§5.2 Bialgebras

$\mathcal{C}$: finite ring cat , $F: \mathcal{C} \to \mathrm{Vec}$ fiber functor.

$$J_{X,Y} : F(X) \otimes F(Y) \longrightarrow F(X \otimes Y) \quad , \quad X, Y \in \mathcal{C}.$$

$H := \mathrm{End}(F)$.

$\mathrm{Vec} \boxtimes \mathrm{Vec} \xrightarrow{\sim} \mathrm{Vec}$

$F(X) \boxtimes F(Y) \longmapsto F(X) \otimes F(Y)$

$\Delta : H \to H \otimes H$ and $\varepsilon : H \to \mathbb{k}$.

$\alpha : \mathrm{End}(F) \otimes \mathrm{End}(F) \to \mathrm{End}(F \otimes F)$

$\alpha : \mathrm{End}(F) \otimes \mathrm{End}(F) \xrightarrow{\sim} \mathrm{End}(F \boxtimes F) \xrightarrow{\text{isom}} \mathrm{End}(F \otimes F)$. $\alpha(\eta_1 \otimes \eta_2)|_{F(X) \otimes F(Y)} := \eta_1|_{F(X)} \otimes \eta_2|_{F(Y)}$.

$\Delta(a) = \alpha^{-1}(\tilde{\Delta}(a))$. $\quad \tilde{\Delta}(a)_{X,Y} = J_{X,Y}^{-1}\, a_{X \otimes Y}\, J_{X,Y} : F(X) \otimes F(Y) \to F(X) \otimes F(Y)$

$: F(X) \otimes F(Y) \xrightarrow{J_{X,Y}} F(X \otimes Y) \xrightarrow{a_{X \otimes Y}} F(X \otimes Y) \xrightarrow{J_{X,Y}^{-1}} F(X) \otimes F(Y)$

$\varepsilon(a) = a_1 \in \mathbb{k} \quad : F(\mathbb{1}) \longrightarrow F(\mathbb{1})$.

Theorem 5.2.1 Let $H = \mathrm{End}(F)$

(i) The alg. H is a coalg with Δ and ε.

proof: (i) Coassociativity of Δ:

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta.$$

$\forall\, a \in \mathrm{End}(F)$, $\Delta(a) = \sum b_i \otimes c_i$. $\quad b_i, c_i \in \mathrm{End}(F)$.

$\Delta(a)_{(X,Y)} := \sum_i b_i|_X \otimes c_i|_Y : F(X) \otimes F(Y) \longrightarrow F(X) \otimes F(Y)$.

$\Delta(a) \in \mathrm{End}(F) \otimes \mathrm{End}(F)$, $\quad \tilde{\Delta}(a) \in \mathrm{End}(F \otimes F)$.

$\tilde{\Delta}(a)_{X,Y} = \Delta(a)_{(X,Y)}$

· Then $((\Delta \otimes \mathrm{id}) \circ \Delta(a))_{(X,Y,Z)} = (\sum \Delta(b_i) \otimes c_i)_{(X,Y,Z)}$

$= \sum \Delta(b_i)|_{(X,Y)} \otimes c_i|_Z = \sum J_{X,Y}^{-1}\, b_i|_{X \otimes Y}\, J_{X,Y} \otimes c_i|_Z$.

$= (J_{X,Y}^{-1} \otimes \mathrm{id}_{F(Z)}) \circ \underline{(\sum b_i|_{X \otimes Y} \otimes c_i|_Z)} \circ (J_{X,Y} \otimes \mathrm{id}_{F(Z)})$

$\Delta(a)_{X \otimes Y, Z}$

$= (J_{X,Y}^{-1} \otimes \mathrm{id}_{F(Z)}) \circ J_{X \otimes Y, Z}^{-1} \circ a_{(X \otimes Y) \otimes Z} \circ J_{X \otimes Y, Z} \circ (J_{X,Y} \otimes \mathrm{id}_{F(Z)})$ $\quad (*)$

$((\mathrm{id} \otimes \Delta) \circ \Delta(a))_{(X,Y,Z)}$

$= (\mathrm{id}_{F(X)} \otimes J_{Y,Z}^{-1}) \circ J_{X, Y \otimes Z}^{-1} \circ a_{X \otimes (Y \otimes Z)} \circ J_{X, Y \otimes Z} \circ (\mathrm{id}_{F(X)} \otimes J_{Y,Z})$ $\quad (**)$.

