变分不等式框架下结构型凸优化的分裂收缩算法

I. 凸优化及其在变分不等式框架下的邻近点算法

中学的数理基础	必要的社会实践
普通的大学数学	一般的优化原理

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天元数学东北中心　2023 年 10 月 17 – 27 日

华罗庚先生普及“双法”对我们的启示

- 华罗庚先生当年普及的双法— 统筹法和优选法。　　普及双法以优选法为主。

- 要“牢记把方法交给群众”。

  —华罗庚《数学工作者要大力为农业服务》

  人民日报 1960 年 10 月 30 日

- 这成为从上世纪60年代开始的近20年间, 华罗庚从事数学普及工作的指导思想。

  — 王元《华罗庚》

- 随着全民族文化水平的提高, 群众有了新的定义. 提供工程师们容易掌握的方法, 可以 作为部分优化学者的工作目标.

能够交给“群众”的方法, 应该是普通大学生能够理解, 掌握的方法.

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Variational Inequalities, Projection and contraction methods for VI,
ADMM-like splitting contraction methods for convex optimization

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我会尽量维护自己的主页，不断修正、更新自己学术的体验.

My Thinkings: 1. 关门感想 2. 说说我的主要研究兴趣 — 兼谈华罗庚推广优选法对我的影响
3. 说说我的主要研究兴趣(续) --- 我们在ADMM类方法的主要工作 4. 古稀回首
5. 两页纸给出ADMM 收敛性证明 6. 两页纸简述我职业生涯中的主要研究工作
7. 两页纸描述古稀之后的研究进展: 从好不容易凑出一个方法到并不费力构造一簇算法
8. 两页PPT讲述求解线性约束凸优化问题预测-校正的广义邻近点(Generaized PPA)算法

《数学文化》2020 年/ 第 11 卷第 2 期刊登的我的自述: 四十年上下求索 一份珍贵的回忆材料

My Talks: 比较系统的知识建议阅读第3个报告. 也建议阅读最近的一些系列报告

For more systematic knowledge, it is recommended to read Talk 3, which is written in English.

19. 2023 年10月在天元数学东北中心八次课程的汇总讲义 前言与目录 I II III IV V VI VII VIII
18. 2023年1月在华南师大《华人数学家论坛》的报告 — 凸优化分裂收缩算法统一框架的最新进展
17. 2022年11月在西电《最优化前沿论坛》的报告 — 从好不容易凑出一个算法到并不费劲构造一簇算法
16. 2022 年7月南理工《计算机》方向暑期班六讲摘要 和六讲的PPT I II III IV V VI
15. 利用预测-校正统一框架构造凸优化的分裂收缩算法(由预测矩阵构造校正矩阵) (ArXiv: 2204.11522)
14. 2022 年元月南师大数科院系列报告B站视频辅助材料 A B C D E F G H I J K L
13. ADMM 类分裂收缩算法的一些最新进展 统一框架下Balanced-ALM 便于向多块推广的ADMM
12. 均困平衡的增广拉格朗日乘子法 — Balanced ALM (一类新的增广拉格朗日乘子法ArXiv: 2108.08554)
11. 一类便于向求解多块问题推广并能处理不等式约束问题的交替方向法 (ArXiv:2107.01897)
10. 瞎子爬山-步步为营—凸优化算法中的变分不等式和邻近点策 略(南京大学数学系本科生论坛上的报告)
9. 被S. Becker 誉为 Very Simple yet Powerful 的 Technique — 应用及新的进展
8. 线性化ALM 线性化ADMM 以及处理三个可分离块问题中缩小有关参数至3/4提高效率的方法

我的报告的 PDF 文件, 一般都可以在我的主页上查到.

连续优化中一些代表性数学模型

1. 鞍点问题 $\min_{x\in\mathcal{X}}\max_{y\in\mathcal{Y}}\{\Phi(x,y)=\theta_1(x)-y^TAx-\theta_2(y)\}$
2. 线性约束的凸优化问题 $\min\{\theta(x)|Ax=b\,(\text{or}\geq b),\ x\in\mathcal{X}\}$
3. 结构型凸优化 $\min\{\theta_1(x)+\theta_2(y)|Ax+By=b,\ x\in\mathcal{X},\ y\in\mathcal{Y}\}$
4. 多块可分离凸优化 $\min\{\sum_{i=1}^p\theta_i(x_i)|\sum_{i=1}^pA_ix_i=b, x_i\in\mathcal{X}_i\}$

变分不等式(VI) 是瞎子爬山的数学表达形式

邻近点算法(PPA) 是步步为营 稳扎稳打的求解方法.

变分不等式和邻近点算法是分析和设计凸优化方法的两大法宝.

分裂是指迭代中子问题都通过分拆求解. 收缩算法有别于可行方向法, 又有别于下降算法, 它的迭代点离优化问题的拉格朗日函数的鞍点越来越近.

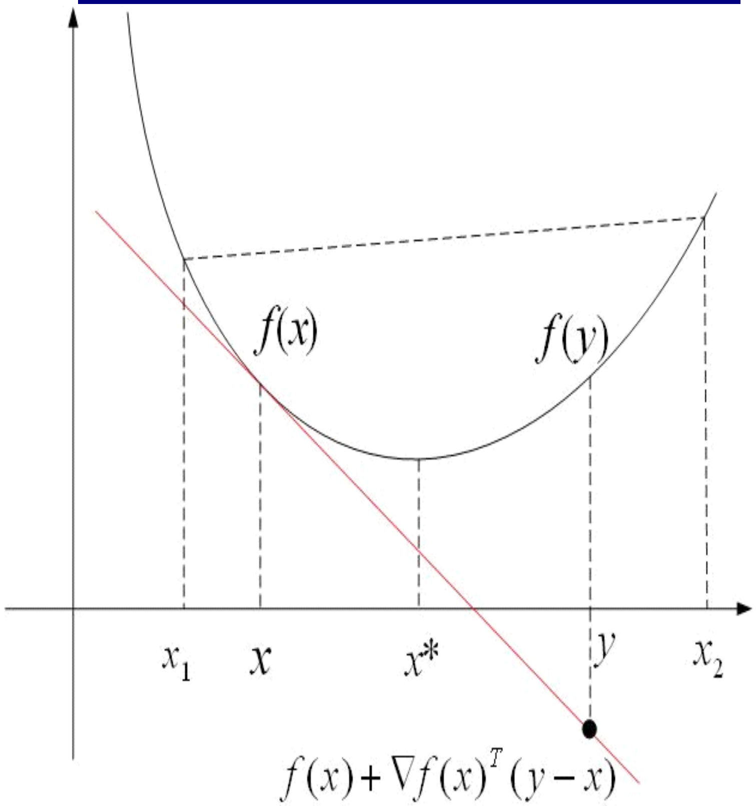
这一讲解释上述问题都可以化为一个单调变分不等式 并介绍什么是邻近点算法

凸函数的定义和基本性质

A function $f(x)$ is convex iff

$$f((1-\theta)x+\theta y) \leq (1-\theta)f(x)+\theta f(y)$$

$$\forall \theta \in [0,1].$$



Convex function

Properties of convex function

- $f \in \mathcal{C}^1$. f is convex iff

  $$f(y) - f(x) \geq \nabla f(x)^T(y-x).$$

  Thus, we have also

  $$f(x) - f(y) \geq \nabla f(y)^T(x-y).$$

- Adding above two inequalities, we get

  $$(y-x)^T(\nabla f(y) - \nabla f(x)) \geq 0.$$

- $f \in \mathcal{C}^1$, ∇f is monotone. $f \in \mathcal{C}^2$, $\nabla^2 f(x)$ is positive semi-definite.

