(Y) \arrow[r, "p_Y"] \arrow[d, "\binom{1}{0}"] & TY \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & C_u \arrow[r, "\binom{f}{w}"] \arrow[d, "w"] & I(Y) \oplus TX \arrow[r, "(p_Y, -Tu)"] \arrow[d, "(0,1)"] & TY \arrow[r] & 0 \\ & TX \arrow[r, equal] \arrow[d] & TX \arrow[d] & & \\ & 0 & 0 & & \end{tikzcd}$$

exact exact

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "m_X"] \arrow[d, "u"] & I(X) \arrow[r] \arrow[d, "I(u)"] & TX \arrow[r] \arrow[d, "T(u)"] & 0 \\ 0 \arrow[r] & Y \arrow[r, "m_Y"] & I(Y) \arrow[r] & T(Y) \arrow[r] & 0 \end{tikzcd}$$

Consider pushout diagram

$$\begin{tikzcd} X \arrow[r, "m_X"] \arrow[d, "u"] & I(X) \arrow[d, "v_u"] \arrow[ddr, bend left, "I(u)"] & \\ Y \arrow[r, "u"] \arrow[drr, bend right, "m_Y"] & C_u = Z \arrow[dr, dashed, "f"] & \\ & & I(Y) \end{tikzcd}$$

then $f: C_u = Z \longrightarrow I(Y)$

s.t $f \cdot i_u = I(u)$ ①

$f \cdot v = m_Y$ ②

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] \arrow[d] & C_u \arrow[r] \arrow[d] & TX \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I(Y) \arrow[r] & I(Y) \oplus T(X) \arrow[r] & TX \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} Y \arrow[r, "v"] \arrow[d, "m_Y"] & C_u \arrow[d, "\binom{f}{w}"] \\ I(Y) \arrow[r, "(1,0)"] & I(Y) \oplus T(X) \end{tikzcd}$$

$\Downarrow$

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "m_Y"] \arrow[d, "v"] & I(Y) \arrow[r, "P_Y"] \arrow[d, "\binom{1}{0}"] & T(Y) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & C_u \arrow[r, "\binom{f}{w}"] & I(Y) \oplus T(X) \arrow[r, "(a,b)"] & T(Y) \arrow[r] & 0 \end{tikzcd}$$

P_Y

$\Rightarrow \underline{a = P_Y}$

aim: $\boxed{b = -Tu}$

$$\underline{b}\ \boxed{w(v,-i_u)} = \underline{-T_u}\ \boxed{w(v,-i_u)}$$

$$\left[\ X \to Y \oplus I(X) \xrightarrow{(r,-i_u)} C_u \to 0\ \right]$$

$$\left[\ Y \to C_u \xrightarrow{w} TX \to 0\ \right]$$

$$\because\ (P_Y, b)\begin{pmatrix} f \\ w \end{pmatrix} i_u = (a, b)\begin{pmatrix} f \\ w \end{pmatrix} i_u = 0$$

$$(P_Y f + bw)\, i_u = 0$$

$$bw\, i_u = -P_Y f\, i_u \overset{①}{=} -P_Y\, I(u)$$

$$bw(v,-i_u) = (0,\ P_Y I(u))$$
$$\overset{6.3}{=} (0,\ T(u) P_X)$$
$$= -T(u)\, W(V,-i_u)$$

Lemma 6.2.4

$(\underline{\mathcal{B}}, T, \mathcal{E})$ and $\mathfrak{R}$ satisfy TR3

Pf:

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & C_u \arrow[r, "w"] \arrow[d, "h"] & TX \arrow[d, "Tf"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & C_{u'} \arrow[r, "w'"] & TX'
\end{tikzcd}$$

$$u'f = gu$$

$$\Rightarrow gu - u'f \in \mathcal{I}(X, Y') : (X \xrightarrow{m_X} I(X) \xrightarrow{t} Y') \Rightarrow gu - u'f = t \cdot m_X$$

$$\begin{tikzcd}
X \arrow[r, "m_X"] \arrow[d, "u"] & I(X) \arrow[d, "i_u"] \arrow[ddr, bend left, "i_{u'} I(f)"] & \\
Y \arrow[r, "v"] \arrow[drr, bend right, "v'g"'] & C_u \arrow[dr, dashed, "h"] & \\
 & & C_{u'}
\end{tikzcd}
\qquad
\begin{tikzcd}
X' \arrow[r, "m_{X'}"] \arrow[d, "u'"] & I(X') \arrow[d, "i_{u'}"] \\
Y' \arrow[r, "v'"] & C_{u'}
\end{tikzcd}$$