· Let $\Phi_{X,Y,Z}$: associat. isom. in $\mathcal{C}$.

By nat. of $a \in \mathrm{End}(F)$, $\Rightarrow$ $F(\Phi_{X,Y,Z}) \circ a_{X \otimes Y \otimes Z} = a_{X \otimes (Y \otimes Z)} \circ F(\Phi_{X,Y,Z})$.

$$\Rightarrow (**) = \underbrace{(Id_{F(X)} \otimes J^{-1}_{Y,Z}) \circ J^{-1}_{X,Y\otimes Z} F(\Phi_{X,Y,Z})}_{①} \circ a_{(X\otimes Y)\otimes Z} \underbrace{F(\Phi^{-1}) J_{X,Y\otimes Z} \circ (Id_{F(X)} \otimes J_{Y,Z})}_{②}$$

By (2.23) P30.

$$\begin{tikzcd}
(F(X)\otimes F(Y))\otimes F(Z) \arrow[r, "\Phi'_{F(X),F(Y),F(Z)}"] \arrow[d, "J_{X,Y}\otimes Id"] & F(X)\otimes(F(Y)\otimes F(Z)) \arrow[d, "Id_{F(X)}\otimes J_{Y,Z}"] \\
F(X\otimes Y)\otimes F(Z) \arrow[d, "J_{X\otimes Y,Z}"] & F(X)\otimes F(Y\otimes Z) \arrow[d, "J_{X,Y\otimes Z}"] \\
F((X\otimes Y)\otimes Z) \arrow[r, "F(\Phi)"] & F(X\otimes(Y\otimes Z))
\end{tikzcd}$$

$$\Rightarrow (**) = \Phi'_{F(X),F(Y),F(Z)} \circ ((\Delta\otimes Id)\circ\Delta(a))_{(X,Y,Z)} \circ \Phi'^{-1}_{F(X),F(Y),F(Z)}$$

$$= (A(\Delta\otimes Id)\circ\Delta(a))_{(X,Y,Z)}$$

$$\begin{tikzcd}
(F(X)\otimes F(Y))\otimes F(Z) \arrow[r, "\Phi'_{F(X),F(Y),F(Z)}"] \arrow[d] & F(X)\otimes(F(Y)\otimes F(Z)) \arrow[d] \\
(F(X)\otimes F(Y)\otimes)F(Z) \arrow[r, "\Phi'_{[illegible]}"] & F(X)\otimes(F(Y)\otimes F(Z))
\end{tikzcd}$$

$$\begin{tikzcd}
End(F) \arrow[r] \arrow[dr] & End(F)\otimes End(F) \arrow[r] & (End(F)\otimes End(F))\otimes End(F) \arrow[d, "A_{End F, End F, End F}"] \\
& End(F)\otimes End(F) \arrow[r] & End(F)\otimes(End(F)\otimes End(F))
\end{tikzcd}$$

· Counity property: $(Id\otimes\varepsilon)\circ\Delta = Id_{End(F)} = (\varepsilon\otimes Id)\circ\Delta$.

$$\begin{tikzcd}
k\otimes End(F) & End(F) \arrow[l] \arrow[d, "\Delta"] \\
& End(F)\otimes End(F) \arrow[ul, "(\varepsilon\otimes Id)"]
\end{tikzcd}$$

$Id_{F(1)}\otimes a \longleftarrow\!\shortmid\ a$

$k\otimes End(F) = Hom(F(1), F(1))$ [illegible]

· $(\varepsilon\otimes Id)\circ\Delta(a) = \sum_i \varepsilon(b_i)\otimes c_i$

$= \sum b_i|_1 \otimes c_i$

we want: $\sum b_i|_1 \otimes c_i = Id_{F(1)}\otimes a$.

i.e. $Id_{F(1)}\otimes a_X \overset{?}{=} \sum_i b_i|_1 \otimes c_i|_X$

$= \Delta(a)_{1,X}$

$= \tilde{\Delta}(a)_{1,X}$

$= J^{-1}_{1,X}\circ a_{1\otimes X}\circ J_{1,X}$

By P31 (2.25)

$$\begin{tikzcd}
k\otimes F(X) \arrow[r, "l'_{F(X)}"] \arrow[d, "\varphi\otimes Id_{F(X)}"] & F(X) \arrow[d, "F(l_X)^{-1}"] \\
F(1)\otimes F(X) \arrow[r, "J_{1,X}"] & F(1\otimes X)
\end{tikzcd}$$

