ADMM 类算法的一些最新进展（I）

凸优化问题分裂收缩算法的统一框架

变分不等式(VI) 是瞎子爬山的数学表达形式

邻近点算法(PPA) 是步步为营 稳扎稳打的求解方法.

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我的几类主要研究工作的分类论文和简要介绍（附阅读建议）

1. 变分不等式的投影收缩算法(Projection-Contraction Methods)
2. 两个可分离函数的乘子交替方向法(ADMM)
3. 多个可分离函数的交替方向类算法(ADMM-Like Methods)
4. 变分不等式框架下的邻近点算法(VI & PPA)

My Thinkings:

1. 关门感想
2. 说说我的主要研究兴趣 — 兼谈华罗庚推广优选法对我的影响
3. 说说我的主要研究兴趣(续) --- 我们在ADMM类方法的主要工作
4. 古稀回首
5. 两页纸简述我职业生涯中的主要研究工作

My Talks:

1. 从变分不等式的投影收缩算法到凸规划的分裂收缩算法 — 我研究生涯的来龙去脉
2. 生活理念对设计优化分裂算法的帮助 — 以改造 ADMM 求解三个可分离算子问题为例
3. 凸优化的分裂收缩算法 — 变分不等式为工具的统一框架 (适合打印的 综合文本)
4. 从商业谈判的角度看一些优化方法的设计 — 从 min-max 问题的求解谈起
5. 我和乘子交替方向法(ADMM) 的20年 — 2017 年5月全国数学规划会议报告 综述版本
6. 图像处理中的凸优化问题及其相应的分裂收缩算法 — ISICDM会议报告I 报告II 报告III
7. 介绍：构造求解凸优化的分裂收缩算法—用好变分不等式和邻近点算法两大法宝
8. 线性化ALM-ADMM等方法中的"替代"参数严重影响收敛速度—提升空间有多少？
9. 被 S. Becker 誉为 Very Simple yet Powerful Technique 的主要思想-应用及新的进展

我的报告的 PDF 文件, 都可以在我的主页上查到.

连续优化中一些代表性数学模型

1. 简单约束问题 $\min\{f(x) \mid x \in \mathcal{X}\}$ 其中 $\mathcal{X}$ 是一个凸集.

2. 线性约束的凸优化问题 $\min\{\theta(x) | Ax = b\,(\text{or} \geq b),\, x \in \mathcal{X}\}$

3. min-max 问题 $\min_{x\in\mathcal{X}} \max_{y\in\mathcal{Y}}\{\Phi(x, y = \theta_1(x) - y^T Ax - \theta_2(y)\}$

4. 结构型凸优化 $\min\{\theta_1(x) + \theta_2(y) | Ax + By = b,\, x \in \mathcal{X},\, y \in \mathcal{Y}\}$

5. 多块可分离凸优化 $\min\{\sum_{i=1}^p \theta_i(x_i) | \sum_{i=1}^p A_i x_i = b, x_i \in \mathcal{X}_i\}$

变分不等式(VI) 是瞎子爬山的数学表达形式

邻近点算法(PPA) 是步步为营 稳扎稳打的求解方法.

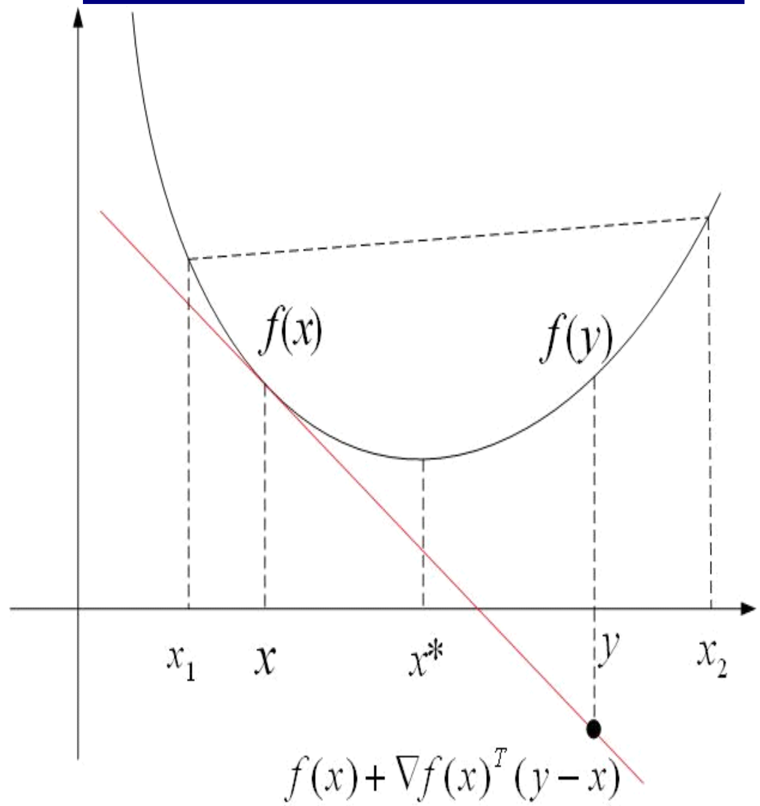
变分不等式和邻近点算法是分析和设计凸优化方法的两大法宝.

凸函数的定义和基本性质

A function $f(x)$ is convex iff

$$f((1-\theta)x+\theta y) \leq (1-\theta)f(x)+\theta f(y)$$

$$\forall \theta \in [0,1].$$



Convex function

Properties of convex function

- $f \in \mathcal{C}^1$. f is convex iff

  $$f(y) - f(x) \geq \nabla f(x)^T(y-x).$$

  Thus, we have also

  $$f(x) - f(y) \geq \nabla f(y)^T(x-y).$$

- Adding above two inequalities, we get

  $$(y-x)^T(\nabla f(y) - \nabla f(x)) \geq 0.$$

- $f \in \mathcal{C}^1$, ∇f is monotone. $f \in \mathcal{C}^2$, $\nabla^2 f(x)$ is positive semi-definite.

- Any local minimum of a convex function is a global minimum.

1 Optimization problem and VI

1.1 Differential convex optimization in Form of VI

Let $\Omega \subset \Re^n$, we consider the convex minimization problem

$$\min\{f(x) \mid x \in \Omega\}. \tag{1.1}$$

What is the first-order optimal condition ?

$x^* \in \Omega^* \quad \Leftrightarrow \quad x^* \in \Omega$ and any feasible direction is not a descent one.

Optimal condition in variational inequality form

- $S_d(x^*) = \{s \in \Re^n \mid s^T \nabla f(x^*) < 0\} \;=\;$ Set of the descent directions.
- $S_f(x^*) = \{s \in \Re^n \mid s = x - x^*,\; x \in \Omega\} \;=\;$ Set of feasible directions.

$x^* \in \Omega^* \quad \Leftrightarrow \quad x^* \in \Omega$ **and** $S_f(x^*) \cap S_d(x^*) = \emptyset$.

瞎子爬山判定山顶的准则是: 所有可行方向都不再是上升方向

The optimal condition can be presented in a variational inequality (VI) form:

$$x^* \in \Omega, \quad (x - x^*)^T F(x^*) \geq 0, \quad \forall x \in \Omega, \tag{1.2}$$

where $F(x) = \nabla f(x)$. For general VI, F is an operator from $\Re^n$ into itself.

