凸优化和单调变分不等式的收缩算法

第四讲: 线性单调变分不等式的一对孪生投影收缩算法的收敛速率

Convergence rate of the twin PC methods for monotone linear variational inequalities

南京大学数学系　　何炳生
hebma@nju.edu.cn

The context of this lecture is based on the publication [2]

For solving the monotone linear variational inequalities, in 1994, we have proposed a class of projection and contraction methods [6, 7]. Before 2015, only convergence results are available to these methods. In this article, we prove the convergence rate of such projection and contraction methods.

1 Introduction

Let Ω be a closed convex subset of $\Re^n$, $M \in \Re^{n\times n}$ and $q \in \Re^n$. The linear variational inequality problem, denoted by LVI(Ω, M, q), is to find a vector $u^* \in \Omega$ such that

$$\text{LVI}(\Omega, M, q) \qquad (u - u^*)^T(Mu^* + q) \geq 0, \qquad \forall u \in \Omega. \tag{1.1}$$

Throughout this paper we assume that the matrix M is positive semi-definite (but not necessary symmetric), *i.e.*,

$$(u - v)^T M(u - v) \geq 0, \quad \forall\, u, v \in \Re^n.$$

Moreover, we assume that the solution set of LVI(Ω, M, q), denoted by Ω^*, is nonempty. The nonempty assumption of the solution set, together with the positivity assumption of the matrix M, implies that Ω^* is closed and convex (see pp. 158 in [3]).

For any $\beta > 0$, the solution set of the following linear variational inequality

$$\text{LVI}(\Omega, M, q) \qquad (u - u^*)^T \beta(Mu^* + q) \geq 0, \quad \forall u \in \Omega.$$

is coincides with the one of (1.1). It is well known that,

$$u^* \text{ is a solution of LVI}(\Omega, M, q) \iff u^* = P_\Omega[u^* - \beta(Mu^* + q)], \tag{1.2}$$

where β is any positive scalar and $P_\Omega(\cdot)$ denotes the projection onto Ω with respect to the Euclidean norm, *i.e.*,

$$P_\Omega(v) = \text{argmin}\{\frac{1}{2}\|u - v\|^2 \mid u \in \Omega\}.$$

For convergence rate investigation, without loss of generality, we fix $\beta = 1$. Throughout this article, we assume that the projection on Ω in the Euclidean-norm has a closed form and it is easy to be carried out. The most important property of the projection mapping is

$$(v - P_\Omega(v))^T (u - P_\Omega(v)) \leq 0, \quad \forall\, v \in \Re^n, \forall\, u \in \Omega. \tag{1.3}$$

A pair of twin projection and contraction methods

For simpleness of the theoretical analysis, without loss the generality, we let $\beta \equiv 1$. For given $u^k \in \Re^n$, the predictor is given by

$$\tilde{u}^k = P_\Omega[u^k - (Mu^k + q)]. \tag{1.4}$$

According to (1.2), u is a solution of LVI(Ω, M, q) if and only if $u^k = \tilde{u}^k$. In [6, 7], we define $e(u^k) = u^k - \tilde{u}^k$ and use $\|e(u^k)\|$ to measure how much u^k fails to be a solution of the linear variational inequality LVI(Ω, M, q).

By using (1.4), the projection and contraction method [6] can be described as

$$\text{(PCM-I)} \qquad u^{k+1} = u^k - \gamma\alpha_k^*(I + M^T)(u^k - \tilde{u}^k), \tag{1.5}$$

where $\gamma \in (0, 2)$ and

$$\alpha_k^* = \frac{\|u^k - \tilde{u}^k\|^2}{\|(I + M^T)(u^k - \tilde{u}^k)\|^2}. \tag{1.6}$$

It was proved that (see Lecture 2 or 3 of this seriers)

$$\|u^{k+1} - u^*\|^2 \le \|u^k - u^*\|^2 - q_k(\gamma), \tag{1.7a}$$

where

$$q_k(\gamma) = \gamma(2-\gamma)\alpha_k^*\|u^k - \tilde{u}^k\|^2. \tag{1.7b}$$

In an earlier preprint [5], it was pointed out that the method

$$\text{(PCM-II)} \qquad u^{k+1} = P_\Omega\{u^k - \gamma\alpha_k^*[(Mu^k + q) + M^T(u^k - \tilde{u}^k)]\}, \tag{1.8}$$

has the same contractive property as (1.7), the result was reported in [7] and detailed proof can be found in [8]. Note that the same step size length is used in (1.5) and (1.8) even if the search directions are different. Thus, this pair of methods are called **a pair of twin projection and contraction methods for linear variational inequalities**.

The projection and contraction methods make one (or two) projection(s) on Ω at each iteration, and the distance of the iterates to the solution set monotonically converges to zero. According to the terminology in [1], these methods belong to the class of Fejér contraction methods. Stimulated by the complexity analysis [14], this article shows the $O(1/t)$ convergence rate of the projection and contraction methods for monotone linear variational inequalities.

2 Preliminaries

Recall that Ω^* can be characterized as (see (2.3.2) in pp. 159 of [3])

$$\Omega^* = \bigcap_{u\in\Omega} \big\{\tilde{u} \in \Omega : (u-\tilde{u})^T (Mu+q) \geq 0\big\}.$$

This implies that $\tilde{u} \in \Omega$ is an approximate solution of LVI(Ω, M, q) with the accuracy ϵ if it satisfies

$$\tilde{u} \in \Omega \quad \text{and} \quad \inf_{u\in\mathcal{D}(\tilde{u})} \big\{(u-\tilde{u})^T (Mu+q)\big\} \geq -\epsilon,$$

where

$$\mathcal{D}(\tilde{u}) = \{u \in \Omega \,|\, \|u-\tilde{u}\| \leq 1\}.$$

In this article, we show that, for given $\epsilon > 0$, in $O(1/\epsilon)$ iterations the pair of twin projection and contraction methods can offer a $\tilde{u}$ such that

$$\tilde{u} \in \Omega \quad \text{and} \quad \sup_{u\in\mathcal{D}(\tilde{u})} \big\{(\tilde{u}-u)^T (Mu+q)\big\} \leq \epsilon. \tag{2.1}$$

Since the term $(\tilde{u}-u)^T (Mu+q)$ is essential in the approximate criterion (2.1) and the search direction in PCM-II is $[(Mu^k+q) + M^T(u^k-\tilde{u}^k)]$, now, we establish a relation between

$$(u-\tilde{u}^k)^T(Mu+q) \quad \text{and} \quad (u-\tilde{u}^k)^T[(Mu^k+q)+M^T(u^k-\tilde{u}^k)].$$