- Any local minimum of a convex function is a global minimum.

1 Optimization problem and VI

1.1 Differential convex optimization in Form of VI

Let $\Omega \subset \Re^n$, we consider the convex minimization problem

$$\min\{f(x) \mid x \in \Omega\}. \tag{1.1}$$

What is the first-order optimal condition ?

$x^* \in \Omega^* \quad \Leftrightarrow \quad x^* \in \Omega$ and any feasible direction is not a descent one.

Optimal condition in variational inequality form

- $S_d(x^*) = \{s \in \Re^n \mid s^T \nabla f(x^*) < 0\} \;=\;$ Set of the descent directions.
- $S_f(x^*) = \{s \in \Re^n \mid s = x - x^*,\ x \in \Omega\} \;=\;$ Set of feasible directions.

$x^* \in \Omega^* \quad \Leftrightarrow \quad x^* \in \Omega$ **and** $S_f(x^*) \cap S_d(x^*) = \emptyset$.

瞎子爬山判定山顶的准则是: 所有可行方向都不再是上升方向

The optimal condition can be presented in a variational inequality (VI) form:

$$x^* \in \Omega, \quad (x - x^*)^T \nabla f(x^*) \geq 0, \quad \forall x \in \Omega. \tag{1.2}$$

Substituting $\nabla f(x)$ with an operator F (from $\Re^n$ into itself), we get a classical VI.

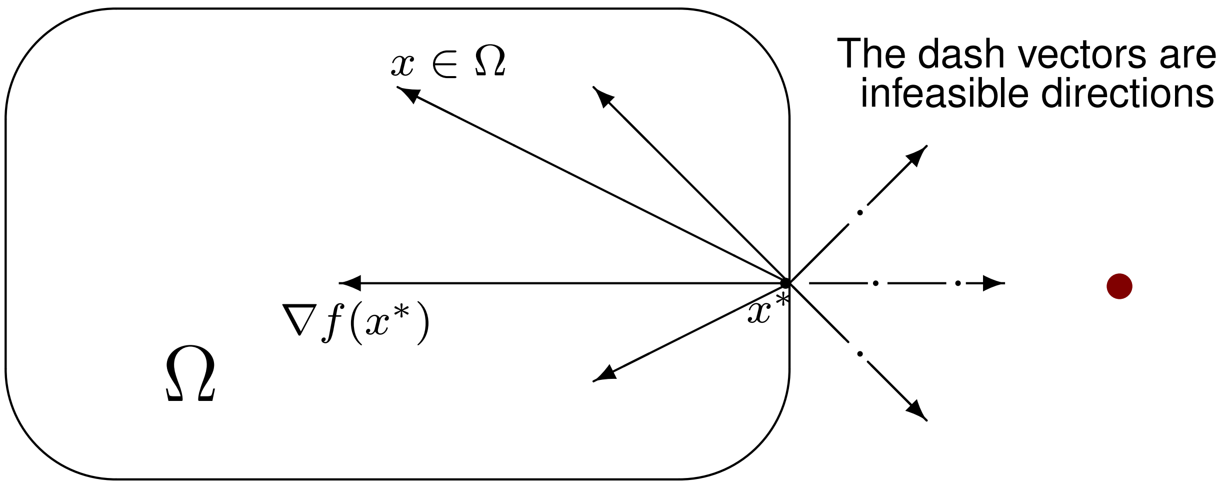


Fig. 1.1 Differential Convex Optimization and VI

Since $f(x)$ is a convex function, we have

$f(y) \geq f(x) + \nabla f(x)^T (y - x)$ and thus $(x - y)^T (\nabla f(x) - \nabla f(y)) \geq 0.$

We say the gradient ∇f of the convex function f is a monotone operator.

通篇我们需要用到的大学数学　主要是基于微积分学的一个引理

$$x^* \in \operatorname{argmin}\{\theta(x)|x \in \mathcal{X}\} \Leftrightarrow \ x^* \in \mathcal{X}, \qquad \theta(x)-\theta(x^*) \ge 0, \quad \forall x \in \mathcal{X};$$

$$x^* \in \operatorname{argmin}\{f(x)|x \in \mathcal{X}\} \Leftrightarrow \ x^* \in \mathcal{X}, \quad (x-x^*)^T \nabla f(x^*) \ge 0, \ \forall x \in \mathcal{X}.$$

上面的凸优化最优性条件是最基本的, 看起来合在一起就是下面的引理:

定理 1 *Let* $\mathcal{X} \subset \Re^n$ *be a closed convex set,* $\theta(x)$ *and* $f(x)$ *be convex functions and* $f(x)$ *is differentiable. Assume that the solution set of the minimization problem* $\min\{\theta(x)+f(x)\,|\,x \in \mathcal{X}\}$ *is nonempty. Then,*

$$x^* \in \arg\min\{\theta(x)+f(x)\,|\,x \in \mathcal{X}\} \tag{1.3a}$$

if and only if 　凸优化最优性条件定理

$$x^* \in \mathcal{X}, \ \ \theta(x)-\theta(x^*)+(x-x^*)^T\nabla f(x^*) \ge 0, \ \ \forall x \in \mathcal{X}. \tag{1.3b}$$

定理把优化问题 (1.3a) 转换成了变分不等式 (1.3b). 下面给出证明.