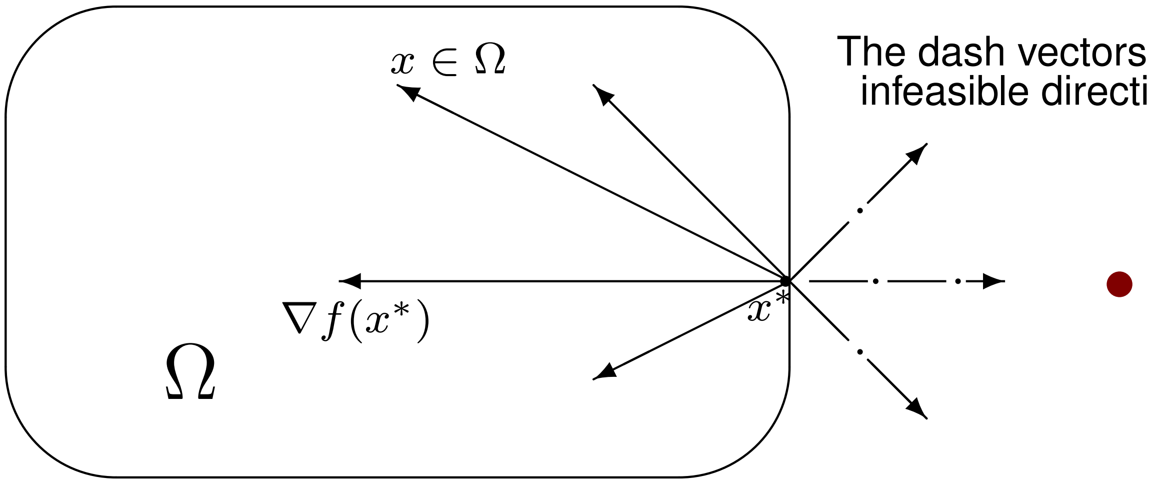


Fig. 1.1 Differential Convex Optimization and VI

Since $f(x)$ is a convex function, we have

$f(y) \geq f(x) + \nabla f(x)^T (y - x)$ and thus $(x - y)^T (\nabla f(x) - \nabla f(y)) \geq 0.$

We say the gradient ∇f of the convex function f is a monotone operator.

通篇我们需要用到的大学数学　主要是基于微积分学的一个引理

$$\min\{\theta(x)|x\in\mathcal{X}\},\quad x^*\in\mathcal{X},\quad \theta(x)-\theta(x^*)\geq 0,\quad \forall x\in\mathcal{X};$$

$$\min\{f(x)|x\in\mathcal{X}\},\quad x^*\in\mathcal{X},\quad (x-x^*)^T\nabla f(x^*)\geq 0,\quad \forall x\in\mathcal{X}.$$

上面的凸优化最优性条件是最基本的, 合在一起就是下面的引理:

Lemma 1 *Let* $\mathcal{X}\subset\Re^n$ *be a closed convex set,* $\theta(x)$ *and* $f(x)$ *be convex functions and* $f(x)$ *is differentiable. Assume that the solution set of the minimization problem* $\min\{\theta(x)+f(x)\,|\,x\in\mathcal{X}\}$ *is nonempty. Then,*

$$x^*\in\arg\min\{\theta(x)+f(x)\,|\,x\in\mathcal{X}\} \tag{1.3a}$$

if and only if

$$x^*\in\mathcal{X},\quad \theta(x)-\theta(x^*)+(x-x^*)^T\nabla f(x^*)\geq 0,\quad \forall x\in\mathcal{X}. \tag{1.3b}$$

这样, 我们就把优化问题 (1.3a), 转换成了变分不等式 (1.3b).

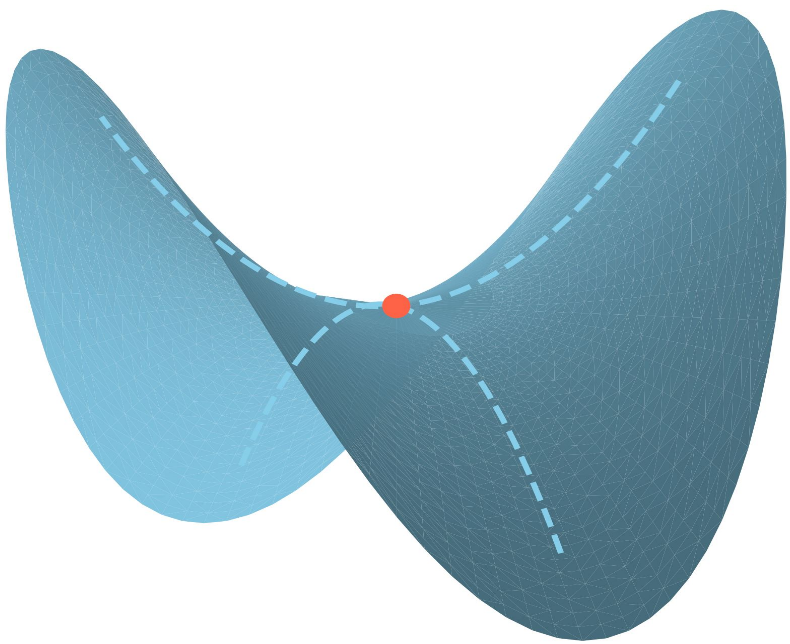
Linearly constrained Optimization in form of VI

We consider the linearly constrained convex optimization problem

$$\min\{\theta(u) \mid \mathcal{A}u = b, \; u \in \mathcal{U}\}. \tag{1.4}$$

The Lagrange function of (4.3) is

$$L(u, \lambda) = \theta(u) - \lambda^T(\mathcal{A}u - b), \qquad (u, \lambda) \in \mathcal{U} \times \Re^m. \tag{1.5}$$



Saddle Point of the Lagrange Function

A pair of $(u^*, \lambda^*) \in \mathcal{U} \times \Re^m$ is called a saddle point if

$$L_{\lambda \in \Re^m}(u^*, \lambda) \le L(u^*, \lambda^*) \le L_{u \in \mathcal{U}}(u, \lambda^*).$$

The above inequalities can be written as

$$\begin{cases} u^* \in \mathcal{U}, \quad L(u, \lambda^*) - L(u^*, \lambda^*) \ge 0, \;\; \forall\, u \in \mathcal{U}, & \text{(1.6a)} \\ \lambda^* \in \Re^m, \; L(u^*, \lambda^*) - L(u^*, \lambda) \ge 0, \;\; \forall\, \lambda \in \Re^m. & \text{(1.6b)} \end{cases}$$

According to the definition of $L(u, \lambda)$ (see(4.4)),

$$\begin{aligned} & L(u, \lambda^*) - L(u^*, \lambda^*) \\ & \quad = \quad [\theta(u) - (\lambda^*)^T(\mathcal{A}u - b)] - [\theta(u^*) - (\lambda^*)^T(\mathcal{A}u^* - b)] \\ & \quad = \quad \theta(u) - \theta(u^*) + (u - u^*)^T(-\mathcal{A}^T\lambda^*) \end{aligned}$$

it follows from (1.6a) that

$$u^* \in \mathcal{U}, \;\; \theta(u) - \theta(u^*) + (u - u^*)^T(-\mathcal{A}^T\lambda^*) \ge 0, \;\; \forall\, u \in \mathcal{U}. \quad \text{(1.7)}$$

Similarly, for (1.6b), since

$$\begin{aligned} & L(u^*, \lambda^*) - L(u^*, \lambda) \\ = \quad & [\theta(u^*) - (\lambda^*)^T(\mathcal{A}u^* - b)] - [\theta(u^*) - (\lambda)^T(\mathcal{A}u^* - b)] \\ = \quad & (\lambda - \lambda^*)^T(\mathcal{A}u^* - b), \end{aligned}$$

we have

$$\lambda^* \in \Re^m, \;\; (\lambda - \lambda^*)^T(\mathcal{A}u^* - b) \geq 0, \;\; \forall\, \lambda \in \Re^m. \tag{1.8}$$

Notice that the above expression is equivalent to

$$\mathcal{A}u^* = b.$$

Writing (1.7) and (1.8) together, we get the following variational inequality:

$$\left\{ \begin{array}{lll} u^* \in \mathcal{U}, & \theta(u) - \theta(u^*) + (u - u^*)^T(-\mathcal{A}^T\lambda^*) \geq 0, & \forall\, u \in \mathcal{U}, \\ \lambda^* \in \Re^m, & (\lambda - \lambda^*)^T(\mathcal{A}u^* - b) \geq 0, & \forall\, \lambda \in \Re^m. \end{array} \right.$$

Using a more compact form, the saddle-point can be characterized as the solution of the following VI:

$$w^* \in \Omega, \quad \theta(u) - \theta(u^*) + (w - w^*)^T F(w^*) \geq 0, \quad \forall\, w \in \Omega. \tag{1.9a}$$

where

$$w = \begin{pmatrix} u \\ \lambda \end{pmatrix}, \quad F(w) = \begin{pmatrix} -\mathcal{A}^T\lambda \\ \mathcal{A}u - b \end{pmatrix} \text{ and } \quad \Omega = \mathcal{U} \times \Re^m. \tag{1.9b}$$

Because F is a affine operator and

$$F(w) = \begin{pmatrix} 0 & -\mathcal{A}^T \\ \mathcal{A} & 0 \end{pmatrix} \begin{pmatrix} u \\ \lambda \end{pmatrix} - \begin{pmatrix} 0 \\ b \end{pmatrix}.$$

The matrix is skew-symmetric, we have

$$(w - \tilde{w})^T (F(w) - F(\tilde{w})) \equiv 0.$$

线性约束的凸优化问题 (4.3), 转换成了混合变分不等式 (1.9).