Lemma 2.1 *For given* u^k, $\tilde{u}^k$ *is given by* (1.4)*. Then for any* u*, the inequality*

$$\begin{aligned} &(u-\tilde{u}^k)^T(Mu+q) \\ &\quad \geq \quad (u-\tilde{u}^k)^T[(Mu^k+q)+M^T(u^k-\tilde{u}^k)]-\frac{1}{2}\|u^k-\tilde{u}^k\|_D^2 \end{aligned} \tag{2.2}$$

is always true, where $D=M^T+M$ *is symmetric and positive semi-definite.*

Proof. By calculating the difference and using the Cauchy-Schwarz Inequality, we get

$$\begin{aligned} &(u-\tilde{u}^k)^T(Mu+q)-(u-\tilde{u}^k)^T[(Mu^k+q)+M^T(u^k-\tilde{u}^k)] \\ &= \quad (u-\tilde{u}^k)^T\{M(u-u^k)-M^T(u^k-\tilde{u}^k)\} \\ &= \quad (u-\tilde{u}^k)^T\{M(u-\tilde{u}^k)-(M+M^T)(u^k-\tilde{u}^k)\} \\ &= \quad \frac{1}{2}\|u-\tilde{u}^k\|_D^2-(u-\tilde{u}^k)^TD(u^k-\tilde{u}^k) \\ &\geq \quad -\frac{1}{2}\|u^k-\tilde{u}^k\|_D^2. \end{aligned}$$

Consequently, we get (2.2) and the assertion is proved. $\square$

3 Important theorems for the twin PC methods

Now, we prove the same key inequality of the twin PC Methods for the complexity analysis. The assertion (2.2) is useful for the proofs of the following theorems, even though the proof is relative simple.

Set $v = u^k - (Mu^k + q)$ in (1.3), since $\tilde{u}^k = P_\Omega[u^k - (Mu^k + q)]$, it follows from (1.3) that

$$\{[u^k - (Mu^k + q)] - \tilde{u}^k\}^T (u - \tilde{u}^k) \le 0, \quad \forall u \in \Omega$$

and consequently,

$$(u - \tilde{u}^k)^T (Mu^k + q) \ge (u - \tilde{u}^k)^T (u^k - \tilde{u}^k), \quad \forall\, u \in \Omega.$$

Adding the term $(u - \tilde{u}^k)^T M^T (u^k - \tilde{u}^k)$ to the both sides of the above inequality, we obtain

$$\begin{aligned} & (u - \tilde{u}^k)^T \{(Mu^k + q) + M^T (u^k - \tilde{u}^k)\} \\ \ge \quad & (u - \tilde{u}^k)^T (I + M^T)(u^k - \tilde{u}^k), \ \forall\, u \in \Omega. \end{aligned} \tag{3.1}$$

Main theorem for PC-Method-I

Theorem 3.1 *For given* $u^k \in \Re^n$, *let* $\tilde{u}^k$ *be defined by* (1.4) *and the new iterate* u^{k+1} *be generated by PCM-I* (1.5) *with any* $\gamma > 0$. *Then we have*

$$\begin{aligned} & \gamma \alpha_k^* (u - \tilde{u}^k)^T (Mu + q) \\ \geq \quad & \frac{1}{2}\big(\|u - u^{k+1}\|^2 - \|u - u^k\|^2\big) + \frac{1}{2} q_k(\gamma), \;\; \forall\, u \in \Omega, \end{aligned} \tag{3.2}$$

where $q_k(\gamma)$ *is defined in* (1.7b).

Proof. Substituting (3.1) in the right hand side of (2.2), we get

$$\begin{aligned} & (u - \tilde{u}^k)^T (Mu + q) \\ \geq \quad & (u - \tilde{u}^k)^T (I + M^T)(u^k - \tilde{u}^k) - \frac{1}{2}\|u^k - \tilde{u}^k\|_D^2. \end{aligned}$$

Thus, we have

$$\begin{aligned} & \gamma \alpha_k^* (u - \tilde{u}^k)^T (Mu + q) \\ \geq \quad & (u - \tilde{u}^k)^T \gamma \alpha_k^* (I + M^T)(u^k - \tilde{u}^k) - \frac{\gamma \alpha_k^*}{2}\|u^k - \tilde{u}^k\|_D^2, \end{aligned}$$

Because $\gamma\alpha_k^*(I+M^T)(u^k-\tilde{u}^k)=u^k-u^{k+1}$ (see (1.5)), we get

$$\begin{aligned}&\gamma\alpha_k^*(u-\tilde{u}^k)^T(Mu+q)\\ &\quad\geq\quad (u-\tilde{u}^k)^T(u^k-u^{k+1})-\frac{\gamma\alpha_k^*}{2}\|u^k-\tilde{u}^k\|_D^2. \qquad (3.3)\end{aligned}$$

To the crossed term in the right hand side of (3.3), namely $(u-\tilde{u}^k)^T(u^k-u^{k+1})$, using an identity

$$(a-b)^T(c-d)=\frac{1}{2}\big(\|a-d\|^2-\|a-c\|^2\big)+\frac{1}{2}\big(\|c-b\|^2-\|d-b\|^2\big),$$

we obtain

$$\begin{aligned}(u-\tilde{u}^k)^T(u^k-u^{k+1}) \quad &= \quad \frac{1}{2}\big(\|u-u^{k+1}\|^2-\|u-u^k\|^2\big)\\ &\quad +\frac{1}{2}\big(\|u^k-\tilde{u}^k\|^2-\|u^{k+1}-\tilde{u}^k\|^2\big). \qquad (3.4)\end{aligned}$$

By using $u^{k+1} = u^k - \gamma\alpha_k^*(I + M^T)(u^k - \tilde{u}^k)$ and (1.6), we get

$$\begin{aligned}
&\|u^k - \tilde{u}^k\|^2 - \|u^{k+1} - \tilde{u}^k\|^2 \\
&\quad = \quad \|u^k - \tilde{u}^k\|^2 - \|(u^k - \tilde{u}^k) - \gamma\alpha_k^*(I + M^T)(u^k - \tilde{u}^k)\|^2 \\
&\quad = \quad 2\gamma\alpha_k^*(u^k - \tilde{u}^k)^T(I + M^T)(u^k - \tilde{u}^k) \\
&\qquad\qquad -\gamma^2\alpha_k^*\big(\alpha_k^*\|(I + M^T)(u^k - \tilde{u}^k)\|^2\big) \\
&\quad = \quad 2\gamma\alpha_k^*\|u^k - \tilde{u}^k\|^2 + \gamma\alpha_k^*\|u^k - \tilde{u}^k\|_D^2 - \gamma^2\alpha_k^*\|u^k - \tilde{u}^k\|^2 \\
&\quad = \quad \gamma(2 - \gamma)\alpha_k^*\|u^k - \tilde{u}^k\|^2 + \gamma\alpha_k^*\|u^k - \tilde{u}^k\|_D^2.
\end{aligned}$$

