Chapter 7. Gorenstein projective obj

- 1950. D. Gorenstein (ph.D.) $\to$
- 1962. H. Bass: Comm. Gorenstein ring R (双边Noether环称为Gorenstein $\Leftrightarrow$ $_RR, R_R$ 左右内射维数 $<\infty$) ☆
- 1966–1967. Auslander

R : G–类: (i.e satisfy)

(1) $\mathrm{Ext}_R^i(M, R) = 0 = \mathrm{Ext}_R^i(\mathrm{Hom}_R(M, R), R)$ $\forall i \geq 1$

(2) $M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ natural iso.

Thm: Com. local ring R is Gorenstein (Gdim) $\Leftrightarrow$ $\mathrm{G\text{-}dim}(N \in R\text{-mod}) < \infty$

1995. Enochs – Jenda

08. Gao – Zhang

$\mathcal{A}$: abelian category

$\mathcal{A} = R\text{-mod}$ (R: left Nother ring) $\quad \mathcal{P}(\mathcal{A}) := \mathcal{P}(R\text{-mod})$,

$\quad\; R\text{-Mod}$ $\quad \mathcal{P}(\mathcal{A}) := \mathcal{P}(R\text{-Mod})$

$M \in R\text{-mod}$

$\mathrm{add}(M) := \{$ direct summands of the finite sum $\}$

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$\mathcal{P}(R) = \mathrm{add}({}_RR)$

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§ 7.1.

$\mathcal{A}$: Abelian category with enough projective objects

$\mathcal{X} \subseteq \mathcal{A}$ be a full subcategory of $\mathcal{A}$

$\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{X})$

Define a Hom$_{\mathcal{A}}(-, \mathcal{X})$ exact sequ. $E^{\cdot}$ meas that

(1) $E^{\cdot}$ itself is exact

(2) $\mathrm{Hom}_{\mathcal{A}}(E, X)$ remains to be exact for $\forall x \in \mathcal{X}$

dually, we have terminology $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, -)$ -exact sequence.

$\mathcal{X} = \mathcal{P}(\mathcal{A})$

Def 7.1 : $\mathcal{A}$ Abelian category with enough projective objects $\mathcal{P}(\mathcal{A})$:

(i) A $\boxed{\text{totally projective resolution in } \mathcal{A}}$ is $\boxed{\mathrm{Hom}_{\mathcal{A}}(-\ \mathcal{P}(\mathcal{A}))}$-exact sequence

$(P.d) = \cdots\ P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \to P^2 \to \cdots$

with $P^i \in \mathcal{P}(\mathcal{A})$ $\quad \mathrm{Ker} d^0 = \mathrm{Im} d^{-1} \cong M$

(ii) An obj $\boxed{M}$ is called Gorenstein projective object. $\exists$ $(P^{\cdot}, d)$

s.t $M \cong \mathrm{Im} d^{-1}$,

Denote by $\mathcal{GP}(\mathcal{A})$, the full subcategory of Goren... proj ... in $\mathcal{A}$

R- left Noether ring, $\mathcal{GP}(R) := \mathcal{GP}(R\text{-mod})$

$\mathcal{P}(A)$

$\mathrm{Hom}(-, \mathcal{P}(A)) \Leftrightarrow \mathrm{Hom}_{A}(P, R)$

$\mathcal{A} = R\text{-Mod} \leftarrow \mathcal{GP}(R)$

Dually. (Gorenstein-injective obj).

(1) $\mathrm{Hom}_{\mathcal{A}}(\mathcal{I}(\mathcal{A}), -)$-exact sequen $(I, d) = \cdots\ I^{-1} \xrightarrow{d^{-1}} I^0 \xrightarrow{d^0} I^1 \to \cdots$

(totally inj. resolution)

$I^i \in \mathcal{I}(\mathcal{A})$

(2) $M \cong \mathrm{Im} d^{-1}$ is called Gorenstein-inj obj.