Convex optimization problem with two separable functions

$$\min\{\theta_1(x) + \theta_2(y) \mid Ax + By = b,\ x \in \mathcal{X}, y \in \mathcal{Y}\}. \tag{1.10}$$

This is a special problem of (4.3) with

$$u = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathcal{U} = \mathcal{X} \times \mathcal{Y}, \quad \mathcal{A} = (A, B).$$

The Lagrangian function of the problem (1.10) is

$$L^{[2]}(x, y, \lambda) = \theta_1(x) + \theta_2(y) - \lambda^T(Ax + By - b).$$

The same analysis tells us that the saddle point is a solution of the following VI:

$$w^* \in \Omega, \quad \theta(u) - \theta(u^*) + (w - w^*)^T F(w^*) \geq 0, \quad \forall\, w \in \Omega. \tag{1.11}$$

where

$$u = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \theta(u) = \theta_1(x) + \theta_2(y), \quad w = \begin{pmatrix} x \\ y \\ \lambda \end{pmatrix}, \tag{1.12a}$$

$$F(w) = \begin{pmatrix} -A^T\lambda \\ -B^T\lambda \\ Ax + By - b \end{pmatrix}, \quad \text{and} \quad \Omega = \mathcal{X} \times \mathcal{Y} \times \Re^m. \tag{1.12b}$$

The affine operator $F(w)$ has the form

$$F(w) = \begin{pmatrix} 0 & 0 & -A^T \\ 0 & 0 & -B^T \\ A & B & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ \lambda \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}.$$

Again, we have $(w - \tilde{w})^T (F(w) - F(\tilde{w})) \equiv 0$.

线性约束的凸优化问题 (1.10), 转换成了变分不等式 (1.11)–(1.12).

Convex optimization problem with three separable functions

$$\min\{\theta_1(x)+\theta_2(y)+\theta_3(z)\,|\,Ax+By+Cz=b,\ x\in\mathcal{X},y\in\mathcal{Y},z\in\mathcal{Z}\},$$

is a special problem of (4.3) with three blocks. The Lagrangian function is

$$L^{[3]}(x,y,z,\lambda)=\theta_1(x)+\theta_2(y)+\theta_3(z)-\lambda^T(Ax+By+Cz-b).$$

The same analysis tells us that the saddle point is a solution of the following VI:

$$w^*\in\Omega,\quad \theta(u)-\theta(u^*)+(w-w^*)^TF(w^*)\ge 0,\quad \forall\, w\in\Omega.$$

where $\theta(u)=\theta_1(x)+\theta_2(y)+\theta_3(z)$,

$$w=\begin{pmatrix} x\\ y\\ z\\ \lambda\end{pmatrix},\quad u=\begin{pmatrix} x\\ y\\ z\end{pmatrix},\quad F(w)=\begin{pmatrix} -A^T\lambda\\ -B^T\lambda\\ -C^T\lambda\\ Ax+By+Cz-b\end{pmatrix},$$

and $$\Omega=\mathcal{X}\times\mathcal{Y}\times\mathcal{Z}\times\Re^m.$$

2 Proximal point algorithms and its Beyond

Lemma 2 *Let the vectors* $a, b \in \Re^n$, $H \in \Re^{n\times n}$ *be a positive definite matrix. If* $b^T H(a-b) \geq 0$, *then we have*

$$\|b\|_H^2 \leq \|a\|_H^2 - \|a-b\|_H^2. \tag{2.1}$$

The assertion follows from $\|a\|_H^2 = \|b+(a-b)\|_H^2 \geq \|b\|_H^2 + \|a-b\|_H^2$.

2.1 Proximal point algorithms for convex optimization

Convex Optimization Now, let us consider the *simple* convex optimization

$$\min\{\theta(x) + f(x) \mid x \in \mathcal{X}\}, \tag{2.2}$$

where $\theta(x)$ and $f(x)$ are convex but $\theta(x)$ is not necessary smooth, $\mathcal{X}$ is a closed convex set.

For solving (2.2), the k-th iteration of the proximal point algorithm (abbreviated to

PPA) [14, 16] begins with a given x^k, offers the new iterate x^{k+1} via the recursion

$$x^{k+1} = \text{Argmin}\{\theta(x) + f(x) + \frac{r}{2}\|x - x^k\|^2 \mid x \in \mathcal{X}\}. \tag{2.3}$$

Since x^{k+1} is the optimal solution of (2.3), it follows from Lemma 1 that

$$\theta(x) - \theta(x^{k+1}) + (x - x^{k+1})^T \{\nabla f(x^{k+1}) + r(x^{k+1} - x^k)\} \ge 0, \ \forall\, x \in \mathcal{X}. \tag{2.4}$$

Setting $x = x^*$ in the above inequality, it follows that

$$(x^{k+1} - x^*)^T r(x^k - x^{k+1}) \ge \theta(x^{k+1}) - \theta(x^*) + (x^{k+1} - x^*)^T \nabla f(x^{k+1}).$$

Since $(x^{k+1} - x^*)^T \nabla f(x^{k+1}) \ge (x^{k+1} - x^*)^T \nabla f(x^*) \ge 0$, it follows that

$$(x^{k+1} - x^*)^T (x^k - x^{k+1}) \ge 0. \tag{2.5}$$

Let $a = x^k - x^*$ and $b = x^{k+1} - x^*$ and using Lemma 2, we obtain

$$\|\boldsymbol{x^{k+1} - x^*}\|^{\mathbf{2}} \le \|\boldsymbol{x^k - x^*}\|^{\mathbf{2}} - \|\boldsymbol{x^k - x^{k+1}}\|^{\mathbf{2}}, \tag{2.6}$$

which is the nice convergence property of Proximal Point Algorithm.

We write the problem (2.2) and its PPA (2.3) in VI form

For the optimization problem (2.2) , namely, $\min\{\theta(x) + f(x) \,|\, x \in \mathcal{X}\}$,
the equivalent variational inequality form is

$$x^* \in \mathcal{X}, \;\; \theta(x) - \theta(x^*) + (x - x^*)^T \nabla f(x^*) \geq 0, \;\; \forall\, x \in \mathcal{X}. \tag{2.7}$$

For solving the problem (2.2), the PPA is

$$x^{k+1} = \text{Argmin}\{\theta(x) + f(x) + \frac{r}{2}\|x - x^k\|^2 \mid x \in \mathcal{X}\}.$$

variational inequality form of the k-th iteration of the PPA (see (2.4)) is:

$$\begin{aligned} x^{k+1} \in \mathcal{X}, \;\; \theta(x) - \theta(x^{k+1}) + (x - x^{k+1})^T \nabla f(x^{k+1}) & \\ \geq (x - x^{k+1})^T r(x^k - x^{k+1}), \;\; \forall\, x \in \mathcal{X}. & \end{aligned} \tag{2.8}$$

PPA 通过求解一系列的 (2.3), 求得 (2.2) 的解, 采用的是步步为营的策略.

The solution of (2.8) is Proximal Point, it has the contraction property (2.6).