Substituting it in the right hand side of (3.4) and using the definition of $q_k(\gamma)$, we obtain

$$\begin{aligned}
(u - \tilde{u}^k)^T(u^k - u^{k+1}) \quad = \quad & \frac{1}{2}\big(\|u - u^{k+1}\|^2 - \|u - u^k\|^2\big) + \frac{1}{2}q_k(\gamma) \\
& + \frac{\gamma\alpha_k^*}{2}\|u^k - \tilde{u}^k\|_D^2. \qquad (3.5)
\end{aligned}$$

Adding (3.3) and (3.5), the theorem is proved. $\square$

Main theorem for PCM-II

The both sequences $\{\tilde{u}^k\}$ and $\{u^k\}$ in the PC method II belong to Ω. In the following lemma we prove the same assertion for PC method II as in Theorem 3.1.

Theorem 3.2 *For given $u^k \in \Re^n$, let $\tilde{u}^k$ be defined by* (1.4) *and the new iterate u^{k+1} be generated by PCM-II* (1.8) *with any $\gamma > 0$. Then we have*

$$\begin{aligned} &\gamma\alpha_k^*(u-\tilde{u}^k)^T(Mu+q) \\ \geq\quad & \frac{1}{2}\big(\|u-u^{k+1}\|^2-\|u-u^k\|^2\big)+\frac{1}{2}q_k(\gamma),\ \ \forall\, u\in\Omega, \end{aligned} \tag{3.6}$$

where $q_k(\gamma)$ is defined in (1.7b)*.*

Proof. By using the assertion in Lemma 2.1, we need only to show that

$$\begin{aligned} &(u-\tilde{u}^k)^T\gamma\alpha_k^*[(Mu^k+q)+M^T(u^k-\tilde{u}^k)] \\ \geq\quad & \frac{1}{2}\big(\|u-u^{k+1}\|^2-\|u-u^k\|^2\big)+\frac{1}{2}q_k(\gamma) \\ & +\frac{\gamma\alpha_k^*}{2}\|u^k-\tilde{u}^k\|_D^2,\ \ \forall\, u\in\Omega. \end{aligned} \tag{3.7}$$

For investigating the term in the left hand side of (3.7), we divide it in the terms

$$(u^{k+1} - \tilde{u}^k)^T \gamma \alpha_k^* [(Mu^k + q) + M^T (u^k - \tilde{u}^k)] \tag{3.8a}$$

and

$$(u - u^{k+1})^T \gamma \alpha_k^* [(Mu^k + q) + M^T (u^k - \tilde{u}^k)]. \tag{3.8b}$$

First, we deal with the term (3.8a). Since $u^{k+1} \in \Omega$, substituting $u = u^{k+1}$ in (3.1) we get

$$\begin{aligned}
&(u^{k+1} - \tilde{u}^k)^T \gamma \alpha_k^* [(Mu^k + q) + M^T (u^k - \tilde{u}^k)] \\
&\quad \geq \quad \gamma \alpha_k^* (u^{k+1} - \tilde{u}^k)^T (I + M^T)(u^k - \tilde{u}^k) \\
&\quad = \quad \gamma \alpha_k^* (u^k - \tilde{u}^k)^T (I + M^T)(u^k - \tilde{u}^k) \\
&\qquad\quad - \gamma \alpha_k^* (u^k - u^{k+1})^T (I + M^T)(u^k - \tilde{u}^k) \\
&\quad = \quad \gamma \alpha_k^* \|u^k - \tilde{u}^k\|^2 + \frac{\gamma \alpha_k^*}{2} \|u^k - \tilde{u}^k\|_D^2 \\
&\qquad\quad - \gamma \alpha_k^* (u^k - u^{k+1})^T (I + M^T)(u^k - \tilde{u}^k).
\end{aligned} \tag{3.9}$$

To the crossed term of the right hand side of (3.9), using the Cauchy-Schwarz Inequality and (1.6) we get

$$
\begin{aligned}
&-\gamma\alpha_k^*(u^k - u^{k+1})^T(I + M^T)(u^k - \tilde{u}^k) \\
&\quad \geq \quad -\frac{1}{2}\|u^k - u^{k+1}\|^2 - \frac{1}{2}\gamma^2(\alpha_k^*)^2\|(I + M^T)(u^k - \tilde{u}^k)\|^2 \\
&\quad = \quad -\frac{1}{2}\|u^k - u^{k+1}\|^2 - \frac{1}{2}\gamma^2\alpha_k^*\|u^k - \tilde{u}^k\|^2.
\end{aligned}
$$

Substituting them in the right hand side of (3.9) and using the notation of $q_k(\gamma)$, we obtain

$$
\begin{aligned}
&(u^{k+1} - \tilde{u}^k)^T\gamma\alpha_k^*[(Mu^k + q) + M^T(u^k - \tilde{u}^k)] \\
&\quad \geq \quad \frac{1}{2}q_k(\gamma) + \frac{\gamma\alpha_k^*}{2}\|u^k - \tilde{u}^k\|_D^2 - \frac{1}{2}\|u^k - u^{k+1}\|^2. \qquad (3.10)
\end{aligned}
$$

Now, we turn to treat of the term (3.8b). The update form of PC Method II (1.8) means that u^{k+1} is the projection of $\big(u^k - \gamma\alpha_k^*[(Mu^k + q) + M^T(u^k - \tilde{u}^k)]\big)$ on Ω. Thus, it follows from (1.3) that

$$
\big\{\big(u^k - \gamma\alpha_k^*[(Mu^k + q) + M^T(u^k - \tilde{u}^k)]\big) - u^{k+1}\big\}^T\big(u - u^{k+1}\big) \leq 0, \quad \forall\, u \in \Omega,
$$

and consequently

$$\left(u-u^{k+1}\right)^T\gamma\alpha_k^*[(Mu^k+q)+M^T(u^k-\tilde{u}^k)] \geq \left(u-u^{k+1}\right)^T\left(u^k-u^{k+1}\right), \;\; \forall\, u\in\Omega.$$

For the the last inequality, by using the identity

$$b^T(b-a) = \frac{1}{2}\left(\|b\|^2-\|a\|^2\right)+\frac{1}{2}\|a-b\|^2$$

with $a=u-u^k$ and $b=u-u^{k+1}$ to the right hand side, we obtain

$$\begin{aligned}&\left(u-u^{k+1}\right)^T\gamma\alpha_k^*[(Mu^k+q)+M^T(u^k-\tilde{u}^k)]\\ \geq\quad & \frac{1}{2}\left(\|u-u^{k+1}\|^2-\|u-u^k\|^2\right)+\frac{1}{2}\|u^k-u^{k+1}\|^2. \qquad (3.11)\end{aligned}$$

Adding (3.10) and (3.11), we get (3.7) and the proof is complete. □

The assertion in Theorem 3.1 (see inequality (3.2)) coincides with the one in Theorem 3.2 (see inequality (3.6)). The only difference is the sequence $\{u^k\}$ in $\Re^n$ or Ω.