Then
$$\begin{tikzcd}
F(1)\otimes F(X) \arrow[rrr, "J_{1,X}"] \arrow[ddd, "id_{F(1)}\otimes a_X"'] & & & F(1\otimes X) \arrow[ddd, "a_{1\otimes X}"] \\
& 1k\otimes F(X) \arrow[ul, "y\otimes id_{F(X)}"'] \arrow[r, "l'_{F(X)}"] \arrow[d, "id\otimes a_X"'] \arrow[ull, phantom, "\cong \text{ by bifunctor}"] \arrow[ur, phantom, "\cong\ (2.25)"] & F(X) \arrow[ur, "F(l_X)^{-1}"] \arrow[d, "a_X"] \arrow[urr, phantom, "\cong \text{ by nat. of } a"] & \\
& 1k\otimes F(X) \arrow[r, "l'_{F(X)}"] \arrow[dl] \arrow[dr, phantom, "\cong\ (2.25)"] \arrow[ur, phantom, "\cong \text{ by nat. of } l'"] & F(X) \arrow[dr] & \\
F(1)\otimes F(X) \arrow[rrr, "J_{1,X}"'] & & & F(1\otimes X)
\end{tikzcd}$$

#

(ii) Δ, ε are unital algebra homomorphisms.

proof: $\Delta(id_F)_{X,Y} = \tilde{\Delta}(id_F)_{X,Y} = J_{X,Y}^{-1}\circ id_{F(X\otimes Y)}\circ J_{X,Y} = id_{F(X)\otimes F(Y)}$.

$\Delta(id_F)_{(X,Y)} = \alpha^{-1}(id_{F(X)\otimes F(Y)}) = id_{F(X)}\otimes id_{F(Y)} = (id_F)|_X \otimes (id_F)|_Y$.

$\Delta(y\nu) \overset{?}{=} \Delta(y)\Delta(\nu)$.

since $\tilde{\Delta}(y\nu)_{X,Y} = J_{X,Y}^{-1}\, y_{X\otimes Y}\circ \nu_{X\otimes Y}\circ J_{X,Y} = \tilde{\Delta}(y)_{X,Y}\circ\tilde{\Delta}(\nu)_{X,Y}$.

$\Rightarrow \Delta$ ....

$\varepsilon(id_F)$

$\varepsilon(y\nu)$

#

Th. 5.2.b. (Reconstruction theorem)

The assignments

$(\mathcal{C}, F) \xmapsto{\ \theta\ } H = End(F)$. $\quad H \xmapsto{\ \xi\ } (Rep(H), Forget)$

are mutually inverse bijections between (1) and (2).

(1) finite ring cat. $\mathcal{C}$ with fiber functor $F: \mathcal{C} \to Vec$,

up to tensor eq. and ~~isom of tensor functors~~

如果去这个条件，那么(2)中is bialg是tensor eq.

(2) isomorphism classes of f.d. bialg. H over k!

proof: (1) $End(F)$ is f.d.

Pro. proj. generator P of $\mathcal{C}$.

Consider $End(F) \xhookrightarrow{\ i\ } Hom_{Vec}(F(P), F(P))$. $\quad a \longmapsto a_P$.

for $\forall X \in \mathcal{C}$, $\exists\ P^{\oplus I} \xrightarrow{\pi} X$ for some index set I.

$$
\begin{tikzcd}
0 \arrow[r] & F(\ker\pi) \arrow[r] \arrow[d, "a_{\ker\pi}"] \arrow[dr, phantom, "\wr \text{ by nat of } a."] & F(P^{\oplus I}) \arrow[r, "F(\pi)"] \arrow[d, "(a_P)^{\oplus I}"] \arrow[dr, phantom, "\wr"] & F(X) \arrow[r] \arrow[d, dashed, "\exists!\, \beta\ \ a_X"] & 0 \\
0 \arrow[r] & F(\ker\pi) \arrow[r] & F(P^{\oplus I}) \arrow[r, "F(\pi)"] & F(X) \arrow[r] & 0
\end{tikzcd}
$$

· (2) mutually inverse.