2.2 Preliminaries of PPA for Variational Inequalities

The optimal condition of the linearly constrained convex optimization is characterized as a mixed monotone variational inequality:

$$w^* \in \Omega, \quad \theta(u) - \theta(u^*) + (w - w^*)^T F(w^*) \geq 0, \quad \forall\, w \in \Omega. \qquad (2.9)$$

PPA for VI (2.9) in H-norm For given w^k and $H \succ 0$, find w^{k+1},

$$w^{k+1} \in \Omega, \quad \theta(u) - \theta(u^{k+1}) + (w - w^{k+1})^T F(w^{k+1}) \geq (w - w^{k+1})^T H(w^k - w^{k+1}), \quad \forall\, w \in \Omega. \qquad (2.10)$$

w^{k+1} is called the proximal point of the k-th iteration for the problem (2.9).

✠ **w^k is the solution of (2.9) if and only if $w^k = w^{k+1}$** ✠

Setting $w = w^*$ in (2.10), we obtain

$$(w^{k+1} - w^*)^T H(w^k - w^{k+1}) \geq \theta(u^{k+1}) - \theta(u^*) + (w^{k+1} - w^*)^T F(w^{k+1})$$

Note that (see the structure of $F(w)$ in (1.9b))

$$(w^{k+1} - w^*)^T F(w^{k+1}) = (w^{k+1} - w^*)^T F(w^*),$$

and consequently (by using (2.9)) we obtain

$$(w^{k+1} - w^*)^T H(w^k - w^{k+1}) \ge \theta(u^{k+1}) - \theta(u^*) + (w^{k+1} - w^*)^T F(w^*) \ge 0.$$

Thus, we have

$$(w^{k+1} - w^*)^T H(w^k - w^{k+1}) \ge 0. \quad (2.11)$$

By setting $\boldsymbol{a = w^k - w^*}$ and $\boldsymbol{b = w^{k+1} - w^*}$,

the inequality (2.11) means that $\boldsymbol{b^T H(a - b) \ge 0}$.

By using Lemma 2, we obtain

$$\|\boldsymbol{w^{k+1} - w^*}\|_H^2 \le \|\boldsymbol{w^k - w^*}\|_H^2 - \|\boldsymbol{w^k - w^{k+1}}\|_H^2. \quad (2.12)$$

We get the nice convergence property of Proximal Point Algorithm.

3 From PDHG to Customized-PPA

We consider the $\min - \max$ problem

$$\min_x \max_y \{\Phi(x,y) = \theta_1(x) - y^T A x - \theta_2(y) \,|\, x \in \mathcal{X}, y \in \mathcal{Y}\}. \tag{3.1}$$

Let (x^*, y^*) be the solution of (3.1), then we have

$$\begin{cases} x^* \in \mathcal{X}, \;\; \Phi(x, y^*) - \Phi(x^*, y^*) \geq 0, \;\; \forall\, x \in \mathcal{X}, & \text{(3.2a)} \\ y^* \in \mathcal{Y}, \;\; \Phi(x^*, y^*) - \Phi(x^*, y) \geq 0, \;\; \forall\, y \in \mathcal{Y}. & \text{(3.2b)} \end{cases}$$

Using the notation of $\Phi(x, y)$, it can be written as

$$\begin{cases} x^* \in \mathcal{X}, \quad \theta_1(x) - \theta_1(x^*) + (x - x^*)^T(-A^T y^*) \geq 0, \quad \forall\, x \in \mathcal{X}, \\ y^* \in \mathcal{Y}, \quad \theta_2(y) - \theta_2(y^*) \; + \; (y - y^*)^T(\, Ax^*) \;\; \geq 0, \quad \forall\, y \in \mathcal{Y}. \end{cases}$$

Furthermore, it can be written as a variational inequality in the compact form:

$$u \in \Omega, \quad \theta(u) - \theta(u^*) + (u - u^*)^T F(u^*) \geq 0, \; \forall\, u \in \Omega, \tag{3.3}$$

where

$$u = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \theta(u) = \theta_1(x) + \theta_2(y), \quad F(u) = \begin{pmatrix} -A^T y \\ Ax \end{pmatrix}, \quad \Omega = \mathcal{X} \times \mathcal{Y}.$$

Since $F(u) = \begin{pmatrix} -A^T y \\ Ax \end{pmatrix} = \begin{pmatrix} 0 & -A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, we have

$$(u - v)^T (F(u) - F(v)) \equiv 0.$$

3.1 Original primal-dual hybrid gradient algorithm [17]

For given (x^k, y^k), PDHG [17] produces a pair of (x^{k+1}, y^{k+1}). First,

$$x^{k+1} = \text{argmin}\{\Phi(x, y^k) + \frac{r}{2}\|x - x^k\|^2 \mid x \in \mathcal{X}\}, \tag{3.4a}$$

and then we obtain y^{k+1} via

$$y^{k+1} = \text{argmax}\{\Phi(x^{k+1}, y) - \frac{s}{2}\|y - y^k\|^2 \mid y \in \mathcal{Y}\}. \tag{3.4b}$$

Ignoring the constant term in the objective function, the subproblems (3.4) are reduced to

$$\begin{cases} x^{k+1} = \text{argmin}\{\theta_1(x) - x^T A^T y^k + \frac{r}{2}\|x - x^k\|^2 \mid x \in \mathcal{X}\}, & \text{(3.5a)} \\ y^{k+1} = \text{argmin}\{\theta_2(y) + y^T A x^{k+1} + \frac{s}{2}\|y - y^k\|^2 \mid y \in \mathcal{Y}\}. & \text{(3.5b)} \end{cases}$$

According to Lemma 1, the optimality condition of (5.8a) is $x^{k+1} \in \mathcal{X}$ and

$$\theta_1(x) - \theta_1(x^{k+1}) + (x - x^{k+1})^T \{-A^T y^k + r(x^{k+1} - x^k)\} \geq 0, \ \forall\, x \in \mathcal{X}. \tag{3.6}$$

Similarly, from (5.8b) we get $y \in \mathcal{Y}$ and

$$\theta_2(y) - \theta_2(y^{k+1}) + (y - y^{k+1})^T \{Ax^{k+1} + s(y^{k+1} - y^k)\} \geq 0, \ \forall\, y \in \mathcal{Y}. \quad (3.7)$$

Combining (3.6) and (3.7), we have

$$\theta(u) - \theta(u^{k+1}) + \begin{pmatrix} x - x^{k+1} \\ y - y^{k+1} \end{pmatrix}^T \left\{ \begin{pmatrix} -A^T y^{k+1} \\ Ax^{k+1} \end{pmatrix} + \begin{pmatrix} r(x^{k+1} - x^k) + A^T(y^{k+1} - y^k) \\ s(y^{k+1} - y^k) \end{pmatrix} \right\} \geq 0, \quad \forall (x, y) \in \Omega.$$

The compact form is $u^{k+1} \in \Omega$,

$$\theta(u) - \theta(u^{k+1}) + (u - u^{k+1})^T \{F(u^{k+1}) + Q(u^{k+1} - u^k)\} \geq 0, \ \forall u \in \Omega, \quad (3.8)$$

where

$$Q = \begin{pmatrix} rI_n & A^T \\ 0 & sI_m \end{pmatrix} \qquad \text{is not symmetric.}$$

It does not be the PPA form (2.10), and we can not expect its convergence.

The following example of linear programming indicates the original PDHG (3.4) is not necessary convergent.

Consider a pair of the primal-dual linear programming:

$$\text{(Primal)} \quad \begin{array}{ll} \min & c^T x \\ \text{s. t.} & Ax = b \\ & x \geq 0. \end{array} \qquad \text{(Dual)} \quad \begin{array}{ll} \max & b^T y \\ \text{s. t.} & A^T y \leq c. \end{array}$$

We take the following example

$$\text{(P)} \quad \begin{array}{ll} \min & x_1 + 2x_2 \\ \text{s. t.} & x_1 + x_2 = 1 \\ & x_1, \ x_2 \geq 0. \end{array} \qquad \text{(D)} \quad \begin{array}{ll} \max & y \\ \text{s. t.} & \begin{bmatrix} 1 \\ 1 \end{bmatrix} y \leq \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{array}$$

where $A = [1, 1], \ b = 1, c = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and the vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

The optimal solutions of this pair of linear programming are $x^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $y^* = 1$.

Note that its Lagrange function is

$$L(x, y) = c^T x - y^T (Ax - b) \tag{3.9}$$

which defined on $\Re^2_+ \times \Re$. (x^*, y^*) is the unique saddle point of the Lagrange function.