4 Convergence rate of the PC methods for LVI

For the different projection and contraction methods, we have the same key inequality which is shown in Theorem 3.1 (see (3.2)) and Theorem 3.2 (see (3.6)), respectively. The contraction property (1.7) of the PC methods is the consequent result of Theorem 3.1 and Theorem 3.2, respectively.

Theorem 4.1 *The sequence* $\{u^k\}$ *generated by PC Method-I or PC Method II satisfies*

$$\|u^{k+1} - u^*\|^2 \le \|u^k - u^*\|^2 - \gamma(2-\gamma)\alpha_k^*\|u^k - \tilde{u}^k\|^2, \quad \forall\, u^* \in \Omega^*. \tag{4.1}$$

Proof. By setting $u = u^*$ in (3.2) and (3.6), we get

$$\|u^k - u^*\|^2 - \|u^{k+1} - u^*\|^2 \ge 2\gamma\alpha_k^*(\tilde{u}^k - u^*)^T(Mu^* + q) + q_k(\gamma).$$

Because $(\tilde{u}^k - u^*)^T(Mu^* + q) \ge 0$, it follows from the last inequality that

$$\|u^{k+1} - u^*\|^2 \le \|u^k - u^*\|^2 - q_k(\gamma).$$

Using the notation of $q_k(\gamma)$ (see (1.7b)), the theorem is proved. $\Box$.

The following is the assertion of the convergence rate in ergodic sense.

Theorem 4.2 *For any integer $t > 0$, we have a $\tilde{u}_t \in \Omega$ which satisfies*

$$(\tilde{u}_t - u)^T F(u) \le \frac{\|I + M^T\|^2}{2\gamma(t+1)} \|u - u^0\|^2, \quad \forall u \in \Omega, \tag{4.2}$$

where

$$\tilde{u}_t = \frac{1}{\Upsilon_t} \sum_{k=0}^{t} \alpha_k^* \tilde{u}^k \qquad \textit{and} \qquad \Upsilon_t = \sum_{k=0}^{t} \alpha_k^*. \tag{4.3}$$

Proof. For the convergence rate proof, we allow $\gamma \in (0, 2]$. In this case, we still have $q_k(\gamma) \ge 0$. By using the monotonicity of F, from (3.2) and (3.6) we get

$$(u - \tilde{u}^k)^T \alpha_k^* (Mu + q) + \frac{1}{2\gamma}\|u - u^k\|^2 \ge \frac{1}{2\gamma}\|u - u^{k+1}\|^2, \ \ \forall\, u \in \Omega.$$

Summing the above inequality over $k = 0, \ldots, t$, we obtain

$$\Big((\sum_{k=0}^{t} \alpha_k^*) u - \sum_{k=0}^{t} \alpha_k^* \tilde{u}^k \Big)^T (Mu + q) + \frac{1}{2\gamma}\|u - u^0\|^2 \ge 0, \quad \forall u \in \Omega.$$

Using the notations of Υ_t and $\tilde{u}_t$ in the above inequality, we derive

$$(\tilde{u}_t - u)^T(Mu + q) \le \frac{\|u - u^0\|^2}{2\gamma\Upsilon_t}, \quad \forall u \in \Omega. \tag{4.4}$$

Indeed, $\tilde{u}_t \in \Omega$ because it is a convex combination of $\tilde{u}^0, \tilde{u}^1, \ldots, \tilde{u}^t$. Because $\alpha_k^* \ge 1/\|I + M\|^2$ for all $k > 0$ (see (1.4)), it follows from (4.3) that

$$\Upsilon_t \ge \frac{t+1}{\|I + M\|^2},$$

The proof is complete. $\square$

and thus the PC methods have $O(1/t)$ convergence rate. For any substantial set $\mathcal{D} \subset \Omega$, the PC methods reach

$$(\tilde{u}_t - u)^T F(u) \le \epsilon, \quad \forall\, u \in \mathcal{D}(\tilde{u}_t), \quad \text{in at most} \quad t = \left\lceil \frac{\|I + M\|^2 d^2}{2\gamma\epsilon} \right\rceil$$

iterations, where $\tilde{u}_t$ is defined in (4.3) and $d = \sup\{\|u - u^0\| \mid u \in \mathcal{D}(\tilde{u})\}$. This convergence rate is in the ergodic sense, the statement (4.2) suggests us to take a larger parameter $\gamma \in (0, 2]$ in the correction steps of the PC methods.

5 PC Method for convex quadratic optimization

This section instigate the convergence rate of the PC method for the convex quadratic programming. Let Ω be a closed convex subset of $\Re^n$, $H \in \Re^{n\times n}$ is positive semidefinite and $c \in \Re^n$. Solving the convex quadratic optimization

$$\min\{\tfrac{1}{2}x^T Hx + c^T x \,|\, x \in \Omega\}$$

is equivalent to finding a vector $x^* \in \Omega$ such that

$$\text{SLVI}(\Omega, H, c) \qquad (x - x^*)^T (Hx^* + c) \geq 0, \qquad \forall x \in \Omega. \tag{5.1}$$

The problem (5.1) is called symmetric linear variational inequality problem and denoted by LVI(Ω, H, c). For given $x^k \in \Re^n$, let

$$\tilde{x}^k = P_\Omega[x^k - \beta(Hx^k + c)]. \tag{5.2}$$

For solving the monotone variational inequalities [7] are developed. For monotone linear variational inequality, SLVI(Ω, H, c), one of the method in [7] (see Method 1 therein) generates the new iterate by

$$\text{(PC Method)} \qquad x^{k+1} = x^k - \gamma\alpha_k^*(x^k - \tilde{x}^k), \tag{5.3}$$

where

$$\alpha_k^* = \frac{\|x^k - \tilde{x}^k\|^2}{\|x^k - \tilde{x}^k\|_G^2}, \qquad G = I + \beta H \qquad \text{and} \qquad \gamma \in (0,2). \tag{5.4}$$

It was proved [7] that the sequence $\{x^k\}$ generated by PC Method satisfies

$$\|x^{k+1} - x^*\|_G^2 \le \|x^k - x^*\|_G^2 - \gamma(2-\gamma)\alpha_k^*\|x^k - \tilde{x}^k\|^2. \tag{5.5}$$

Because a projection is needed in each iteration and the distance of the iterates to the solution set is monotonically decreasing, according to the terminology in [1], such methods are called *Projection and Contraction Methods* (see [4]).