· $O\xi(H) = \theta(\mathrm{Rep}(H), \mathrm{Forget}) = \mathrm{End}(\mathrm{Forget}) \overset{\text{bialg.}}{\cong} H$

$\mathrm{End}(\mathrm{Forget}) \longrightarrow H.$

$a \overset{\alpha}{\longmapsto} a_H(1_H)$

$a' \overset{\beta}{\longleftarrow\!\!\shortmid} h$

$a_H : \mathrm{Forg}(H) \longrightarrow \mathrm{Forg}(H).$

s.t. for $\forall$ H-module V,

$a'_V(v) = h.v$ $\quad$ a' is a nat. transf.

$$
\begin{tikzcd}
v \arrow[rrr, mapsto] & & & h.v \\
\mathrm{Forg}({}_H V) \arrow[rrr, "a'_V"] \arrow[d, "\mathrm{Forg}(\phi)"] \arrow[rrrd, phantom, "\wr"] & & & \mathrm{Forg}({}_H V) \arrow[d, "\mathrm{Forg}(\phi)"] \\
\mathrm{Forg}({}_H W) \arrow[rrr, "a'_W"] & & & \mathrm{Forg}({}_H W) \\
\phi(v) \arrow[rrr, mapsto] & & & h\phi(v) = \phi(h.v)
\end{tikzcd}
$$

① $\alpha\beta(h) = a'_H(1) = h.1 = h.$

② $\beta\alpha(a) = \beta(a_H(1)) = \beta(h') = a'$

$a_H(1) = h'$

for any ${}_H V$, $\forall v \in {}_H V$, define $\phi_v : {}_H H \longrightarrow {}_H V$, $h \longmapsto h.v$.

$a_V(v) = a_V(\mathrm{Forg}(\phi_v)(1))$

$= (a_v \circ \mathrm{Forg}(\phi_v))(1) = (\mathrm{Forg}(\phi_v) \circ a_H)(1)$

$= \mathrm{Forg}(\phi_v)(a_H(1)) = h'.v$ $\quad$ $(= a_H(1).v)$

$$
\begin{tikzcd}
\mathrm{Forg}(H) \arrow[r, "a_H"] \arrow[d, "\mathrm{Forg}(\phi_v)"] \arrow[dr, phantom, "\wr"] & \mathrm{Forg}(H) \arrow[d, "\mathrm{Forg}(\phi_v)"] \\
\mathrm{Forg}(V) \arrow[r, "a_v"] & \mathrm{Forg}(V)
\end{tikzcd}
$$

$\Rightarrow a' = a.$

③ α is a bialg. hom.

α is a coalg. hom.

(1) $\varepsilon_H \circ \alpha = \varepsilon_{\mathrm{End}(\mathrm{Forg})} \iff \varepsilon_H \circ \alpha(a) = \varepsilon_{\mathrm{End}}(a). \iff \varepsilon_H(a_H(1)) = a_k.$

$$
\begin{tikzcd}
\mathrm{Forg}(H) \arrow[r, "a_H"] \arrow[d] \arrow[dr, phantom, "\wr"] & \mathrm{Forg}(H) \arrow[d] \\
k \arrow[r, "a_k"] & k
\end{tikzcd}
$$

$$a \overset{\alpha}{\longmapsto} a_H(1_H)$$

(2) $\Delta_H \circ \alpha = (\alpha \otimes \alpha)\circ \Delta_{End}$. $\Leftrightarrow$ $\Delta_H \circ \alpha(a) = (\alpha\otimes\alpha)\circ\Delta_{End}(a)$.

$\Delta_{End}(a)_{(X,Y)} = \tilde{\alpha}(a)_{X,Y} = a_{X\otimes Y}$

$$\begin{tikzcd}
Forg(H) \arrow[r, "a_H"] \arrow[d, "Forg(\Delta)"'] \arrow[dr, phantom, "\circlearrowright"] & Forg(H) \arrow[d, "Forg(\Delta)"] \\
Forg(H\otimes H) \arrow[r, "a_{H\otimes H}"] & Forg(H\otimes H)
\end{tikzcd}$$

$\Delta_H(a_H(1)) = a_{H\otimes H}(1\otimes 1)$

$= (\alpha\otimes\alpha)\Delta_{End}(a)$.

$\Delta_{End}(a) = \sum a_{(1)}\otimes a_{(2)}$

$(\alpha\otimes\alpha)(\ ) = a_{(1)H}(1)\otimes a_{(2)H}(1)$.

$(\Delta_{End}(a)_{(H,H)})(1\otimes 1)$

$\cdot$ $\xi\theta((\mathcal{C},F)) = \xi(End(F)) = (Rep(End(F)), Forget) \overset{\otimes}{\simeq} (\mathcal{C}, F)$

Let $\tilde{F} : \mathcal{C} \longrightarrow Rep(End(F))$.