For the convex optimization problem $\min\{\theta(x) \mid Ax = b, x \in \mathcal{X}\}$, its Lagrangian function is $$L(x, y) = \theta(x) - y^T (Ax - b),$$ which defined on $\mathcal{X} \times \Re^m$. Find the saddle point of the Lagrangian function is a special $\min - \max$ problem (3.1) whose $\theta_1(x) = \theta(x)$, $\theta_2(y) = -b^T y$ and $\mathcal{Y} = \Re^m$.

For solving the $\min$-$\max$ problem (3.9), by using (3.4), the iterative formula is

$$\begin{cases} x^{k+1} = \max\{(x^k + \frac{1}{r}(A^T y^k - c)), 0\}, \\ y^{k+1} = y^k - \frac{1}{s}(Ax^{k+1} - b). \end{cases}$$

We use $(x_1^0, x_2^0; y^0) = (0, 0; 0)$ as the start point. For this example, the method is not convergent.

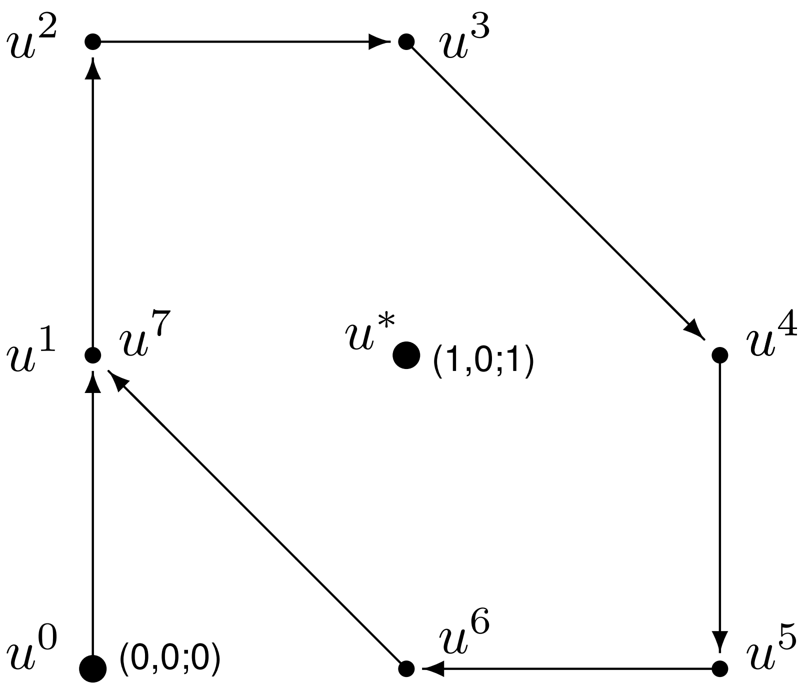
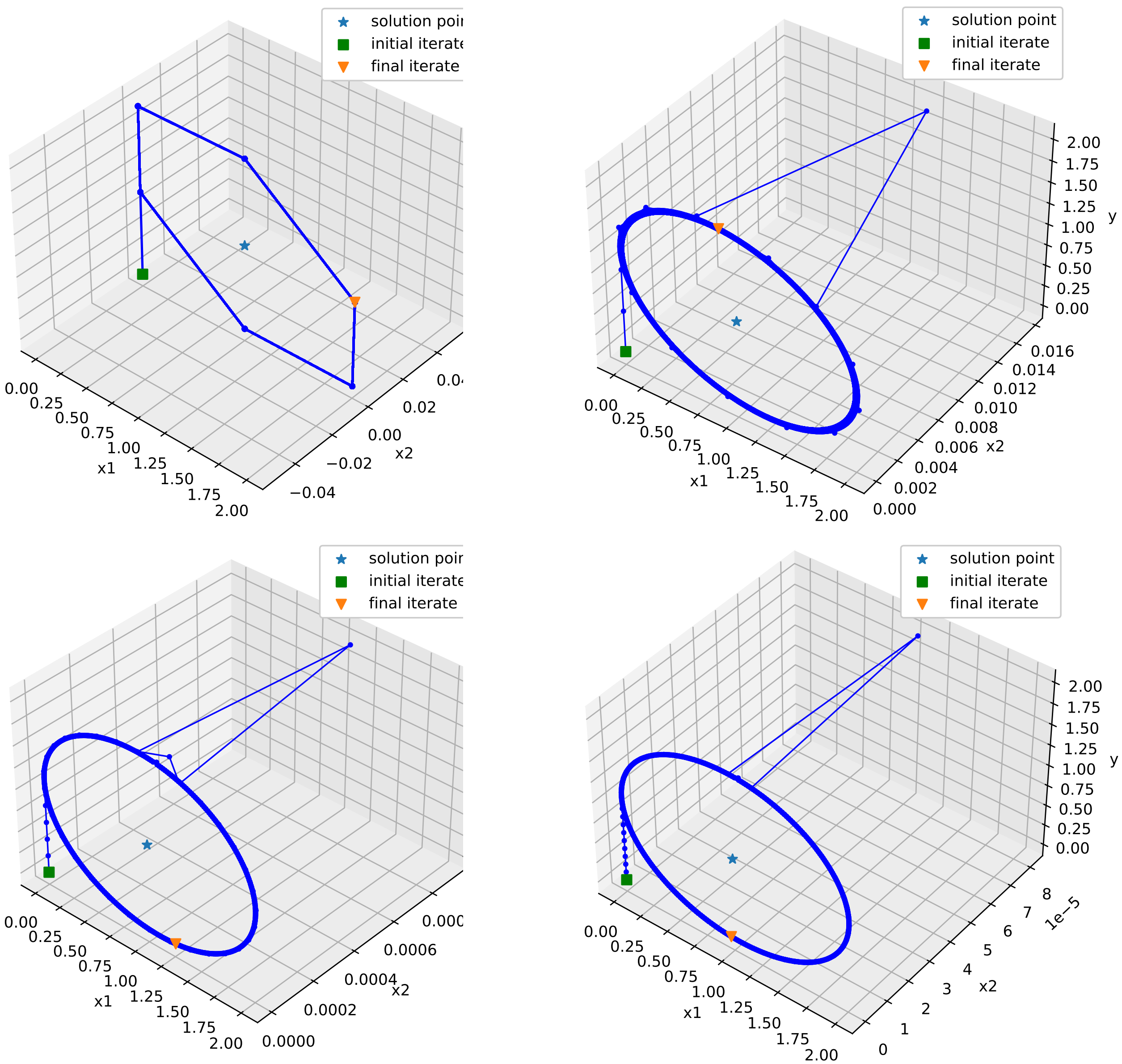


Fig. 4.1 The sequence generated by PDHG Method with $r = s = 1$

$$u^0 = (0, 0; 0)$$
$$u^1 = (0, 0; 1)$$
$$u^2 = (0, 0; 2)$$
$$u^3 = (1, 0; 2)$$
$$u^4 = (2, 0; 1)$$
$$u^5 = (2, 0; 0)$$
$$u^6 = (1, 0; 0)$$
$$u^7 = (0, 0; 1)$$

$$\boldsymbol{u^{k+6} = u^k}$$



对 $r = s = 1, 2, 5, 10$, PDHG 方法都不收敛

3.2 Customized Proximal Point Algorithm-Classical Version

If we change the non-symmetric matrix Q to a symmetric matrix H such that

$$Q = \begin{pmatrix} rI_n & A^T \\ 0 & sI_m \end{pmatrix} \quad \Rightarrow \quad H = \begin{pmatrix} rI_n & A^T \\ A & sI_m \end{pmatrix},$$

then the variational inequality (3.8) will become the following desirable form:

$$\theta(u) - \theta(u^{k+1}) + (u - u^{k+1})^T \{F(u^{k+1}) + H(u^{k+1} - u^k)\} \geq 0, \ \forall u \in \Omega.$$