Proof : First, if (1.3a) is true, then for any $x \in \mathcal{X}$, we have

$$\frac{\theta(x_\alpha) - \theta(x^*)}{\alpha} + \frac{f(x_\alpha) - f(x^*)}{\alpha} \geq 0, \tag{1.4}$$

where

$$x_\alpha = (1-\alpha)x^* + \alpha x, \quad \forall \alpha \in (0,1].$$

Because $\theta(\cdot)$ is convex, it follows that

$$\theta(x_\alpha) \leq (1-\alpha)\theta(x^*) + \alpha\theta(x),$$

and thus

$$\theta(x) - \theta(x^*) \geq \frac{\theta(x_\alpha) - \theta(x^*)}{\alpha}, \;\; \forall \alpha \in (0,1].$$

Substituting the last inequality in the left hand side of (1.4), we have

$$\theta(x) - \theta(x^*) + \frac{f(x_\alpha) - f(x^*)}{\alpha} \geq 0, \quad \forall \alpha \in (0,1].$$

Using $f(x_\alpha) = f(x^* + \alpha(x - x^*))$ and letting $\alpha \to 0_+$, from the above inequality we get

$$\theta(x) - \theta(x^*) + \nabla f(x^*)^T (x - x^*) \geq 0, \;\; \forall\, x \in \mathcal{X}.$$

Thus (1.3b) follows from (1.3a). Conversely, since f is convex, it follow that

$$f(x_\alpha) \le (1-\alpha)f(x^*) + \alpha f(x)$$

and it can be rewritten as

$$f(x_\alpha) - f(x^*) \le \alpha(f(x) - f(x^*)).$$

Thus, we have

$$f(x) - f(x^*) \ge \frac{f(x_\alpha) - f(x^*)}{\alpha} = \frac{f(x^* + \alpha(x - x^*)) - f(x^*)}{\alpha},$$

for all $\alpha \in (0,1]$. Letting $\alpha \to 0_+$, we get

$$f(x) - f(x^*) \ge \nabla f(x^*)^T(x - x^*).$$

Substituting it in the left hand side of (1.3b), we get

$$x^* \in \mathcal{X}, \quad \theta(x) - \theta(x^*) + f(x) - f(x^*) \ge 0, \quad \forall\, x \in \mathcal{X},$$

and (1.3a) is true. The proof is complete. $\square$

可微约束优化问题的最优性必要条件

设 $f(x)$, $\varphi_i(x), i = 1, \ldots, m$, 都是从 $\Re^n \to \Re$ 的连续可微函数, 研究问题

$$
\begin{array}{rl}
\min & f(x) \\
\text{s.t} & \varphi_1(x) = 0, \\
& \vdots \\
& \varphi_m(x) = 0
\end{array}
$$

相应的Lagrange函数

$$
L(x, \lambda) = f(x) - \sum_{i=1}^{m} \lambda_i \varphi_i(x).
$$

最优性必要条件是:

$$
\left\{
\begin{array}{l}
\begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}
-
\begin{pmatrix}
\frac{\partial \varphi_1}{\partial x_1} & \frac{\partial \varphi_2}{\partial x_1} & \cdots & \frac{\partial \varphi_m}{\partial x_1} \\
\frac{\partial \varphi_1}{\partial x_2} & \frac{\partial \varphi_2}{\partial x_2} & \cdots & \frac{\partial \varphi_m}{\partial x_2} \\
\vdots & & & \\
\frac{\partial \varphi_1}{\partial x_n} & \frac{\partial \varphi_2}{\partial x_n} & \cdots & \frac{\partial \varphi_m}{\partial x_n}
\end{pmatrix}
\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{pmatrix}
= 0. \\
\varphi_i(x) = 0, \quad i = 1, \ldots, m.
\end{array}
\right.
$$

1.2 Linear constrained convex optimization and VI

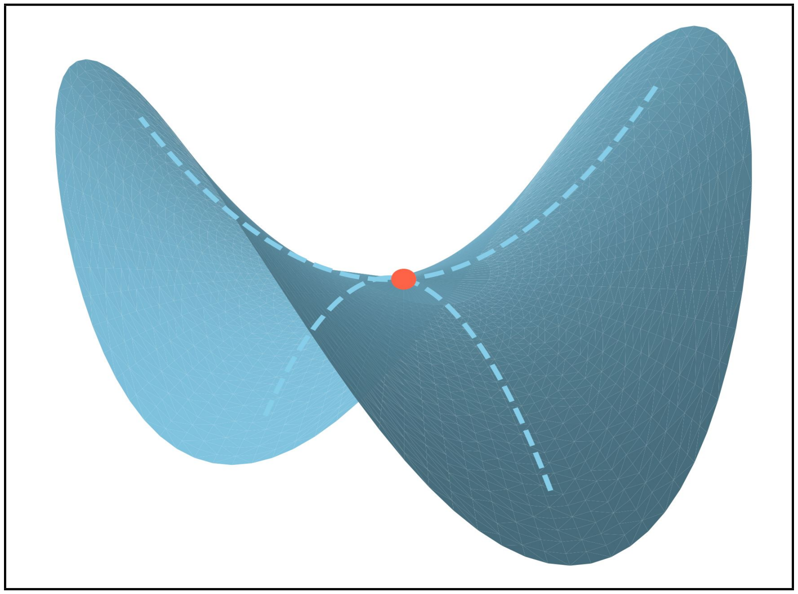
We consider the linearly constrained convex optimization problem

$$\min\{\theta(u) \mid \mathcal{A}u = b,\ u \in \mathcal{U}\}. \qquad (1.5)$$

The Lagrangian function of the problem (1.5) is

$$L(u,\lambda) = \theta(u) - \lambda^T(\mathcal{A}u - b), \qquad (1.6)$$

which is defined on $\mathcal{U} \times \Re^m$.



Example 1 of the problem (1.5): Finding the nearest correlation matrix

A positive semi-definite matrix, whose each diagonal element is equal 1, is called the correlation matrix. For given symmetric $n \times n$ matrix C, the mathematical form of finding the nearest correlation matrix X is

$$\min\{\tfrac{1}{2}\|X - C\|_F^2 \mid \mathrm{diag}(X) = e,\ X \in S_+^n\}, \qquad (1.7)$$

where S_+^n is the positive semi-definite cone and e is a n-vector whose each element is equal 1. The problem (1.7) is a concrete problem of type (1.5).

Example 2 of the problem (1.5): The matrix completion problem

Let M be a given $m \times n$ matrix, Π is the elements indices set of M,

$$\Pi \subset \{(ij) | i \in \{1, \ldots, m\},\ j \in \{1, \ldots, n\}\}.$$

The mathematical form of the matrix completion problem is relaxed to

$$\min\{\|X\|_* \mid X_{ij} = M_{ij},\ (ij) \in \Pi\}, \qquad (1.8)$$

where $\|\cdot\|_*$ is the nuclear norm–the sum of the singular values of a given matrix. The problem (1.8) is a convex optimization of form (1.5). The matrix A in (1.5) for the linear constraints

$$X_{ij} = M_{ij},\ (ij) \in \Pi,$$

is a projection matrix, and thus $\|A^T A\| = 1$.