$\mathcal{GI}(\mathcal{A})$

$\mathcal{X} \subseteq \mathcal{A}$: full subcategory of $\mathcal{A}$, Put

$^{\perp}\mathcal{X} = \{ M \in \mathcal{A} \mid \mathrm{Ext}^i_{\mathcal{A}}(M, X) = 0 \mid \forall X \in \mathcal{X}\ \forall i \geq 1 \}$

• Fact 7.13. $\mathcal{A}$: Abelian category with enough projective objects

(i) $\underline{P(A) \subseteq GP(A)} \underset{\Delta}{\subseteq} {}^{\perp}P(A)$

$0 \to P \xrightarrow{id_M} P \to 0$
$\qquad\;\; {}^{-1} \qquad\quad {}^{0}$

$\forall P \in P(A)$

R：自入射环

$(GP(A) = R\text{-mod}) \Leftarrow$

$\cdots \to P^1 \to P^0 \to M \to I^0 \to I^1 \to \cdots$

$\mathrm{Hom}(\_, P(A))$
$\qquad\qquad\quad\;\; \cup$
$\qquad\qquad\quad\; I(A)$

Then $\cdots \to P^{-1} \xrightarrow{d^{-1}} \mathrm{Im}\, d^{-1} \to 0$ is a projective resolution of M
$\qquad\qquad\qquad\qquad\qquad\;\; \wr\!\!\mid$
$\qquad\qquad\qquad\qquad\qquad\;\; M$

$\mathrm{Hom}(-, P(A))$

$\mathrm{Ext}^i_{\mathcal{A}}(M, \underline{P(A)}) = 0$

$\underline{0 \to \mathrm{Hom}_{\mathcal{A}}(d^{-1}, P(A))} \xrightarrow{0} \mathrm{Hom}_{\mathcal{A}}(P^{-1}, \underline{P(A)}) \to \cdots$

then $\mathrm{Ext}^k(M, P(A)) = H^k(\qquad) = 0$

(ii) $\left\{ \begin{array}{l} \mathcal{A}: \text{ proj} = \text{inj} \\ GP(A) = \mathcal{A} \end{array} \right.$

(iii) $(P^\cdot, d)$: totally Gorenstein pro. resolution

$\mathrm{Ext}^i_{\mathcal{A}}(\mathrm{Im}\, d^i, P(A)) = 0$

$\Rightarrow \left\{ \begin{array}{l} \underline{\mathrm{Im}\, d^i \in GP(A)} \\ \cdots \to \underline{P^i \to \mathrm{Im}\, d^i \to 0} \end{array} \right.$

$0 \to$

✓ $0 \to \mathrm{Im}\, d^i \to P^{i+1} \to \cdots$

✓ $0 \to \mathrm{Im}\, d^i \to P^{i+1} \to \cdots P^j \to \mathrm{Im}\, d^j \to 0$

$\Bigg\}$ $\underline{\mathrm{Hom}_{\mathcal{A}}(-, P(A))\text{-exact}}$

$\cancel{\mathcal{A}}$

$\to \cdots \qquad \cdots \to \mathrm{Hom}(P^{i+1}, P(A)) \underset{满}{\to} \mathrm{Hom}(\mathrm{Im}\, d^i, P(A)) \to 0$

$0 \to \mathrm{Im}\, d^i \to P^{i+1} \to \mathrm{Im}\, d^{i+1} \to 0 \quad \to \mathrm{Hom}-\mathrm{Ext}$

$\mathrm{Hom}(\_, P(A))$

$\mathrm{Hom}_{\mathcal{A}}(P^{i+1}, P(A)) \underset{满}{\to} \mathrm{Hom}_{\mathcal{A}}(\mathrm{Im}\, d^i, P(A)) \to \underline{\mathrm{Ext}^1_{\mathcal{A}}(\mathrm{Im}\, d^{i+1}, P(A))}$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\; \| \; 0$

$GP(A)$

(iv) $M \in GP(\mathcal{A})$. $L \in \mathcal{A}$ $\mathrm{Pdim}(L) = n < \infty$,