For this purpose, we need only to change (3.7) in PDHG, namely,

$$\theta_2(y) - \theta_2(y^{k+1}) + (y - y^{k+1})^T \{Ax^{k+1} + s(y^{k+1} - y^k)\} \geq 0, \ \forall\, y \in \mathcal{Y}.$$

to

$$\theta_2(y) - \theta_2(y^{k+1}) + (y - y^{k+1})^T \{A[2x^{k+1} - x^k] + s(y^{k+1} - y^k)\} \geq 0, \ \forall\, y \in \mathcal{Y}. \tag{3.10}$$

$$\theta_2(y) - \theta_2(y^{k+1}) + (y - y^{k+1})^T \{Ax^{k+1} + A(x^{k+1} - x^k) + s(y^{k+1} - y^k)\} \geq 0.$$

Thus, for given (x^k, y^k), producing a proximal point (x^{k+1}, y^{k+1}) via (3.4a) and (3.10) can be summarized as:

$$x^{k+1} = \text{argmin}\Big\{\Phi(x, y^k) + \frac{r}{2}\big\|x - x^k\big\|^2 \;\big|\; x \in \mathcal{X}\Big\}. \tag{3.11a}$$

$$y^{k+1} = \text{argmax}\Big\{\Phi\big([2x^{k+1} - x^k], y\big) - \frac{s}{2}\big\|y - y^k\big\|^2\Big\} \tag{3.11b}$$

By ignoring the constant term in the objective function, getting x^{k+1} from (3.11a) is equivalent to obtaining x^{k+1} from

$$x^{k+1} = \text{argmin}\Big\{\theta_1(x) + \frac{r}{2}\big\|x - \big[x^k + \frac{1}{r}A^T y^k\big]\big\|^2 \;\big|\; x \in \mathcal{X}\Big\}.$$

The solution of (3.11b) is given by

$$y^{k+1} = \text{argmin}\Big\{\theta_2(y) + \frac{s}{2}\big\|y - \big[y^k + \frac{1}{s}A(2x^{k+1} - x^k)\big]\big\|^2 \;\big|\; y \in \mathcal{Y}\Big\}.$$

According to the assumption, there is no difficulty to solve (3.11a)-(3.11b).

In the case that $rs > \|A^T A\|$, the matrix

$$H = \begin{pmatrix} rI_n & A^T \\ A & sI_m \end{pmatrix} \quad \text{is positive definite.}$$

Theorem 1 *The sequence* $\{u^k = (x^k, \lambda^k)\}$ *generated by the customized PPA* (3.11) *satisfies*

$$\|u^{k+1} - u^*\|_H^2 \le \|u^k - u^*\|_H^2 - \|u^k - u^{k+1}\|_H^2. \tag{3.12}$$

For the minimization problem $\min\{\theta(x) \mid Ax = b, x \in \mathcal{X}\}$,

the iterative scheme is

$$x^{k+1} = \text{argmin}\Big\{\theta(x) + \frac{r}{2}\Big\|x - \big[x^k + \frac{1}{r}A^T y^k\big]\Big\|^2 \,\Big|\, x \in \mathcal{X}\Big\}. \tag{3.13a}$$

$$y^{k+1} = y^k - \frac{1}{s}\big[A(2x^{k+1} - x^k) - b\big]. \tag{3.13b}$$

For solving the $\min$-$\max$ problem (3.9), by using (3.11), the iterative formula is

$$\begin{cases} x^{k+1} = \max\{(x^k + \frac{1}{r}(A^T y^k - c)), 0\}, \\ y^{k+1} = y^k - \frac{1}{s}[A(2x^{k+1} - x^k) - b]. \end{cases}$$

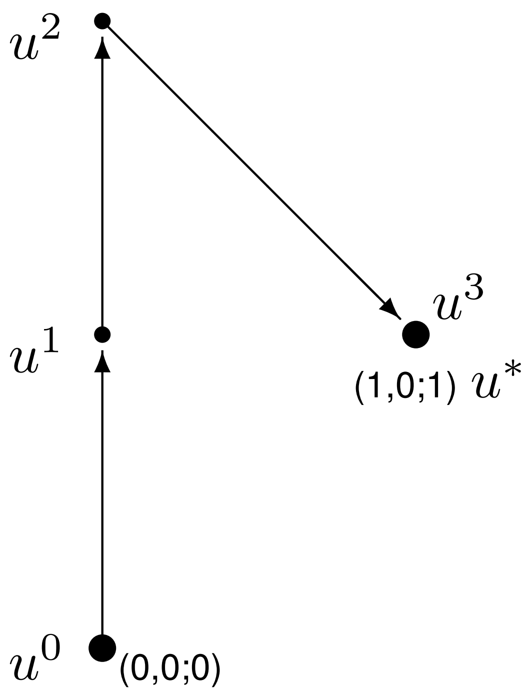
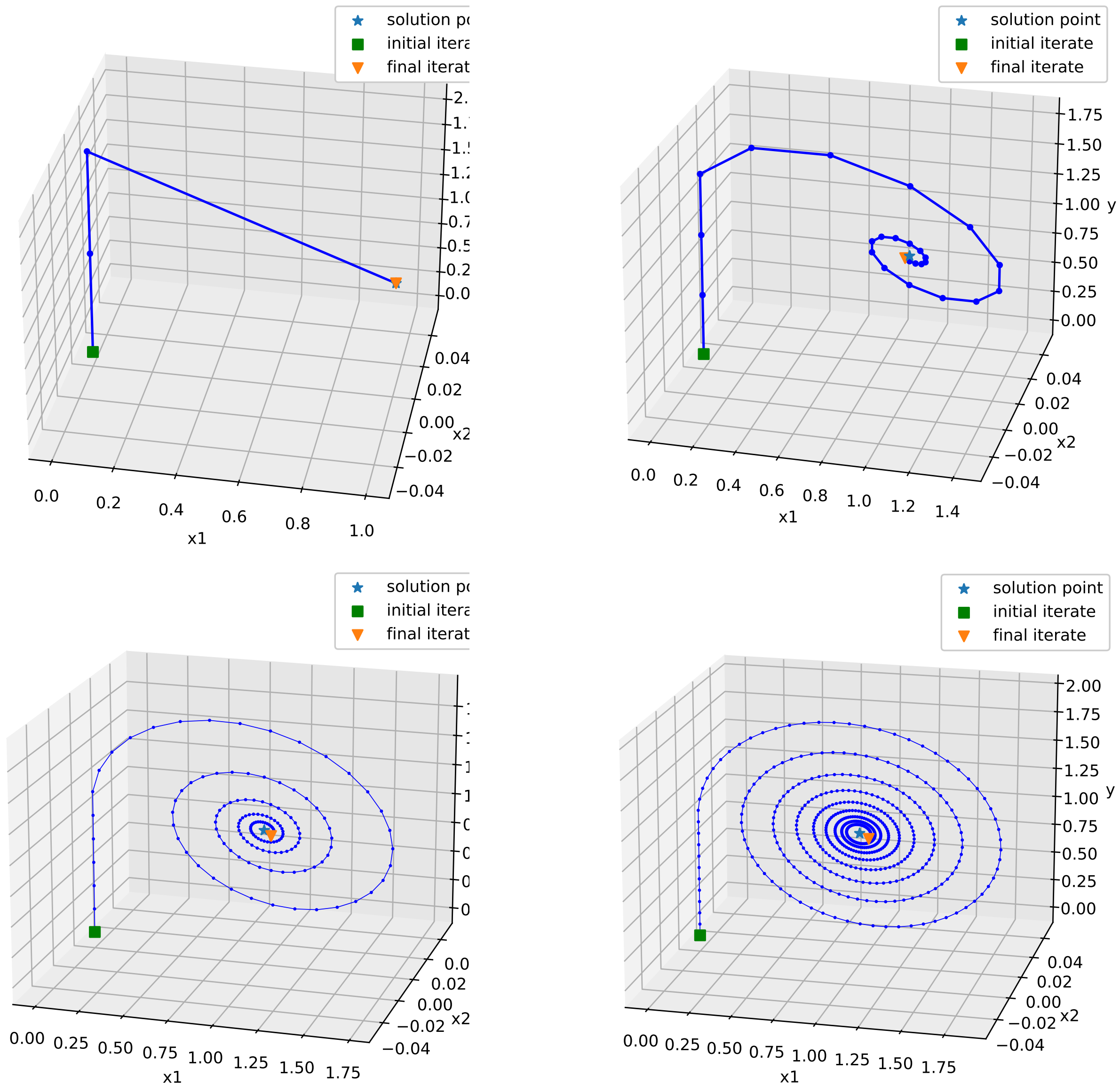


Fig. 4.2 The sequence generated by
C-PPA Method with $r = s = 1$

$$u^0 = (0,0;0)$$
$$u^1 = (0,0;1)$$
$$u^2 = (0,0;2)$$
$$u^3 = (1,0;1)$$
$$\boldsymbol{u^3 = u^*}.$$



对 $r = s = 1, 2, 5, 10$, C-PPA 方法都收敛. 参数越大, 收敛越慢

Remark Let the linear constraints become to a system of inequalities.

$$\min\{\theta(x) \mid Ax = b,\ x \in \mathcal{X}\} \quad \Rightarrow \quad \min\{\theta(x) \mid Ax \ge b,\ x \in \mathcal{X}\}$$

In this case, the Lagrange multiplier λ should be nonnegative. $\Omega = \mathcal{X} \times \Re^m_+$.