M is low Rank, only some elements of M are known.

$$\begin{array}{cccccccccc}
* & & & * & & * & & & * & \\
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\end{array}$$

A pair of $(u^*, \lambda^*) \in \mathcal{U} \times \Re^m$ is called a saddle point of the Lagrange function (1.6), if

$$L_{\lambda\in\Re^m}(u^*, \lambda) \le L(u^*, \lambda^*) \le L_{u\in\mathcal{U}}(u, \lambda^*).$$

The above inequalities can be written as

$$\left\{ \begin{array}{ll} u^* \in \mathcal{U}, \quad L(u, \lambda^*) - L(u^*, \lambda^*) \ge 0, \ \ \forall\, u \in \mathcal{U}, & \text{(1.9a)} \\ \lambda^* \in \Re^m, \ L(u^*, \lambda^*) - L(u^*, \lambda) \ge 0, \ \ \forall\, \lambda \in \Re^m. & \text{(1.9b)} \end{array} \right.$$

According to the definition of $L(u, \lambda)$ (see(1.6)),

$$\begin{array}{rl} & L(u, \lambda^*) - L(u^*, \lambda^*) \\ = & [\theta(u) - (\lambda^*)^T(\mathcal{A}u - b)] - [\theta(u^*) - (\lambda^*)^T(\mathcal{A}u^* - b)] \\ = & \theta(u) - \theta(u^*) + (u - u^*)^T(-\mathcal{A}^T\lambda^*) \end{array}$$

it follows from (1.9a) that

$$u^* \in \mathcal{U}, \ \ \theta(u) - \theta(u^*) + (u - u^*)^T(-\mathcal{A}^T\lambda^*) \ge 0, \ \ \forall\, u \in \mathcal{U}. \quad (1.10)$$

Similarly, for (1.9b), since

$$\begin{aligned} & L(u^*, \lambda^*) - L(u^*, \lambda) \\ & \quad = \quad [\theta(u^*) - (\lambda^*)^T(\mathcal{A}u^* - b)] - [\theta(u^*) - (\lambda)^T(\mathcal{A}u^* - b)] \\ & \quad = \quad (\lambda - \lambda^*)^T(\mathcal{A}u^* - b), \end{aligned}$$

thus we have

$$\lambda^* \in \Re^m, \quad (\lambda - \lambda^*)^T(\mathcal{A}u^* - b) \geq 0, \quad \forall\, \lambda \in \Re^m. \tag{1.11}$$

Notice that the expression (1.11) (the inner product of the vector $(\mathcal{A}u^* - b)$ with any vector is nonnegative) is equivalent to

$$\mathcal{A}u^* = b.$$

Writing (1.10) and (1.11) together, we get the following variational inequality:

$$\begin{cases} u^* \in \mathcal{U}, & \theta(u) - \theta(u^*) + (u - u^*)^T(-\mathcal{A}^T\lambda^*) \geq 0, \quad \forall\, u \in \mathcal{U}, \\ \lambda^* \in \Re^m, & \qquad\qquad\qquad\qquad (\lambda - \lambda^*)^T(\mathcal{A}u^* - b) \geq 0, \quad \forall\, \lambda \in \Re^m. \end{cases}$$

Using a more compact form, the saddle-point can be characterized as the solution of the following VI:

$$w^* \in \Omega, \quad \theta(u) - \theta(u^*) + (w - w^*)^T F(w^*) \geq 0, \quad \forall\, w \in \Omega, \tag{1.12a}$$

where

$$w = \begin{pmatrix} u \\ \lambda \end{pmatrix}, \quad F(w) = \begin{pmatrix} -\mathcal{A}^T \lambda \\ \mathcal{A}u - b \end{pmatrix} \quad \text{and} \quad \Omega = \mathcal{U} \times \Re^m. \tag{1.12b}$$

Setting $w = (u, \lambda^*)$ and $w = (u^*, \lambda)$ in (1.12), respectively, we get (1.10) and (1.11). Because F is a affine operator and

$$F(w) = \begin{pmatrix} 0 & -\mathcal{A}^T \\ \mathcal{A} & 0 \end{pmatrix} \begin{pmatrix} u \\ \lambda \end{pmatrix} - \begin{pmatrix} 0 \\ b \end{pmatrix}.$$

The matrix is skew-symmetric, we have

$$(w - \tilde{w})^T (F(w) - F(\tilde{w})) \equiv 0.$$

线性约束的凸优化问题 (1.5), 转换成了混合变分不等式 (1.12).

Two block separable convex optimization

We consider the following structured separable convex optimization

$$\min\{\theta_1(x) + \theta_2(y) \mid Ax + By = b,\, x \in \mathcal{X}, y \in \mathcal{Y}\}. \tag{1.13}$$

This is a special problem of (1.5) with

$$u = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathcal{U} = \mathcal{X} \times \mathcal{Y}, \quad \mathcal{A} = (A, B).$$

The Lagrangian function of the problem (1.13) is

$$L^{(2)}(x, y, \lambda) = \theta_1(x) + \theta_2(y) - \lambda^T(Ax + By - b).$$

The same analysis tells us that the saddle point is a solution of the following VI:

$$w^* \in \Omega, \quad \theta(u) - \theta(u^*) + (w - w^*)^T F(w^*) \geq 0, \quad \forall\, w \in \Omega. \tag{1.14}$$

where

$$u=\begin{pmatrix} x \\ y \end{pmatrix}, \quad \theta(u)=\theta_1(x)+\theta_2(y), \quad w=\begin{pmatrix} x \\ y \\ \lambda \end{pmatrix}, \tag{1.15a}$$

$$F(w)=\begin{pmatrix} -A^T\lambda \\ -B^T\lambda \\ Ax+By-b \end{pmatrix}, \quad \text{and} \quad \Omega=\mathcal{X}\times\mathcal{Y}\times\Re^m. \tag{1.15b}$$

The affine operator $F(w)$ has the form

$$F(w)=\begin{pmatrix} 0 & 0 & -A^T \\ 0 & 0 & -B^T \\ A & B & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ \lambda \end{pmatrix}-\begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}.$$

Again, due to the skew-symmetry, we have $(w-\tilde{w})^T(F(w)-F(\tilde{w}))\equiv 0$.

可分离线性约束凸优化问题 (1.13), 转换成了变分不等式 (1.14)–(1.15).

Convex optimization problem with three separable functions

$$\min\{\theta_1(x)+\theta_2(y)+\theta_3(z)\,|\,Ax+By+Cz=b,\ x\in\mathcal{X},y\in\mathcal{Y},z\in\mathcal{Z}\},$$

is a special problem of (1.5) with three blocks. The Lagrangian function is

$$L^{(3)}(x,y,z,\lambda)=\theta_1(x)+\theta_2(y)+\theta_3(z)-\lambda^T(Ax+By+Cz-b).$$

The same analysis tells us that the saddle point is a solution of the following VI:

$$w^*\in\Omega,\quad \theta(u)-\theta(u^*)+(w-w^*)^T F(w^*)\ge 0,\quad \forall\, w\in\Omega.$$

where $\theta(u)=\theta_1(x)+\theta_2(y)+\theta_3(z)$,

$$w=\begin{pmatrix} x\\ y\\ z\\ \lambda\end{pmatrix},\quad u=\begin{pmatrix} x\\ y\\ z\end{pmatrix},\quad F(w)=\begin{pmatrix} -A^T\lambda\\ -B^T\lambda\\ -C^T\lambda\\ Ax+By+Cz-b\end{pmatrix},$$

and $$\Omega=\mathcal{X}\times\mathcal{Y}\times\mathcal{Z}\times\Re^m.$$

线性约束的凸优化问题, 都转换成了变分不等式. 问题归结为求一个鞍点.