$\underline{\mathrm{Ext}^i_{\mathcal{A}}(M, L) = 0 \quad \forall i \geq 1}$

$\Gamma \; 0 \to P_n \cdots P_1 \xrightarrow{d^1} P_0 \to L \to 0$

$\underline{\mathrm{Ext}^i_{\mathcal{A}}(M, P_i) = 0}$ $(i \geq 1)$

$\downarrow$

$\mathrm{Ext}^i_{\mathcal{A}}(M, L) = 0$ $(\forall i \geq 1)$

$0 \to \ker d^0 \to P_0 \xrightarrow{d^0} L \to 0$

$\mathrm{Im}\, d^1$

$0 \to \mathrm{Hom}_{\mathcal{A}}(M, \ker d^0) \to \mathrm{Hom}_{\mathcal{A}}(M, P_0) \to \mathrm{Hom}(M, L) \to \mathrm{Ext}^1_{\mathcal{A}}(M, \ker d^0)$

$\mathrm{Im}\, P_{n-1}$

$\to \mathrm{Ext}^1_{\mathcal{A}}(M, P_0) \to \underline{\mathrm{Ext}^1_{\mathcal{A}}(M, L)}$

$\mathrm{Ext}^1_{\mathcal{A}}(M, P_0) = 0$

$\to \mathrm{Ext}_{\mathcal{A}}(M, \ker d^0) \to \cdots$

$\mathrm{Ext}_{\mathcal{A}}(M, \ker d^0) = 0 \, ?$

$0 \to P_n \to P_{n-1} \to \mathrm{Im}\, d_{n-1} \to 0$

$\underline{\mathrm{Ext}^1(M, \mathrm{Im}\, d_{n-1}) = 0}$ ✓

$\mathrm{Ext}(M, \ \ , \ p(A) = 0$

$M \in \mathrm{GP}(A)$

Def 7.14. $\mathcal{A}$: (enough proj obj) abelian category.

$\mathcal{X} \subseteq \mathcal{A}$: full subcategory.

(1) A $\mathcal{X}$-resolution of $\underline{M}$ in $\underline{\mathcal{X}} \subseteq \mathcal{A}$

$$\cdots \quad X_1 \to X_0 \to M \to 0$$

with $X_i \in \mathcal{X}$

(2) A $\mathcal{X}$-coresolution of M in $\mathcal{X}$

$$0 \to M \to X^0 \to X^1 \to \cdots$$

Def: $\mathcal{X}_{\omega} = \{ M \in \mathcal{A} \mid 0 \to M \to T^0 \xrightarrow{d^0} T^1 \xrightarrow{d^1} \cdots \}$

$\cap$

$\mathcal{A}$ with $T^i \in \omega$, $\underline{\ker d^i} \in {}^{\perp}\omega$ $\forall i \geq 0$, $(M \in {}^{\perp}\omega)$

$M = \ker d^0$

$\omega \subseteq \mathcal{A}$

full subcategory

aim $\underline{(\mathrm{GP}(\mathcal{A}), \mathcal{E})}$

Prop 7.15: $\mathcal{A}$: abelian category with enough projective objts.

(i) $M \in \mathcal{GP}(\mathcal{A}) \Leftrightarrow M \in {}^{\perp}\mathcal{P}(\mathcal{A})$ and $\exists$ a $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{P}(\mathcal{A}))$-exact co-resolution.

(ii) $M \in \mathcal{GP}(\mathcal{A}) \Leftrightarrow M \in \mathcal{X}_{\omega}$, $\omega = \mathcal{P}(\mathcal{A})$

i.e. $\mathcal{X}_{\mathcal{P}(\mathcal{A})} = \mathcal{GP}(\mathcal{A}) = \{ M \in \mathcal{A} \mid 0 \to M \to T^0 \to T^1 \to \cdots$, $\mathrm{Ker}\, d^i \in {}^{\perp}\mathcal{P}(\mathcal{A})$, $T^i \in \mathcal{P}(\mathcal{A}) \}$

(iii) [EJ, prop 10.2.3] $M \in \mathcal{GP}(\mathcal{A})$, $M \notin \mathcal{P}(\mathcal{A})$. then $\mathrm{pdim}(M) = \infty$

$\mathrm{Pdim}(M) = 0$ or ∞

Thus, $\dfrac{\mathrm{gl.dim}\, R < \infty}{}$ $\dfrac{\mathcal{GP}(R) = \mathcal{P}(R)}{\mathrm{GP}(R) = \mathcal{P}(R)}$

$(\mathrm{Pdim} R < \infty)$

(iv). $\forall$ Gorenstein projective module M.