We need only to make a slight change in the prediction procedure:

In the primal-dual order:

$$y^{k+1} = y^k - \tfrac{1}{s}\big[A(2x^{k+1} - x^k) - b\big] \qquad \Rightarrow$$

$$\Rightarrow \quad y^{k+1} = \Big\{y^k - \tfrac{1}{s}\big[A(2x^{k+1} - x^k\big) - b\big]\Big\}_+$$

3.3 Simplicity recognition

Frame of VI is recognized by some Researcher in Image Science

Diagonal preconditioning for first order primal-dual algorithms in convex optimization*

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- T. Pock and A. Chambolle, IEEE ICCV, 1762-1769, 2011
- A. Chambolle, T. Pock, A first-order primal-dual algorithms for convex problem with applications to imaging, J. Math. Imaging Vison, 40, 120-145, 2011.

preconditioned algorithm. In very recent work [10], it has been shown that the iterates (2) can be written in form of a proximal point algorithm [14], which greatly simplifies the convergence analysis.

From the optimality conditions of the iterates (4) and the convexity of G and F^* it follows that for any $(x, y) \in X \times Y$ the iterates x^{k+1} and y^{k+1} satisfy

$$\left\langle \begin{pmatrix} x - x^{k+1} \\ y - y^{k+1} \end{pmatrix}, F\begin{pmatrix} x^{k+1} \\ y^{k+1} \end{pmatrix} + M\begin{pmatrix} x^{k+1} - x^k \\ y^{k+1} - y^k \end{pmatrix} \right\rangle \geq 0 , \tag{5}$$

where

$$F\begin{pmatrix} x^{k+1} \\ y^{k+1} \end{pmatrix} = \begin{pmatrix} \partial G(x^{k+1}) + K^T y^{k+1} \\ \partial F^*(y^{k+1}) - K x^{k+1} \end{pmatrix}$$

and

$$M = \begin{bmatrix} \mathrm{T}^{-1} & -K^T \\ -\theta K & \Sigma^{-1} \end{bmatrix} . \tag{6}$$

It is easy to check, that the variational inequality (5) now takes the form of a proximal point algorithm [10, 14, 16].

作者 C-P 说到我们的 PPA 解释极大地简化了收敛性分析.

我们依然认为, 只有当左边 (6) 式的矩阵 M 对称正定, 才是收敛的 PPA 方法.

否则, 就像我们前面给出的例子, 方法是不一定收敛的.

由 CP 方法演译得来的矩阵M, 当 $\theta = 0$, 方法不能保证收敛.

对 $\theta \in (0, 1)$, 收敛性没有证明, 至今还是一个 Open Problem.

[9] L. Ford and D. Fulkerson. *Flows in Networks*. Princeton University Press, Princeton, New Jersey, 1962.

[10] B. He and X. Yuan. Convergence analysis of primal-dual algorithms for total variation image restoration. Technical report, Nanjing University, China, 2010.

Later, the Reference [10] is published in SIAM J. Imaging Science [11].

Math. Program., Ser. A
DOI 10.1007/s10107-015-0957-3

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FULL LENGTH PAPER

On the ergodic convergence rates of a first-order primal–dual algorithm

Antonin Chambolle[1] · **Thomas Pock**[2,3]

The paper published by Chambolle and Pock in Math. Progr. uses the VI framework

1 Introduction

In this work we revisit a first-order primal–dual algorithm which was introduced in [15, 26] and its accelerated variants which were studied in [5]. We derive new estimates for the rate of convergence. In particular, exploiting a proximal-point interpretation due to [16], we are able to give a very elementary proof of an ergodic $O(1/N)$ rate of convergence (where N is the number of iterations), which also generalizes to non-

Algorithm 1: $O(1/N)$ Non-linear primal–dual algorithm

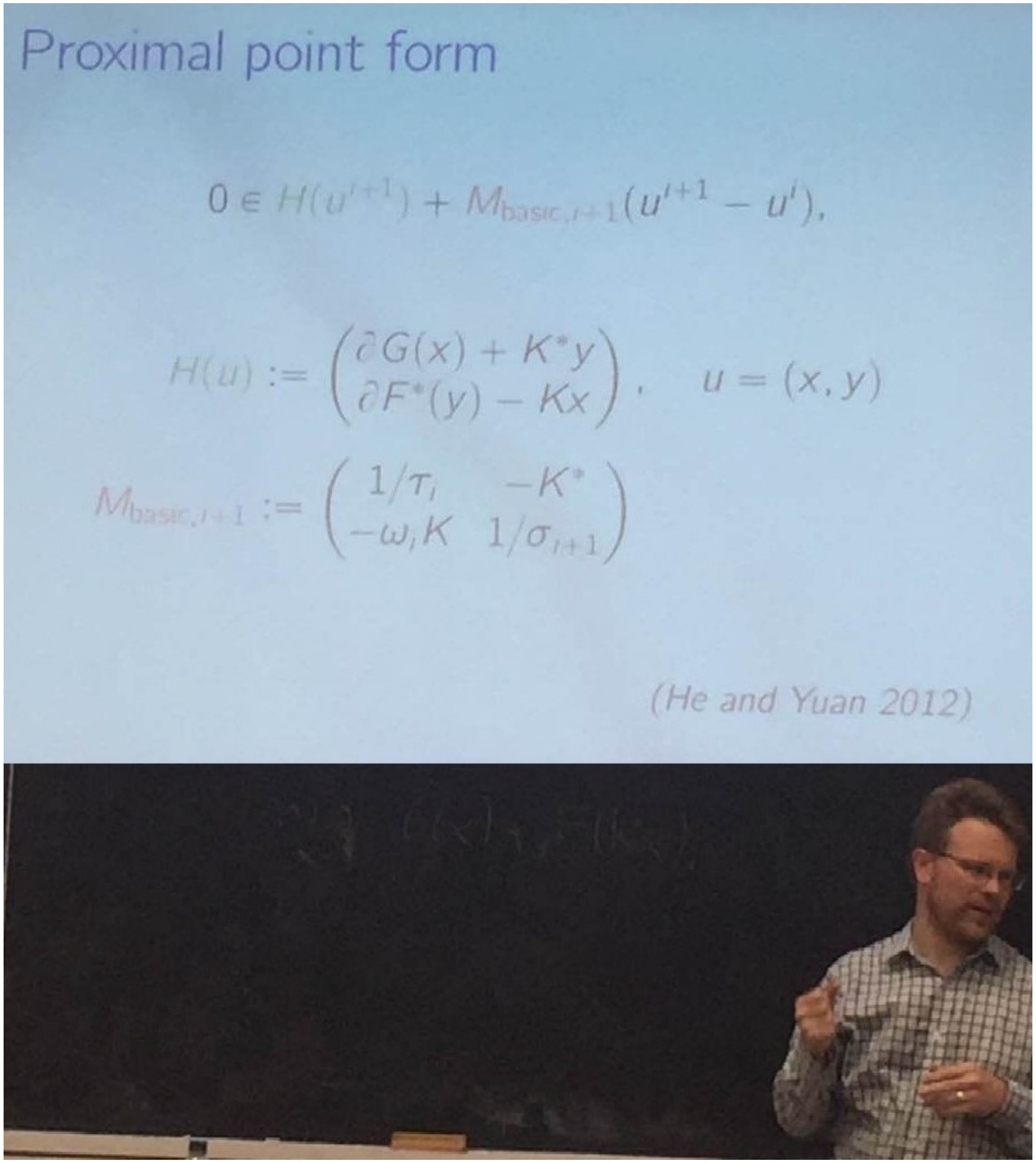
- Input: Operator norm $L := \|K\|$, Lipschitz constant L_f of ∇f, and Bregman distance functions D_x and D_y.
- Initialization: Choose $(x^0, y^0) \in \mathcal{X} \times \mathcal{Y}$, $\tau, \sigma > 0$
- Iterations: For each $n \geq 0$ let

$$(x^{n+1}, y^{n+1}) = \mathcal{PD}_{\tau,\sigma}(x^n, y^n, 2x^{n+1} - x^n, y^n) \qquad (11)$$