2 Proximal point algorithms and its Beyond

引理 1 *Let the vectors* $a, b \in \Re^n$, $H \in \Re^{n\times n}$ *be a positive definite matrix. If* $b^T H(a-b) \geq 0$, *then we have*

$\|x\|^2 = x^T x, \quad \|x\|_H^2 = x^T H x.$

$$\|b\|_H^2 \leq \|a\|_H^2 - \|a-b\|_H^2. \tag{2.1}$$

The assertion follows from $\|a\|_H^2 = \|b+(a-b)\|_H^2 \geq \|b\|_H^2 + \|a-b\|_H^2.$

2.1 Proximal point algorithms for convex optimization

Convex Optimization Now, let us consider the *simple* convex optimization

$$\min\{\theta(x) + f(x) \mid x \in \mathcal{X}\}, \tag{2.2}$$

where $\theta(x)$ and $f(x)$ are convex but $\theta(x)$ is not necessary smooth, $\mathcal{X}$ is a closed convex set. For solving (2.2), the k-th iteration of the proximal point algorithm (abbreviated to PPA) [8, 10] begins with a given x^k, offers the new iterate x^{k+1} via the recursion

邻近点算法
$$x^{k+1} = \operatorname{argmin}\{\theta(x) + f(x) + \frac{r}{2}\|x - x^k\|^2 \mid x \in \mathcal{X}\}. \tag{2.3}$$

Since x^{k+1} is the optimal solution of (2.3), it follows from Lemma 1 that

$$\theta(x) - \theta(x^{k+1}) + (x - x^{k+1})^T \{\nabla f(x^{k+1}) + r(x^{k+1} - x^k)\} \geq 0, \ \forall x \in \mathcal{X}. \tag{2.4}$$

Setting $x = x^*$ in the above inequality, it follows that

$$(x^{k+1} - x^*)^T r(x^k - x^{k+1}) \geq \theta(x^{k+1}) - \theta(x^*) + (x^{k+1} - x^*)^T \nabla f(x^{k+1}).$$

Because f isconvex, $(x^{k+1} - x^*)^T \nabla f(x^{k+1}) \geq (x^{k+1} - x^*)^T \nabla f(x^*)$, it follows that

$$\begin{aligned} &\theta(x^{k+1}) - \theta(x^*) + (x^{k+1} - x^*)^T \nabla f(x^{k+1}) \\ \geq \quad &\theta(x^{k+1}) - \theta(x^*) + (x^{k+1} - x^*)^T \nabla f(x^*) \geq 0 \end{aligned}$$

and consequently,

$$(x^{k+1} - x^*)^T (x^k - x^{k+1}) \geq 0. \tag{2.5}$$

Let $a = x^k - x^*$ and $b = x^{k+1} - x^*$ and using Lemma 1, we obtain

PPA 算法的收缩性质

$$\|\boldsymbol{x}^{k+1} - \boldsymbol{x}^*\|^2 \leq \|\boldsymbol{x}^k - \boldsymbol{x}^*\|^2 - \|\boldsymbol{x}^k - \boldsymbol{x}^{k+1}\|^2, \tag{2.6}$$

which is the nice convergence property of Proximal Point Algorithm.

The residue sequence $\{\|x^k - x^{k+1}\|\}$ is also monotonically no-increasing.

Proof. Replacing $k+1$ in (2.4) with k, we get

$$\theta(x) - \theta(x^k) + (x - x^k)^T\{\nabla f(x^k) + r(x^k - x^{k-1})\} \geq 0, \quad \forall\, x \in \mathcal{X}.$$

Let $x = x^{k+1}$ in the above inequality, it follows that

$$\theta(x^{k+1}) - \theta(x^k) + (x^{k+1} - x^k)^T\{\nabla f(x^k) + r(x^k - x^{k-1})\} \geq 0. \qquad (2.7)$$

Setting $x = x^k$ in (2.4), we become

$$\theta(x^k) - \theta(x^{k+1}) + (x^k - x^{k+1})^T\{\nabla f(x^{k+1}) + r(x^{k+1} - x^k)\} \geq 0. \qquad (2.8)$$

Adding (2.7) and (2.8) and using $(x^k - x^{k+1})^T[\nabla f(x^k) - \nabla f(x^{k+1})] \geq 0$, we get

$$(x^k - x^{k+1})^T\{(x^{k-1} - x^k) - (x^k - x^{k+1})\} \geq 0. \qquad (2.9)$$

Setting $a = x^{k-1} - x^k$ and $b = x^k - x^{k+1}$ in (2.9) and using (2.1), we obtain

$$\boldsymbol{\|x^k - x^{k+1}\|^2 \leq \|x^{k-1} - x^k\|^2 - \|(x^{k-1} - x^k) - (x^k - x^{k+1})\|^2}. \qquad (2.10)$$

We write the problem (2.2) and its PPA (2.3) in VI form

For the optimization problem (2.2) , namely, $\min\{\theta(x)+f(x)\,|\,x\in\mathcal{X}\}$, the equivalent variational inequality form is

$$x^*\in\mathcal{X},\quad \theta(x)-\theta(x^*)+(x-x^*)^T\nabla f(x^*)\ge 0,\quad \forall\, x\in\mathcal{X}. \tag{2.11}$$

For solving the problem (2.2), the PPA is

$$x^{k+1}=\text{Argmin}\{\theta(x)+f(x)+\frac{r}{2}\|x-x^k\|^2\;|\;x\in\mathcal{X}\}.$$

variational inequality form of the k-th iteration of the PPA (see (2.4)) is:

$$\begin{aligned} x^{k+1}\in\mathcal{X},\quad \theta(x)-\theta(x^{k+1})&+(x-x^{k+1})^T\nabla f(x^{k+1})\\ &\ge (x-x^{k+1})^T r(x^k-x^{k+1}),\quad \forall\, x\in\mathcal{X}. \end{aligned} \tag{2.12}$$

PPA 通过求解一系列的 (2.3), 求得 (2.2) 的解, 采用的是步步为营的策略.

The solution of (2.12) is Proximal Point, it has the contraction property (2.6).