$X \longmapsto (F(X), \rho_{F(X)})$

$Hom_{\mathcal{C}}(X,Y) \ni f \longmapsto F(f) \in Hom_{Rep(End(F))}(F(X), F(Y))$

$F(X) \in Vec$,

$End(F) \xrightarrow{\rho_{F(X)}} Hom_k(F(X), F(X))$

$id_F \longmapsto id_{F(X)}$

$a \longmapsto a_X$.

$\Rightarrow$ $\rho_{F(X)}$ is a rep.

① $\tilde{F}$ is a tensor functor

(1) faithful, exact.

(2) $\tilde{F}(X)\otimes\tilde{F}(Y) \cong \tilde{F}(X\otimes Y)$.

$\tilde{F}(1_{\mathcal{C}}) \cong k$. by $F(1_{\mathcal{C}}) = k$.

i.e. $(F(X), \rho_{F(X)}) \otimes (F(Y), \rho_{F(Y)}) \overset{J_{X,Y}}{\cong} (F(X\otimes Y), \rho_{F(X\otimes Y)})$

$\quad\quad ||$

$(F(X)\otimes F(Y), \rho_{F(X)\otimes F(Y)})$

$\forall a \in End(F)$, $\rho_{F(X)\otimes F(Y)}(a) = (\rho_{F(X)}\otimes\rho_{F(Y)})(\Delta(a)) = \rho_{F(X)}(a_{(1)}) \otimes \rho_{F(Y)}(a_{(2)})$

$= a_{(1)}|_X \otimes a_{(2)}|_Y = J_{X,Y}^{-1} \circ \underbrace{a_{X\otimes Y}}_{= \rho_{F(X\otimes Y)}(a)} \circ J_{X,Y}$.

$\rho_{F(X\otimes Y)}(a) = a_{X\otimes Y}$

(3) The monoidal structure given by F is a fiber functor

⊗ equi.

Pf. Cor. 1.8.11 $F \cong \mathrm{Hom}_{\mathcal{C}}(\vartheta, -)$ for $\vartheta \in \mathcal{C}$.

$\Rightarrow F \cong \mathrm{Hom}_{\mathcal{C}}(P, -)$, P is proj. gen. of $\mathcal{C}$.

$\Rightarrow \mathrm{Hom}_{\mathcal{C}}(P, -) : \mathcal{C} \longrightarrow \mathrm{Rep}(\mathrm{End}(F))$

$\Rightarrow \widetilde{F}$ is an eq.

$\Rightarrow \mathrm{Rep}(\mathrm{End}(F)) \overset{\otimes}{\simeq} \mathcal{C}$.

· Forget $\cong F$ as tensor functor.

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "\widetilde{F}"'] & \arrow[d, phantom, "2."] & \mathrm{Vec} \\
& \mathrm{Rep}(\mathrm{End}(F)) \arrow[ur, "\text{forget}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X) \otimes F(Y) \arrow[r, "J_{X,Y}"] \arrow[d, "\eta_X \otimes \eta_Y"', "\mathrm{id}_{F(X)} \otimes \mathrm{id}_{F(Y)}"] & F(X \otimes Y) \arrow[d, "\eta_{X \otimes Y}"] \\
\widetilde{F}(X) \otimes \widetilde{F}(Y) \arrow[r, "\widetilde{J}_{X,Y}"] & \widetilde{F}(X \otimes Y)
\end{tikzcd}$$

#

J is invertible $J \in H \otimes H$

Left twist

$(J \otimes 1)(\Delta \otimes \mathrm{id})(J) \in H \otimes H \otimes H$. $H^J = (H, m, \mu, \Delta^J, \varepsilon)$

$= (1 \otimes J) \cdot (\mathrm{id} \otimes \Delta)(J)$ $\qquad \Delta^J(h) = J \cdot \Delta(h) \cdot J^{-1}$

· Proposition. 设 H 是 f.d. bialg， $\mathcal{J}\in H\otimes H$ 是(左) twist.

记 $F: \mathrm{Rep}(H)\to \mathrm{Vec}$ 和 $F': \mathrm{Rep}(H^{\mathcal{J}})\to\mathrm{Vec}$ 是忘却(fiber)函子.