In this section, we show that, for given $\epsilon > 0$, in $O(1/\epsilon)$ iterations the projection and contraction method can find a $\tilde{x}$ such that

$$\tilde{x} \in \Omega \quad \text{and} \quad \sup_{x \in \mathcal{D}(\tilde{x})} \left\{ (\tilde{x} - x)^T (Hx + c) \right\} \le \epsilon, \tag{5.6}$$

where

$$\mathcal{D}(\tilde{x}) = \{x \in \Omega \,|\, \|x - \tilde{x}\| \le 1\}.$$

In this sense, we will establish the algorithmic convergence complexity for the pair of geminate projection and contraction method .

5.1 Main theorem for complexity analysis

This section shows the main theorems for the complexity analysis. Now, we prove the key inequality for the complexity analysis of the pair of geminate Algorithm.

Theorem 5.1 *For given $x^k \in \Re^n$, let $\tilde{x}^k$ be defined by* (5.2) *and the new iterate x^{k+1} be generated by PC Method* (5.3) *with any $\gamma \in (0,2)$. Then we have*

$$\begin{aligned} &\gamma\alpha_k^*\beta(x-\tilde{x}^k)^T(Hx+c) \\ &\quad \ge \quad \frac{1}{2}\big(\|x-x^{k+1}\|_G^2 - \|x-x^k\|_G^2\big) + \frac{1}{2}q_k(\gamma), \ \ \forall\, x \in \Omega, \end{aligned} \qquad (5.7)$$

where

$$q_k(\gamma) = \gamma(2-\gamma)\alpha_k^*\|x^k - \tilde{x}^k\|^2. \qquad (5.8)$$

Proof. Set $v = x^k - \beta(Hx^k + c)$ in (1.3), because $\tilde{x}^k = P_\Omega[x^k - \beta(Hx^k + c)]$ and thus $\tilde{x}^k = P_\Omega(v)$, it follows that

$$(x-\tilde{x}^k)^T\{x^k - \beta(Hx^k + c) - \tilde{x}^k\} \le 0 \quad \forall\, x \in \Omega.$$

Thus, we have

$$(x-\tilde{x}^k)^T\beta(Hx^k+c) \ge (x-\tilde{x}^k)^T(x^k-\tilde{x}^k), \quad \forall\, x\in\Omega.$$

Adding the term $(x-\tilde{x}^k)^T\beta H(x^k-\tilde{x}^k)$ to the both sides of the above inequality and using $(I+\beta H)=G$, we obtain

$$\begin{aligned}&(x-\tilde{x}^k)^T\{\beta(Hx^k+c)+\beta H(x^k-\tilde{x}^k)\}\\ &\quad\ge\quad (x-\tilde{x}^k)^T G(x^k-\tilde{x}^k), \quad \forall\, x\in\Omega.\end{aligned}$$

Then, we rewrite the above inequality in our desirable form

$$\begin{aligned}&(x-\tilde{x}^k)^T\{\beta(Hx+c)+\beta H(\tilde{x}^k-x)+2\beta H(x^k-\tilde{x}^k)\}\\ &\quad\ge\quad (x-\tilde{x}^k)^T G(x^k-\tilde{x}^k), \quad \forall\, x\in\Omega.\end{aligned}$$

By using the Cauchy-Schwarz inequality, from the above inequality we obtain

$$\begin{aligned}&(x-\tilde{x}^k)^T\beta(Hx+c)\\ &\quad\ge\quad (x-\tilde{x}^k)^T G(x^k-\tilde{x}^k)+\beta\|x-\tilde{x}^k\|_H-2(x-\tilde{x}^k)^T\beta H(x^k-\tilde{x}^k)\\ &\quad\ge\quad (x-\tilde{x}^k)^T G(x^k-\tilde{x}^k)-\beta\|x^k-\tilde{x}^k\|_H^2, \quad \forall\, x\in\Omega,\end{aligned}$$

and thus

$$\begin{aligned}
&\gamma\alpha_k^*(x-\tilde{x}^k)^T\beta(Hx+c) \\
&\quad\geq \quad (x-\tilde{x}^k)^T G\gamma\alpha_k^*(x^k-\tilde{x}^k) - \gamma\alpha_k^*\beta\|x^k-\tilde{x}^k\|_H^2, \quad \forall\, x\in\Omega.
\end{aligned}$$

Because $\gamma\alpha_k^*(x^k-\tilde{x}^k) = (x^k - x^{k+1})$ (see (1.5)), we get

$$\begin{aligned}
&\gamma\alpha_k^*(x-\tilde{x}^k)^T\beta(Hx+c) \\
&\quad\geq \quad (x-\tilde{x}^k)^T G(x^k-x^{k+1}) - \gamma\alpha_k^*\beta\|x^k-\tilde{x}^k\|_H^2, \quad \forall\, x\in\Omega. \qquad (5.9)
\end{aligned}$$

To the crossed term in the right hand side of (5.9), namely $(x-\tilde{x}^k)^T G(x^k - x^{k+1})$, using an identity

$$(a-b)^T G(c-d) = \frac{1}{2}\big(\|a-d\|_G^2 - \|a-c\|_G^2\big) + \frac{1}{2}\big(\|c-b\|_G^2 - \|d-b\|_G^2\big),$$

we obtain

$$\begin{aligned}
(x-\tilde{x}^k)^T G(x^k-x^{k+1}) \quad = \quad & \frac{1}{2}\big(\|x-x^{k+1}\|_G^2 - \|x-x^k\|_G^2\big) + \\
& \frac{1}{2}\big(\|x^k-\tilde{x}^k\|_G^2 - \|x^{k+1}-\tilde{x}^k\|_G^2\big). \qquad (5.10)
\end{aligned}$$