2.2 Preliminaries of PPA for Variational Inequalities

The optimal condition of the linearly constrained convex optimization is characterized as a mixed monotone variational inequality: 变分不等式

$$w^* \in \Omega, \quad \theta(u) - \theta(u^*) + (w - w^*)^T F(w^*) \ge 0, \quad \forall\, w \in \Omega. \tag{2.13}$$

PPA for VI (2.13) **in H-norm (定义)** For given w^k and $H \succ 0$, find w^{k+1},

$$\begin{aligned} w^{k+1} \in \Omega, \quad \theta(u) - \theta(u^{k+1}) &+ (w - w^{k+1})^T F(w^{k+1}) \\ &\ge (w - w^{k+1})^T H(w^k - w^{k+1}), \quad \forall\, w \in \Omega, \end{aligned} \tag{2.14}$$

邻近点算法

w^{k+1} is called the proximal point of the k-th iteration for the problem (2.13).

(2.14) 是求解 VI (2.13) 的 PPA 算法的定义. 第二讲就会用例子说明这是容易做到的.

✠ w^{k+1} **is the solution of** (2.13) **if and only if** $w^k = w^{k+1}$ ✠

Setting $w = w^*$ in (2.14), we obtain

$$(w^{k+1} - w^*)^T H(w^k - w^{k+1}) \ge \theta(u^{k+1}) - \theta(u^*) + (w^{k+1} - w^*)^T F(w^{k+1}).$$

Note that (see the structure of $F(w)$ in (1.12b))

$$(w^{k+1}-w^*)^T F(w^{k+1}) = (w^{k+1}-w^*)^T F(w^*),$$

and consequently (by using (2.13)) we obtain

$$(w^{k+1}-w^*)^T H(w^k-w^{k+1}) \geq \theta(u^{k+1})-\theta(u^*)+(w^{k+1}-w^*)^T F(w^*) \geq 0.$$

Thus, we have

$$(w^{k+1}-w^*)^T H(w^k-w^{k+1}) \geq 0. \tag{2.15}$$

By setting $\boldsymbol{a = w^k - w^*}$ and $\boldsymbol{b = w^{k+1} - w^*}$,

the inequality (2.15) means that $\boldsymbol{b^T H(a-b) \geq 0}$.

By using Lemma 1, we obtain

$$\|\boldsymbol{w^{k+1}-w^*}\|_{\boldsymbol{H}}^{\boldsymbol{2}} \leq \|\boldsymbol{w^k-w^*}\|_{\boldsymbol{H}}^{\boldsymbol{2}} - \|\boldsymbol{w^k-w^{k+1}}\|_{\boldsymbol{H}}^{\boldsymbol{2}}. \tag{2.16}$$

We get the nice convergence property of Proximal Point Algorithm.

请证明：$\|w^k-w^{k+1}\|^2 \leq \|w^{k-1}-w^k\|^2$, 即序列 $\{\|w^k-w^{k+1}\|_H\}$ 是单调不增的.

2.3 Variants of PPA for Variational Inequalities

Let v be a sub-vector of w. The k-th iteration begins with given v^k . v 核心变量

PPA for VI (2.13) in H-norm For given v^k and $H \succ 0$, find w^{k+1},

$$w^{k+1} \in \Omega, \quad \theta(u) - \theta(u^{k+1}) + (w - w^{k+1})^T F(w^{k+1}) \\ \geq (v - v^{k+1})^T H(v^k - v^{k+1}), \quad \forall\, w \in \Omega, \tag{2.17}$$

w^{k+1} is called the proximal point of the k-th iteration for the problem (2.13).

✠ w^{k+1} **is the solution of** (2.13) **if and only if** $v^k = v^{k+1}$ ✠

In this case, v is called the essential variables of w. In addition, we define

$$\mathcal{V}^* = \{v^* \text{ is a subvector of } w^* \mid w^* \in \Omega^*\}.$$

Setting $w = w^*$ in (2.17), we obtain

$$(v^{k+1} - v^*)^T H(v^k - v^{k+1}) \geq \theta(u^{k+1}) - \theta(u^*) + (w^{k+1} - w^*)^T F(w^{k+1}).$$

Note that (see the structure of $F(w)$ in (1.12b))

$$(w^{k+1}-w^*)^T F(w^{k+1}) = (w^{k+1}-w^*)^T F(w^*),$$

and consequently (by using (2.13)) we obtain

$$(v^{k+1}-v^*)^T H(v^k - v^{k+1}) \ge \theta(u^{k+1}) - \theta(u^*) + (w^{k+1}-w^*)^T F(w^*) \ge 0.$$

Thus, we have

$$(v^{k+1}-v^*)^T H(v^k - v^{k+1}) \ge 0. \tag{2.18}$$

By using Lemma 1, we obtain

$$\|\boldsymbol{v}^{k+1}-\boldsymbol{v}^*\|_{\boldsymbol{H}}^2 \le \|\boldsymbol{v}^k-\boldsymbol{v}^*\|_{\boldsymbol{H}}^2 - \|\boldsymbol{v}^k-\boldsymbol{v}^{k+1}\|_{\boldsymbol{H}}^2. \tag{2.19}$$

We get the nice convergence property of Proximal Point Algorithm.

The residue sequence $\{\|v^k - v^{k+1}\|_H\}$ is also monotonically no-increasing.

序列 $\{\|v^k - v^{k+1}\|_H\}$ 是单调不增的. $\|v^k - v^{k+1}\|_H^2 \le \|v^{k-1} - v^k\|_H^2$.

3 Augmented Lagrangian Method (ALM)

We consider the convex optimization, namely

$$\min\{\theta(u) \mid \mathcal{A}u = b, \; u \in \mathcal{U}\}. \tag{3.1}$$

The related variational inequality of the saddle point of the Lagrangian function is

$$w^* \in \Omega, \quad \theta(u) - \theta(u^*) + (w - w^*)^T F(w^*) \geq 0, \quad \forall\, w \in \Omega. \tag{3.2a}$$

where

$$w = \begin{pmatrix} u \\ \lambda \end{pmatrix}, \quad F(w) = \begin{pmatrix} -\mathcal{A}^T\lambda \\ \mathcal{A}u - b \end{pmatrix} \quad \text{and} \quad \Omega = \mathcal{U} \times \Re^m. \tag{3.2b}$$

Augmented Lagrangian Method

The augmented Lagrangian function of the problem (3.1) is

$$\mathcal{L}_\beta(u, \lambda) = \theta(u) - \lambda^T(\mathcal{A}u - b) + \frac{\beta}{2}\|\mathcal{A}u - b\|^2,$$

The k-th iteration of the **Augmented Lagrangian Method** [7, 9] begins with a given λ^k, obtain $w^{k+1} = (u^{k+1}, \lambda^{k+1})$ via

$$\text{(ALM)} \quad \left\{ \begin{array}{ll} u^{k+1} = \arg\min\big\{\mathcal{L}_\beta(u, \lambda^k) \;\big|\; u \in \mathcal{U}\big\}, & \text{(3.3a)} \\ \lambda^{k+1} = \lambda^k - \beta(\mathcal{A}u^{k+1} - b). & \text{(3.3b)} \end{array} \right.$$

In (3.3), u^{k+1} is only a computational result of (3.3a) from given λ^k, it is called the intermediate variable. In order to start the k-th iteration of ALM, we need only to have λ^k and thus we call it as the essential variable.