$\exists \to \cdots F^{-1} \xrightarrow{d^{-1}} F^0 \xrightarrow{d^0} F^1 \to \cdots$ s.t $M \cong \mathrm{Im}\, d^{-1}$

with F^i are free modules.

⌈ 证 of (i). $\underline{0 \to M \to X^0 \to X^1 \to \cdots \text{ with } X^i \in \mathcal{P}(\mathcal{A})}$

$\exists$ $P^{\bullet} \to M \to 0$: proj. resolution of M

Then $P^{\bullet} \to M \to X^0 \to X^1 \to \cdots$ is a totally projective resolution

$\mathrm{Im}\, d^{-1} \cong M$. ⌋

⌈ (ii) follows from (i) ⌋

⌈ (iii). $\dfrac{\mathrm{Pdim}(M) = n \neq 0 < \infty}{M \in \mathcal{GP}(\mathcal{A})}$

By Fact 7.13. (iv) $M \in \mathcal{GP}(R) \Leftarrow \mathrm{Ext}^i_{\mathcal{A}}(M, \mathcal{P}(\mathcal{A})) = 0$ $(\forall i \geq 0)$

Let $0 \to P_n \xrightarrow{d_n} P_{n-1} \to \cdots P_0 \to M \to 0$ is a projective resolution of M.

$\operatorname{Im} d_i \to$

$\downarrow$ 反证 $\forall i$ $\quad 0\to P'_{n-1} \xrightarrow{t_{n-1}} \cdots \to M \to 0$

$0 \to Hom_A(P_0, P_n) \to Hom_A(P_1, P_n) \to \cdots$

$Hom(P_{n-1}, P_n) \xrightarrow[t_n^*]{满} Hom(P_n, P_n) \to 0$

$Ext^n_A(M, P_n) = 0 \quad S \overset{t_n^*}{\longmapsto} id_{P_n}$

$S t_n = id_{P_n}$

$\Rightarrow$ t_n is split monomorphism i.e $\exists\, P'_{n-1}$ s.t

$P_n = \operatorname{Im} t_n \oplus P'_{n-1}$

$0 \to P_n \to \cdots \quad M \to 0$

$\Rightarrow$

$0 \to P'_{n-1} \xrightarrow{t_{n-1}} P_{n-2} \to \cdots P_0 \to M \to 0$ $\Big\}$ $P\dim(M) = n$

$\Rightarrow P\dim(M) = 0$ or ∞

$M \in GR(A)$

(iv): $M \in GR(A) \Leftarrow Hom_A(-, P(A))$

$F^0 \to M \to 0$

$0 \to M \to P^0 \to P^1 \to \cdots$

Choose Q^0, Q^1, s.t $Q^0 \oplus P^0 = F^0$

$0 \to M \to P^0 \to P^1 \to P^2$

$\oplus$

$0 \to Q^0 \xrightarrow{id_{Q^0}} Q^0 \to 0$

$0 \to Q^1 \xrightarrow{id_{Q^1}} Q^1 \to 0$

$\oplus$

$0 \to Q^2 \to Q^2 \to 0$

$0 \to M \to P^0 \oplus Q^0 \to P^1 \oplus Q^0 \oplus Q^1$

$\to P^2 \oplus Q^1 \oplus Q^2 \cdots \to$

$Hom(-, P(R))$

Def 7.16. $\mathcal{A}$: abelian category with enough proj...