By using $x^{k+1} = x^k - \gamma\alpha_k^*(x^k - \tilde{x}^k)$ and (5.4), we get

$$\begin{aligned}
&\|x^k - \tilde{x}^k\|_G^2 - \|x^{k+1} - \tilde{x}^k\|_G^2 \\
&= \quad \|x^k - \tilde{x}^k\|_G^2 - \|(x^k - \tilde{x}^k) - \gamma\alpha_k^*(x^k - \tilde{x}^k)\|_G^2 \\
&= \quad 2\gamma\alpha_k^*(x^k - \tilde{x}^k)^T G(x^k - \tilde{x}^k) - \gamma^2\alpha_k^*(\alpha_k^*\|x^k - \tilde{x}^k\|_G^2) \\
&= \quad 2\gamma\alpha_k^*\|x^k - \tilde{x}^k\|^2 + 2\gamma\alpha_k^*\beta\|x^k - \tilde{x}^k\|_H^2 - \gamma^2\alpha_k^*\|x^k - \tilde{x}^k\|^2 \\
&= \quad \gamma(2-\gamma)\alpha_k^*\|x^k - \tilde{x}^k\|^2 + 2\gamma\alpha_k^*\|x^k - \tilde{x}^k\|_H^2.
\end{aligned}$$

Substituting it in the right hand side of (5.10) and using the definition of $q_k(\gamma)$, we obtain

$$\begin{aligned}
(x - \tilde{x}^k)^T G(x^k - x^{k+1}) \quad = \quad & \frac{1}{2}\big(\|x - x^{k+1}\|_G^2 - \|x - x^k\|_G^2\big) \\
& + \frac{1}{2}q_k(\gamma) + \gamma\alpha_k^*\beta\|x^k - \tilde{x}^k\|_H^2. \qquad (5.11)
\end{aligned}$$

Adding (5.9) and (5.11), the theorem is proved. $\Box$

By setting $x = x^*$ in (5.7), we get

$$\|x^k - x^*\|_G^2 - \|x^{k+1} - x^*\|_G^2 \geq 2\gamma\alpha_k^*(\tilde{x}^k - x^*)^T(Hx^* + c) + q_k(\gamma).$$

Because $(\tilde{x}^k - x^*)^T(Hx^* + c) \geq 0$, it follows from the last inequality and (5.8) that

$$\|x^{k+1} - x^*\|_G^2 \leq \|x^k - x^*\|_G^2 - \gamma(2-\gamma)\alpha_k^*\|x^k - \tilde{x}^k\|^2.$$

Thus, the contraction property (5.5) of the PC-Method is a byproduct of Theorem 5.1.

5.2 Convergence rate of the PC method

This section uses Theorem 3.1 to show the convergence rate of the projection type algorithm.

Theorem 5.2 *Let the sequence* $\{x^k\}$ *be generated by PC Method, and* $\tilde{x}^k$ *be given by* (1.4). *For any integer* $t > 0$, *it holds that*

$$(\tilde{x}_t - x)^T(Hx + c) \leq \frac{\|I + \beta H\|}{2\gamma\beta(t+1)}\|x - x^0\|_G^2, \quad \forall x \in \Omega, \tag{5.12}$$

where

$$\tilde{x}_t = \frac{1}{\Upsilon_t}\sum_{k=0}^{t}\alpha_k^*\tilde{x}^k \qquad \textit{and} \qquad \Upsilon_t = \sum_{k=0}^{t}\alpha_k^*. \tag{5.13}$$

Proof. For the convergence rate proof, we allow $\gamma \in (0,2]$. In this case, we still have $q_k(\gamma) \geq 0$. From (3.2) we get

$$(x-\tilde{x}^k)^T \alpha_k^*(Hx+c) + \frac{1}{2\gamma\beta}\|x-x^k\|_G^2 \geq \frac{1}{2\gamma\beta}\|x-x^{k+1}\|_G^2, \ \ \forall\, x \in \Omega.$$

Summing the above inequality over $k = 0,\ldots,t$, we obtain

$$\Big((\sum_{k=0}^{t} \alpha_k^*)x - \sum_{k=0}^{t} \alpha_k^* \tilde{x}^k \Big)^T (Hx+c) + \frac{1}{2\gamma\beta}\|x-x^0\|^2 \geq 0, \quad \forall x \in \Omega.$$

Using the notations of Υ_t and $\tilde{x}_t$ in the above inequality, we derive

$$(\tilde{x}_t - x)^T (Hx+c) \leq \frac{\|x-x^0\|^2}{2\gamma\beta\Upsilon_t}, \quad \forall x \in \Omega. \tag{5.14}$$

Indeed, $\tilde{x}_t \in \Omega$ because it is a convex combination of $\tilde{x}^0, \tilde{x}^1, \ldots, \tilde{x}^t$. Because $\alpha_k^* \geq 1/\|I+\beta H\|$ for all $k > 0$ (see (5.2)), it follows from (5.13) that

$$\Upsilon_t \geq \frac{t+1}{\|I+\beta H\|}.$$

Substituting it in (5.14), the proof is complete. $\Box$

Thus the projection and contraction method have $O(1/t)$ convergence rate. For any substantial set $\mathcal{D} \subset \Omega$, the PC Method and PC Method reach

$$(\tilde{x}_t - x)^T (Hx + c) \le \epsilon, \quad \forall\, x \in \mathcal{D}(\tilde{x}_t), \quad \text{in at most} \quad t = \left\lceil \frac{\|I + \beta H\| d^2}{2\gamma\beta\epsilon} \right\rceil$$

iterations, where $\tilde{x}_t$ is defined in (5.13) and $d = \sup\{\|x - x^0\| \mid x \in \mathcal{D}(\tilde{x}_t)\}$. This convergence rate is in the ergodic sense, the statement (5.12) suggests us to take a larger parameter $\gamma \in (0, 2]$ in the correction steps of the projection and contraction method .

5.3 Convergence rate in the point manner

In this subsection, we consider a special version of PC Method which updates the new iterate by

$$x^{k+1} = x^k - (x^k - \tilde{x}^k), \tag{5.15}$$

where the positive definite matrix G satisfies the following condition:

$$G = I + \beta H \preceq 2(1 - \nu) I, \quad \nu \in (0, 1). \tag{5.16}$$

Under the condition (5.16), if we use the original PC Method (1.5), the α_k^* defined in (1.6) satisfies

$$\alpha_k^* = \frac{\|x^k - \tilde{x}^k\|^2}{\|x^k - \tilde{x}^k\|_G^2} \geq \frac{1}{2(1-\nu)}. \tag{5.17}$$

The update form (5.15) can be viewed in PC Method (5.3) by dynamically taking $\gamma_k \equiv 1/\alpha_k^*$. According to (5.5) and (5.17), using

$$\gamma_k \alpha_k^* = 1 \quad \text{and} \quad \|x^k - \tilde{x}^k\|^2 = \alpha_k^* \|G^{-1}(I+H)(x^k - \tilde{x}^k)\|_G^2,$$

we obtain

$$\begin{aligned} q_k(\gamma_k) &= (2-\gamma_k)\|x^k - \tilde{x}^k\|^2 \\ &= (2-\gamma_k)\alpha_k^* \|x^k - \tilde{x}^k\|_G^2 \;\geq\; \frac{\nu}{1-\nu}\|x^k - \tilde{x}^k\|_G^2, \quad \forall\, x^* \in \Omega^*. \end{aligned}$$

Then, from (1.7), the above inequality and (5.15) follows that

$$\|x^{k+1} - x^*\|_G^2 \leq \|x^k - x^*\|_G^2 - \frac{\nu}{1-\nu}\|x^k - x^{k+1}\|_G^2, \quad \forall\, x^* \in \Omega^*. \tag{5.18}$$

In the following we focus on showing that the sequence $\{\|x^k - x^{k+1}\|_G\}$ is monotonically non-increasing. For this purpose, we need to prove the following lemma.