若张量范畴等价 $(\mathcal{E},\mathcal{J}): \mathrm{Rep}(H)\to\mathrm{Rep}(H^{\mathcal{J}})$ 满足：

$F'\circ\mathcal{E}$ 与 F 作为张量函子自然同构，则 $H^{\mathcal{J}}\cong H$ as bialgs.

· Remark. $(\mathcal{E},\mathcal{J})$ 的定义如下：

$$\mathcal{E}:\ V\mapsto V\ (\text{作为 } H\text{-mod}),\quad \begin{array}{c} V\\ f\big\downarrow\\ W\end{array}\ \mapsto\ \begin{array}{c} V\\ f\big\downarrow\\ W\end{array}.$$

monoidal stru $\mathcal{J}: \mathcal{E}(V)\otimes\mathcal{E}(W)\xrightarrow{\cong}\mathcal{E}(V\otimes W)$, $v\otimes w\mapsto \mathcal{J}(v\otimes w)$.

由左 twist 的定义可知(左乘) $\mathcal{J}$ 是 $\mathrm{Rep}(H^{\mathcal{J}})$ 中的 morphism.

· Proof. 设自然同构 $\eta\in\mathrm{Nat}(F'\circ\mathcal{E},F)$ 满足

$\forall\ V,W\in\mathrm{Rep}(H)$, Vec 中的下图交换：

$$\begin{tikzcd}
\mathcal{E}(\cancel{V}H)\otimes\mathcal{E}(\cancel{W}H) \arrow[r, "\mathcal{J}"] \arrow[d, "\eta_V\otimes\eta_W"'] & \mathcal{E}(\cancel{V}H\otimes\cancel{W}H) \arrow[d, "\eta_{V\otimes W}"] \\
\cancel{V}H\otimes\cancel{W}H \arrow[r, "\mathrm{id}", "(\text{作为}F\text{的monoidal stru})"'] & \cancel{V}H\otimes\cancel{W}H
\end{tikzcd}\qquad (\star)$$

$$\begin{tikzcd}
1\otimes 1 \arrow[r, mapsto] \arrow[d, mapsto] & \mathcal{J} \arrow[d, mapsto] \\
h\otimes h \arrow[r, mapsto] & h\otimes h \quad \Delta(h)\mathcal{J}
\end{tikzcd}$$

对象　态　monoidal stru

$$\begin{tikzcd}
\mathrm{Rep}(H) \arrow[rr, "F=(\text{“Id”},\,\mathrm{id},\,\mathrm{id}_)", "\text{忘却}"'] \arrow[dr, "\mathcal{E}=(\mathrm{Id},\,\mathrm{id},\,\mathcal{J})"'] & & \mathrm{Vec} \\
& \mathrm{Rep}(H^{\mathcal{J}}) \arrow[ur, "F'=(\text{“Id”},\,\mathrm{id},\,\mathrm{id}_)"'] &
\end{tikzcd}$$

对象　态　monoidal stru

· 事实上，由于自然的线性同构 η_V 与所有 H-mod map 交换.

即 $\forall\ \begin{array}{c} V\\ f\big\downarrow\\ W\end{array}$ in $\mathrm{Rep}(H)$,
$\begin{tikzcd}
\mathcal{E}(V) \arrow[r, "f"] \arrow[d, "\eta_V"'] \arrow[dr, phantom, "?"] & \mathcal{E}(W) \arrow[d, "\eta_W"] \\
V \arrow[r, "f"'] & W
\end{tikzcd}$，故 $\exists$ 可逆元 $h\in H$, s.t. $\forall\ V\in\mathrm{Rep}(H)$,

$\eta_V: v\mapsto h\cdot v$ (h 与 V 无关).

(理由：先判断 $V=W={}_HH$ 以及自由模的情形，而对一般的 H-mods，
考虑 V 和 W 的 free resolution $\Leftarrow$ η 的自然性)

因此，对 $1\otimes 1\in\mathcal{E}(V)\otimes\mathcal{E}(W)$ 应用 $(\star)$ 的交换性，可得

$\Delta(h)\mathcal{J}=h\otimes h$， 即 $\mathcal{J}=\Delta(h^{-1})(h\otimes h)$.

这说明：左 twists $\mathcal{J}$ 和 $1\otimes 1$ 在 [EGNO15, P115 第三段] 的意义下等价，<u>从而</u> $H^{\mathcal{J}}\cong H^{1\otimes 1}=H$ (as bialgs).

留到 §5.14 解决吧～