(1) $\mathcal{X} \subseteq \mathcal{A}$ is resolving subcategory, $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{X}$

$\mathcal{X}$ is closed under extension, kernel of kernel, direct summand of direct sum

(2) $\mathcal{W} \subseteq \mathcal{A}$ $\quad \mathrm{Ext}^i_{\mathcal{A}}(X,Y) = 0, \forall X, Y \in \mathcal{W}, \forall i \geq 1$

(3) dual to (1)

Thm 7.17. $\mathcal{A}$: abelian category with enough proj.

(1) $\mathcal{W} \subseteq \mathcal{A}$, satisfy (2) $\Rightarrow$ $\mathcal{X}_{\mathcal{W}} = \{ M \in \mathcal{A} \mid 0 \to T^0 \to T^1 \to \cdots, \ T^i \in \mathcal{W}, \ \mathrm{Ker}\, d^i \in {}^{\perp}\mathcal{W}, \ \mathrm{Hom}_{\mathcal{A}}(-, \mathcal{W})\text{-exact} \}$ is a resolving subcategory,

(2) $\mathcal{GP}(\mathcal{A})$ is a resolving category

(3) $(\mathcal{GP}(\mathcal{A}), \mathcal{S})$ is Frobenius Category

$\mathcal{S} = \{ 0 \to X \to Y \to Z \to 0 \mid X, Y, Z \in \mathcal{GP}(\mathcal{A}) \}$

① $\mathcal{GP}(\mathcal{A}) \subseteq \mathcal{A}$ closed under extension ☆ ☆ ✓

$\Rightarrow$ $\mathcal{GP}(\mathcal{A})$ is exact category

Since $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{GP}(\mathcal{A}) \subseteq {}^{\perp}\mathcal{P}(\mathcal{A})$

$\Rightarrow$ The projective objects are $\mathcal{S}$-injective objts in $\mathcal{GP}(\mathcal{R})$ and proj objts

$\Rightarrow$ $\mathcal{GP}(\mathcal{A})$ is Frobenius category

(i) $\mathcal{X}_{\mathcal{W}}$ (is closed under extension)

Let $0 \to X \to Y \to Z \to 0$ be a s.e.s.

$(X, Y, Z \in \mathcal{GP}(\mathcal{R})$

By 7.15 (i) (ii)

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "i_X"] & Y \arrow[r, "g"] \arrow[d, "\binom{a}{i_Z \circ g}"] & Z \arrow[r] \arrow[d, "i_Z"] & 0 \\
0 \arrow[r] & P \arrow[r] \arrow[d] & P \oplus Q \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
 & X' & Y' & Z' &
\end{tikzcd}$$

$X, Z \in \mathcal{X}_w$

$(w = P(A))$

$\mathcal{X}_w \overset{w=P(A)}{=} \mathcal{X}_{P(A)} = \mathcal{G}P(A)$

① $P, Q \in w\,(P(A))$

$\mathrm{Coker}(i_X) = X' \in \mathcal{X}_w$

$\mathrm{Coker}(i_Z) = Z' \in \mathcal{X}_w$

claim : $\mathrm{Ext}^1_A(Z, P) = 0$ ✓

$Z \in \mathcal{X}_w$

$\Rightarrow Z \in {}^{\perp}w \qquad P \in w$

$$0 \to \mathrm{Hom}_A(Z, P) \to \mathrm{Hom}_A(Y, P) \xrightarrow[\text{满}]{f^*} \mathrm{Hom}_A(X, P) \to \mathrm{Ext}^1_A(Z, P) \to \mathrm{Ext}^1_A(Y, P)$$

$$a \longmapsto a \circ f = i_X$$

By the snake lemma, $\exists$ cokernel of $\binom{a}{i_Z \circ g} = Y'$

Since X', Z' are Gorenstein projective

$X', Z' \in {}^{\perp}w \ ({}^{\perp}P(A)) \Rightarrow Y' \in {}^{\perp}P(A)$

$\Rightarrow$ we have s.e.s $0 \to X' \to Y' \to Z' \to 0$

$\vdots$

$\Rightarrow Y \in \mathcal{G}P(A)$

Closed kernel of epimorphism ... #