Lemma 5.1 *For given x^k, let $\tilde{x}^k$ be offered by* (1.4). *If the algorithm uses the update form*

(5.15) *to generate the new iterate, then we have*

$$\begin{aligned}(x^k - x^{k+1})^T G\{(x^k - \tilde{x}^k) - (x^{k+1} - \tilde{x}^{k+1})\} \\ \geq \quad \|(x^k - \tilde{x}^k) - (x^{k+1} - \tilde{x}^{k+1})\|^2 + \beta\|x^k - x^{k+1}\|_H^2. \end{aligned} \qquad (5.19)$$

Proof. First, set $y = x^k - \beta(Hx^k + c)$ in (1.3), since $\tilde{x}^k = P_\Omega[x^k - \beta(Hx^k + c)]$, it follows that

$$(x - \tilde{x}^k)^T\{\beta(Hx^k + c) - (x^k - \tilde{x}^k)\} \geq 0, \quad \forall\, x \in \Omega.$$

We rewrite it as

$$(x - \tilde{x}^k)^T\{\beta(H\tilde{x}^k + c) + 2\beta H(x^k - \tilde{x}^k) - (I + \beta H)(x^k - \tilde{x}^k)\} \geq 0, \quad \forall\, x \in \Omega,$$

and thus

$$\begin{aligned}(x - \tilde{x}^k)^T\beta(H\tilde{x}^k + c) \\ \geq\ (x - \tilde{x}^k)^T G(x^k - \tilde{x}^k) - 2\beta(x - \tilde{x}^k)^T H(x^k - \tilde{x}^k),\ \forall x \in \Omega. \end{aligned} \qquad (5.20)$$

Set $x = \tilde{x}^{k+1}$ in (5.20), we have

$$\begin{aligned}(\tilde{x}^{k+1} - \tilde{x}^k)^T \beta(H\tilde{x}^k + c) \quad &\geq \quad (\tilde{x}^{k+1} - \tilde{x}^k)^T G(x^k - \tilde{x}^k) \\ &\quad -2\beta(\tilde{x}^{k+1} - \tilde{x}^k)^T H(x^k - \tilde{x}^k). \quad (5.21)\end{aligned}$$

Note that (5.20) is also true for $k := k+1$ and thus we have

$$\begin{aligned}(x - \tilde{x}^{k+1})^T \beta(H\tilde{x}^{k+1} + c) \quad &\geq \quad (x - \tilde{x}^{k+1})^T G(x^{k+1} - \tilde{x}^{k+1}) \\ &\quad -2\beta(x - \tilde{x}^{k+1})^T H(x^{k+1} - \tilde{x}^{k+1}), \ \forall\, x \in \Omega.\end{aligned}$$

Set $x = \tilde{x}^k$ in the above inequality, we obtain

$$\begin{aligned}(\tilde{x}^k - \tilde{x}^{k+1})^T \beta(H\tilde{x}^{k+1} + c) \quad &\geq \quad (\tilde{x}^k - \tilde{x}^{k+1})^T G(x^{k+1} - \tilde{x}^{k+1}) \\ &\quad -2\beta(\tilde{x}^k - \tilde{x}^{k+1})^T H(x^{k+1} - \tilde{x}^{k+1}). \quad (5.22)\end{aligned}$$

Adding (5.21) and (5.22), we get

$$\begin{aligned}&(\tilde{x}^k - \tilde{x}^{k+1})^T G\{(x^k - \tilde{x}^k) - (x^{k+1} - \tilde{x}^{k+1})\} \\ &\quad \geq \quad \beta\|\tilde{x}^k - \tilde{x}^{k+1}\|_H^2 \\ &\qquad\quad +2\beta(\tilde{x}^k - \tilde{x}^{k+1})^T H\{(x^k - \tilde{x}^k) - (x^{k+1} - \tilde{x}^{k+1})\}. \quad (5.23)\end{aligned}$$

Moreover, by adding the identity

$$\begin{aligned}&\{(x^k-\tilde{x}^k)-(x^{k+1}-\tilde{x}^{k+1})\}^T G\{(x^k-\tilde{x}^k)-(x^{k+1}-\tilde{x}^{k+1})\}\\ &\quad = \quad \beta\|(x^k-\tilde{x}^k)-(x^{k+1}-\tilde{x}^{k+1})\|_H^2+\|(x^k-\tilde{x}^k)-(x^{k+1}-\tilde{x}^{k+1})\|^2\end{aligned}$$

to the both sides of (5.23), we obtain

$$\begin{aligned}&(x^k-x^{k+1})^T G\{(x^k-\tilde{x}^k)-(x^{k+1}-\tilde{x}^{k+1})\}\\ &\quad\geq\quad \beta\|\tilde{x}^k-\tilde{x}^{k+1}\|_H^2+2\beta(\tilde{x}^k-\tilde{x}^{k+1})^T H\{(x^k-\tilde{x}^k)-(x^{k+1}-\tilde{x}^{k+1})\}\\ &\qquad\quad +\beta\|(x^k-\tilde{x}^k)-(x^{k+1}-\tilde{x}^{k+1})\|_H^2+\|(x^k-\tilde{x}^k)-(x^{k+1}-\tilde{x}^{k+1})\|^2\\ &\quad=\quad \beta\|x^k-x^{k+1}\|_H^2+\|(x^k-\tilde{x}^k)-(x^{k+1}-\tilde{x}^{k+1})\|^2. \qquad (5.24)\end{aligned}$$