The subproblem (3.3a) is a problem of mathematical form

$$\min\{\theta(u) + \tfrac{\beta}{2}\|\mathcal{A}u - p^k\|^2 \mid u \in \mathcal{U}\} \tag{3.4}$$

where $\beta > 0$ is a given scalar and $p^k = b + \frac{1}{\beta}\lambda^k$.

Assumption: The solution of problem (3.4) has closed-form solution or can be efficiently computed with a high precision.

Changing the constant term in the objective function does not affect the solution of the optimization problem. Thus,

$$\begin{aligned} u^{k+1} &\in \operatorname{argmin}\{\mathcal{L}_\beta(u,\lambda^k) \mid u \in \mathcal{U}\} \\ &= \operatorname{argmin}\{\theta(u) - (\lambda^k)^T \mathcal{A}u + \tfrac{\beta}{2}\|\mathcal{A}u - b\|^2 \mid u \in \mathcal{U}\} \\ &= \operatorname{argmin}\{\theta(u) + \tfrac{\beta}{2}\|(\mathcal{A}u - b) - \tfrac{1}{\beta}\lambda^k\|^2 \mid u \in \mathcal{U}\} \end{aligned}$$

According to Lemma 1, the optimal condition of (3.3a) is $u^{k+1} \in \mathcal{U}$ and

$$\theta(u) - \theta(u^{k+1}) + (u - u^{k+1})^T\{-\mathcal{A}^T\lambda^k + \beta\mathcal{A}^T(\mathcal{A}u^{k+1} - b)\} \ge 0, \ \forall u \in \mathcal{U}.$$

Because $\lambda^k - \beta(\mathcal{A}u^{k+1} - b) = \lambda^{k+1}$, the above VI can be written as

$$u^{k+1} \in \mathcal{U}, \quad \theta(u) - \theta(u^{k+1}) + (u - u^{k+1})^T\{-\mathcal{A}^T\lambda^{k+1}\} \ge 0, \ \forall u \in \mathcal{U}. \tag{3.5}$$

The update form (3.3b) is

$$(\mathcal{A}u^{k+1} - b) + \frac{1}{\beta}(\lambda^{k+1} - \lambda^k) = 0.$$

and it is equivalent to

$$(\lambda - \lambda^{k+1})^T(\mathcal{A}u^{k+1} - b) \ge (\lambda - \lambda^{k+1})^T\frac{1}{\beta}(\lambda^k - \lambda^{k+1}), \quad \forall \lambda \in \Re^m. \tag{3.6}$$

Combining VI's (3.5) and (3.6), we get

$$\theta(u) - \theta(u^{k+1}) + \begin{pmatrix} u - u^{k+1} \\ \lambda - \lambda^{k+1} \end{pmatrix}^T \begin{pmatrix} -\mathcal{A}^T\lambda^{k+1} \\ \mathcal{A}u^{k+1} - b \end{pmatrix} \geq (\lambda - \lambda^{k+1})^T \frac{1}{\beta}(\lambda^k - \lambda^{k+1}),$$

for all $w = (u, \lambda) \in \Omega$. Using the notations in (3.2), we get the compact form

$$\begin{aligned} &\theta(u) - \theta(u^{k+1}) + (w - w^{k+1})^T F(w^{k+1}) \\ &\quad \geq \quad (\lambda - \lambda^{k+1})^T \frac{1}{\beta}(\lambda^k - \lambda^{k+1}), \ \forall w \in \Omega. \end{aligned} \tag{3.7}$$

This is the PPA form (2.17) in which

$$v = \lambda \qquad \text{and} \qquad H = \frac{1}{\beta} I_m.$$

The related contraction inequality (2.19) becomes

$$\|\lambda^{k+1} - \lambda^*\|^2_{\frac{1}{\beta}I_m} \leq \|\lambda^k - \lambda^*\|^2_{\frac{1}{\beta}I_m} - \|\lambda^k - \lambda^{k+1}\|^2_{\frac{1}{\beta}I_m}$$

or

$$\|\lambda^{k+1} - \lambda^*\|^2 \leq \|\lambda^k - \lambda^*\|^2 - \|\lambda^k - \lambda^{k+1}\|^2. \tag{3.8}$$

The above inequality is the key for the convergence proof of the ALM.

4 The relaxed PPA （延伸的邻近点算法）

We shall maintain our focus on the monotone variational inequality (2.13), namely,

$$w^* \in \Omega, \quad \theta(u) - \theta(u^*) + (w - w^*)^T F(w^*) \geq 0, \quad \forall\, w \in \Omega.$$

The PPA form (2.17) reads as

$$\begin{aligned} w^{k+1} \in \Omega, \quad \theta(u) - \theta(u^{k+1}) + (w - w^{k+1})^T F(w^{k+1}) \\ \geq (v - v^{k+1})^T H(v^k - v^{k+1}), \quad \forall\, w \in \Omega. \end{aligned}$$

Set the output of the above VI as $\tilde{w}^k$, we have

$$\begin{aligned} \tilde{w}^k \in \Omega, \quad \theta(u) - \theta(\tilde{u}^k) + (w - \tilde{w}^k)^T F(\tilde{w}^k) \\ \geq (v - \tilde{v}^k)^T H(v^k - \tilde{v}^k), \quad \forall\, w \in \Omega. \end{aligned} \tag{4.1}$$

Setting $w = w^*$ in (4.1), we obtain

$$(\tilde{v}^k - v^*)^T H(v^k - \tilde{v}^k) \geq \theta(\tilde{u}^k) - \theta(u^*) + (\tilde{w}^k - w^*)^T F(\tilde{w}^k). \tag{4.2}$$

Applying (see (1.12b)) the identity

$$(\tilde{w}^k - w^*)^T F(\tilde{w}^k) \equiv (\tilde{w}^k - w^*)^T F(w^*)$$

to (4.2), we obtain

$$(\tilde{v}^k - v^*)^T H(v^k - \tilde{v}^k) \geq \theta(\tilde{u}^k) - \theta(u^*) + (\tilde{w}^k - w^*)^T F(w^*).$$

Because RHS of the above inequality is , we have

$$(\tilde{v}^k - v^*)^T H(v^k - \tilde{v}^k) \geq 0.$$

We write it as

$$\{(v^k - v^*) - (v^k - \tilde{v}^k)\}^T H(v^k - \tilde{v}^k) \geq 0$$

and thus

$$(v^k - v^*)^T H(v^k - \tilde{v}^k) \geq \|v^k - \tilde{v}^k\|_H^2, \quad \forall v^* \in \mathcal{V}^*. \tag{4.3}$$