This is just the inequality (5.19) and the lemma is proved. □

Theorem 5.3 *For given x^k, let $\tilde{x}^k$ be offered by* (1.4)*. If the algorithm uses the update form* (5.15) *to generate the new iterate and the the condition* (5.16) *is satisfied, then for all integer $k \geq 0$, it holds that*

$$\|x^{k+1}-x^{k+2}\|_G^2 \leq \|x^k-x^{k+1}\|_G^2. \qquad (5.25)$$

Proof. Using $\tilde{x}^k = x^{k+1}$ and $\tilde{x}^{k+1} = x^{k+2}$ (see (5.15)) in (5.19), we obtain

$$\begin{aligned} &(x^k - x^{k+1})^T G\{(x^k - x^{k+1}) - (x^{k+1} - x^{k+2})\} \\ \geq \quad & \|(x^k - x^{k+1}) - (x^{k+1} - x^{k+2})\|^2. \end{aligned} \tag{5.26}$$

Setting $a = (x^k - x^{k+1})$ and $b = (x^{k+1} - x^{k+2})$ in the identity

$$\|a\|_G^2 - \|b\|_G^2 = 2a^T G(a - b) - \|a - b\|_G^2,$$

we obtain

$$\begin{aligned} &\|x^k - x^{k+1}\|_G^2 - \|x^{k+1} - x^{k+2}\|_G^2 \\ = \quad & 2(x^k - x^{k+1})^T G\{(x^k - x^{k+1}) - (x^{k+1} - x^{k+2})\} \\ & -\|(x^k - x^{k+1}) - (x^{k+1} - x^{k+2})\|_G^2. \end{aligned}$$

Inserting (5.26) into the first term of the right-hand side of the last equality, we obtain

$$\begin{aligned} &\|x^k - x^{k+1}\|_G^2 - \|x^{k+1} - x^{k+2}\|_G^2 \\ \geq \quad & 2\|(x^k - x^{k+1}) - (x^{k+1} - x^{k+2})\|^2 \\ & -\|(x^k - x^{k+1}) - (x^{k+1} - x^{k+2})\|_G^2. \end{aligned} \tag{5.27}$$

Thus,

$$\begin{aligned} & \|x^k - x^{k+1}\|_G^2 - \|x^{k+1} - x^{k+2}\|_G^2 \\ \geq \quad & \|(x^k - x^{k+1}) - (x^{k+1} - x^{k+2})\|_{(2I-G)}^2. \end{aligned} \tag{5.28}$$

According to (5.16), the matrix $(2I - G)$ is positive definite, and thus the right hand side of (5.28) is non-negative. The assertion (5.25) follows immediately. □

With (5.18) and (5.25), we can prove the main result for the convergence rate in the non-ergodic sense.

Theorem 5.4 *For given x^k, let $\tilde{x}^k$ be offered by* (1.4)*. If the algorithm uses the update form* (5.15) *to generate the new iterate and the the condition* (5.16) *is satisfied, then for any integer $t > 0$, we have*

$$\|x^t - x^{t+1}\|_G^2 \leq \frac{1 - \nu}{(t + 1)\nu} \|x^0 - x^*\|_G^2. \tag{5.29}$$

Proof. First, it follows from (5.18) that

$$\frac{\nu}{1-\nu}\sum_{k=0}^{\infty}\|x^k - x^{k+1}\|_G^2 \le \|x^0 - x^*\|_G^2, \quad \forall\, x^* \in \Omega^*. \tag{5.30}$$

According to Theorem 5.3, the sequence $\{\|x^k - x^{k+1}\|_G^2\}$ is monotonically non-increasing. Therefore, we have

$$(t+1)\|x^t - x^{t+1}\|_G^2 \le \sum_{k=0}^{t}\|x^k - x^{k+1}\|_G^2. \tag{5.31}$$

The assertion (5.29) follows from (5.30) and (5.31) immediately. □

Notice that Ω^* is convex and closed (see (2.3.2) of [3]). Let

$$d := \inf\{\|x^0 - x^*\|_G \,|\, x^* \in \Omega^*\}.$$

Then, for any given $\epsilon > 0$, Theorem 5.4 shows that the it needs at most $\lfloor d^2/c_0\epsilon \rfloor$ iterations to ensure that $\|x^k - x^{k+1}\|_G^2 \le \epsilon$.

Recall that x^k is a solution of SLVI(Ω, H, c) if $\|x^k - x^{k+1}\|_G^2 = 0$ (see (5.15)). A worst-case $O(1/t)$ convergence rate in a non-ergodic sense is thus established.

References

[1] E. Blum and W. Oettli, *Mathematische Optimierung: Grundlagen und Verfahren. Ökonometrie und Unternehmensforschung*. Berlin-Heidelberg-New York, Springer-Verlag, 1975.

[2] C. H. Chen, X. L. Fu, B.S. He and X. M. Yuan, On the Iteration Complexity of Some Projection Methods for Monotone Linear Variational Inequalities, JOTA, 172(2017) 914-928.

[3] F. Facchinei and J.S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems, Vol. I and II*. Springer Series in Operations Research. Springer Verlag, New York, 2003.

[4] B.S. He, A projection and contraction method for a class of linear complementarity problems and its application in convex quadratic programming, *Applied Mathematics and Optimization*, **25**, pp. 247–262, 1992.

[5] B.S. He, A globally linearly convergent projection and contraction method for a class of linear complementarity problems. Schwerpunktprogramm der Deutschen Forschungsgemeinschaft Anwen- dungsbezogene Optimierung und Steuerung, Report No. 352, 1992.

[6] B.S. He, A new method for a class of linear variational inequalities, *Mathematical Programming* **66**, pp. 137–144, 1994.

[7] B.S. He, Solving a class of linear projection equations, *Numerische Mathematik* **68**, pp. 71–80, 1994.

[8] B.S. He, A modified projection and contraction method for a class of linear complementarity problems, *Journal of Computational Mathematics* **14**, pp. 54–63, 1996.

[9] B.S. He, A class of projection and contraction methods for monotone variational inequalities, *Applied Mathematics and optimization* **35**, pp. 69–76, 1997.

[10] B.S. He and M.-H. Xu, A general framework of contraction methods for monotone variational inequalities, *Pacific J. Optimization* **4**, pp. 195-212, 2008.

[11] G.M. Korpelevich, The extragradient method for finding saddle points and other problems, *Ekonomika i Matematicheskie Metody* **12**, pp. 747-756, 1976.

[12] E.N. Khobotov, Modification of the extragradient method for solving variational inequalities and certain optimization problems, *USSR Computational Mathematics and Mathematical Physics*, Vol. 27, pp.120-127, 1987.

[13] A. Nemirovski, Prox-method with rate of convergence $O(1/t)$ for variational inequality with Lipschitz continuous monotone operators and smooth convex-concave saddle point problems, *SIAM J. Optimization* **15**, pp. 229-251, 2005.

[14] P. Tseng, On accelerated proximal gradient methods for convex-concave optimization, Department of Mathematics, University of Washington, Seattle, WA 98195, USA, 2008.