The inequality (4.3) means that $(v^k - \tilde{v}^k)$ is the ascent direction of the unknown distance function $\frac{1}{2}\|v - v^*\|_H^2$ at the point v^k.

$$\Big\langle \nabla\big(\tfrac{1}{2}\|v - v^*\|_H^2\big)\Big|_{v=v^k},\ (v^k - \tilde{v}^k)\Big\rangle \ge \|v^k - \tilde{v}^k\|_H^2, \quad \forall\, v^* \in \mathcal{V}^*.$$

The task of the algorithm is to produce a decreasing sequence $\{\|v^k - v^*\|_H^2\}$.
Set

$$v^{k+1}(\alpha) = v^k - \alpha(v^k - \tilde{v}^k) \tag{4.4}$$

which is an α dependent new iterate. It is clear we want to maximize

$$\vartheta(\alpha) = \|v^k - v^*\|_H^2 - \|v^{k+1}(\alpha) - v^*\|_H^2. \tag{4.5}$$

Note that

$$\begin{aligned} \vartheta(\alpha) &= \|v^k - v^*\|_H^2 - \|(v^k - v^*) - \alpha(v^k - \tilde{v}^k)\|_H^2 \\ &= 2\alpha(v^k - v^*)^T H(v^k - \tilde{v}^k) - \alpha^2\|v^k - \tilde{v}^k\|_H^2 \end{aligned} \tag{4.6}$$

is a quadratic function of α.

We can not directly maximize $\vartheta(\alpha)$ in (4.6) because the coefficient of the linear term $2(v^k - v^*)^T H(v^k - \tilde{v}^k)$ contains the unknown solution v^*.

Using (4.3), from (4.6) we get

$$\vartheta(\alpha) \geq 2\alpha\|v^k - \tilde{v}^k\|_H^2 - \alpha^2\|v^k - \tilde{v}^k\|_H^2 \tag{4.7}$$

Set

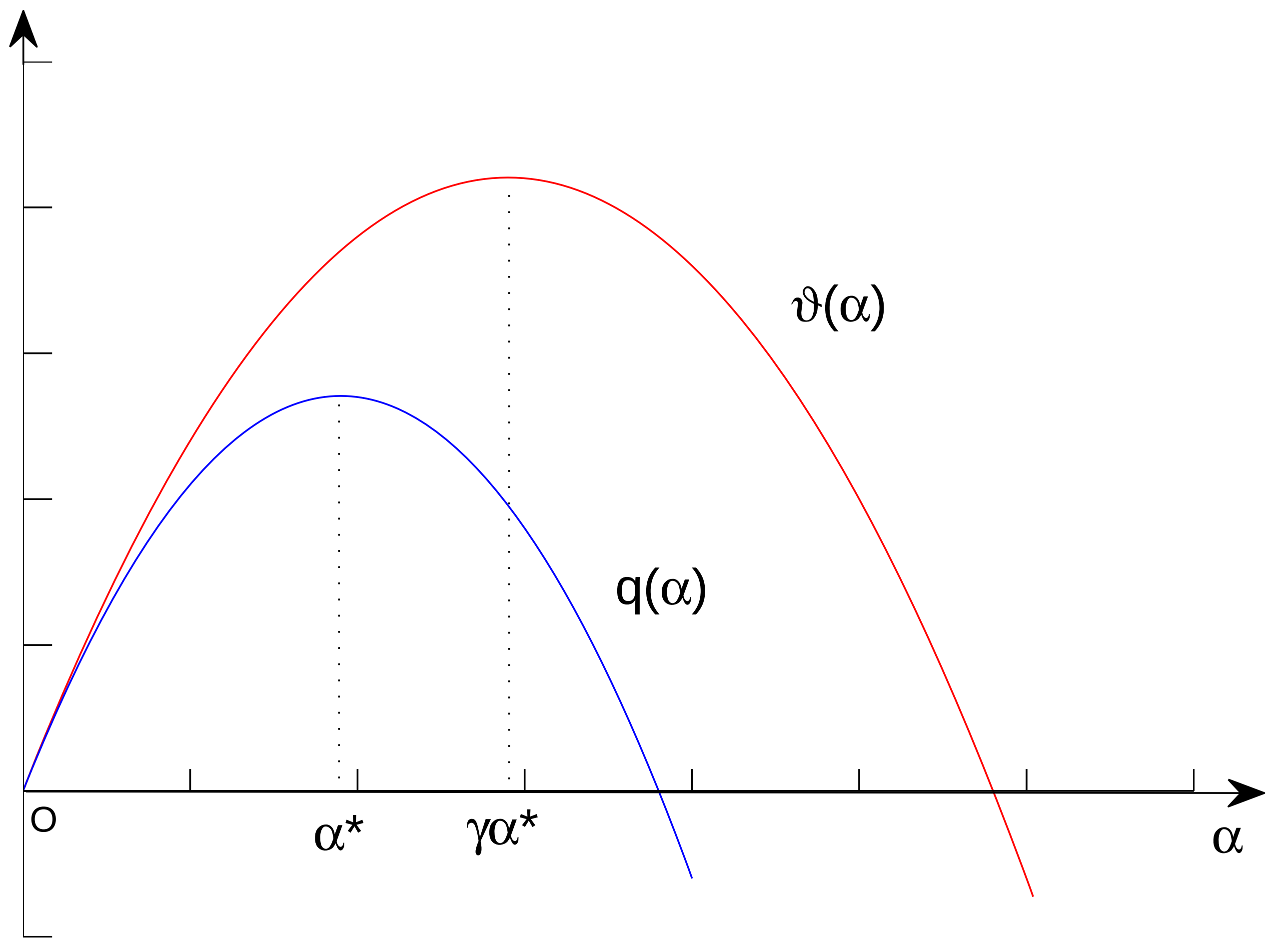
$$q(\alpha) = (2\alpha - \alpha^2)\|v^k - \tilde{v}^k\|_H^2, \tag{4.8}$$

which is a quadratic lower-bound function of $\vartheta(\alpha)$. The quadratic function $q(\alpha)$ reaches its maximum at $\alpha^* \equiv 1$.

$$v^{k+1} = v^k - \gamma(v^k - \tilde{v}^k), \quad \gamma \in (0,2) \tag{4.9}$$

The generated sequence $\{v^k\}$ satisfies

$$\|v^{k+1} - v^*\|_H^2 \leq \|v^k - v^*\|_H^2 - \gamma(2-\gamma)\|v^k - \tilde{v}^k\|_H^2. \tag{4.10}$$



取 $\gamma \in [1, 2)$ 的示意图

这一讲是预备知识. 要求读者理解（或者是先承认）优化问题拉格朗日函数的鞍点和变分不等式 (VI) 解点的等价的关系, 以及 PPA 算法的定义及收缩性质.

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