The elegant interpretation in [16] shows that by writing the algorithm in this form

♣ 该文的文献 [16] 是我们发表在 SIAM J. Imaging Science 上的文章.

B.S. He and X.M. Yuan, Convergence analysis of primal-dual algorithms for a saddle -point problem: From contraction perspective, *SIAM J. Imag. Science* **5**(2012), 119-149.



2017 年 7 月, 南方科技大学数学系的一位副主任去英国访问. 在他参加的一个学术会议上, 首位报告人讲: 用 He and Yuan 提出的邻近点形式 (PPF), 处理图像问题。

见到一幅幻灯片介绍我们的工作, 我的同事抢拍了一张照片发给我。

这也说明, 只有简单的思想才容易得到传播, 被人接受。

The Chen-Teboulle algorithm is the proximal point algorithm

Stephen Becker *

November 22, 2011; posted August 13, 2019

Abstract

We revisit the
on the step-size p

Recent works such as [HY12] have proposed a very simple yet powerful technique for analyzing optimization methods.

1 Background

Recent works such as [HY12] have proposed a very simple yet powerful technique for analyzing optimization methods. The idea consists simply of working with a different norm in the *product* Hilbert space. We fix an inner product $\langle x, y\rangle$ on $\mathcal{H} \times \mathcal{H}^*$. Instead of defining the norm to be the induced norm, we define the primal norm as follows (and this induces the dual norm)

$$\|x\|_V = \sqrt{\langle Vx, x\rangle} = \sqrt{\langle x, x\rangle_V}, \quad \|y\|_V^* = \|y\|_{V^{-1}} = \sqrt{\langle y, V^{-1}y\rangle} = \sqrt{\langle y, y\rangle_{V^{-1}}}$$

for any Hermitian positive definite $V \in \mathcal{B}(\mathcal{H}, \mathcal{H})$; we write this condition as $V \succ 0$. For finite dimensional spaces $\mathcal{H}$, this means that V is a positive definite matrix.

3.4 Relationship to Chambolle-Pock Method

Chambolle and Pock [3] have proposed a method for solving the convex-concave $\min - \max$ problem, in short, C-P method. Applied C-P method to the problem (3.1), it is also required $rs > \|A^T A\|$.

CP method. For given (x^k, λ^k), C-P method obtains x^{k+1} via

$$x^{k+1} = \arg\min\{\Phi(x, y^k) + \frac{r}{2}\|x - x^k\|^2 \mid x \in \mathcal{X}\}. \tag{3.14a}$$

Then, λ^{k+1} is given by

$$y^{k+1} = \arg\max\{\Phi([x^{k+1} + \tau(x^{k+1} - x^k)], y) - \frac{s}{2}\|y - y^k\|^2 \mid y \in \mathcal{Y}\} \tag{3.14b}$$

where $\tau \in [0, 1]$.

- 原始-对偶混合梯度法(PDHG) (3.4) 和按需定制的邻近点算法(C-PPA) (3.11) 都是 Chambolle-Pock 方法 [3] 分别取 $\tau = 0$ 和 $\tau = 1$ 的特例.
- 对 $\tau = 0$ 的 PDHG 方法 (3.4), §3.1 中已经说明不能保证收敛. 对 $\tau = 1$ 的 CPPA 方法 (3.11), 其收敛性在 §3.2 中有了结论.
- 根据我们的知识, 对于 $\tau \in (0, 1)$ 的 CP 方法 (3.14), 收敛性还没有定论.

CP 方法十年记 2020 年9 月

- Chambolle 和 Pock 在 2010 年提出的求解 $\min - \max$ 问题的原始-对偶方法, 在图像处理领域有着广泛的应用和很大的影响, 被称为CP 方法。
- Chambolle 和Pock 方法的第一个版本公布于2010 年6 月. 他们的方法中有个 $[0, 1]$ 之间的参数, 但在文章中, 只对参数为1 的方法给了证明. 读了他们的这篇文章以后, 我们对这类方法的收敛性进行了研究.
- 由于我们多年研究单调变分不等式的求解方法, 很快发现, 参数为1 的 CP 方法, 可以解释为变分不等式H-模(H为对称正定矩阵) 的邻近点算法 (PPA), 因此收敛性证明特别简单. 五个月后的 2010 年 11 月 4 日, 我们把相关证

明的第一稿, OO-2790, 公布在 Optimization Online 上. 同时, 对参数为 0 的 CP 方法, 我们找到了不收敛的例子

- 参数在 $(0, 1)$ 间的CP 方法, 能不能保证收敛, 这个问题至今没有解决.
- Chambolle 和 Pock 很快发现了我们的工作, 一个多月后的 2010 年 12 月 21 日, 他们的文章在 J. MIV online 正式发表. 我们高兴地看到, Chambolle 和 Pock 这么快就注意到并引用了我们的文章, 也提到了我们的证明. 我们的文章正式发表以后, CP 后来就不再提参数在 $[0, 1)$ 间的方法了.
- 特别感谢CP 方法的原创者认可我们给出的简单证明. 他们在2011年的IEEE ICCV 会议论文中, 称赞我们的工作极大地简化了收敛性分析 (which greatly simplifies the convergence analysis).
- 后来CP 方法的作者又有多篇相关的文章发表(后面的文章他们都只讨论参数为 1 的方法). 他们于2016 年在Math. Progr. 发表的文章中, 继续利用我们的 PPA 解释, 文章的引言中就开诚布公(In particular, exploiting a proximal-point interpretation due to [16], we are able to give a very elementary proof). 这里的$[16]$ 是我们 2010 年的预印本 OO-2790, 2012 年春发表在 SIAM Imaging Science.

4 Special prediction-correction methods

We study the optimization algorithms using the guidance of variational inequality.

$$w^* \in \Omega, \quad \theta(u) - \theta(u^*) + (w - w^*)^T F(w^*) \ge 0, \quad \forall\, w \in \Omega. \tag{4.1}$$

4.1 Algorithms $Q = H$, H is positive definite

[Prediction Step.] With given v^k, find a vector $\tilde{w}^k \in \Omega$ such that

$$\theta(u) - \theta(\tilde{u}^k) + (w - \tilde{w}^k)^T F(\tilde{w}^k) \ge (v - \tilde{v}^k)^T H (v^k - \tilde{v}^k), \ \forall w \in \Omega, \tag{4.2a}$$

where the matrix H is symmetric and positive definite.

[Correction Step.] The new iterate v^{k+1} by

$$v^{k+1} = v^k - \alpha(v^k - \tilde{v}^k), \quad \alpha \in (0, 2) \tag{4.2b}$$

H is a symmetric positive definite matrix.

Since $G = (2-\alpha)H, \quad \alpha\|v^k - \tilde{v}^k|_G^2 = \alpha(2-\alpha)\|v^k - \tilde{v}^k|_H^2.$

The sequence $\{v^k\}$ generated by the prediction-correction method (4.2) satisfies

$$\|v^{k+1} - v^*\|_H^2 \le \|v^k - v^*\|_H^2 - \alpha(2-\alpha)\|v^k - \tilde{v}^k\|_H^2. \quad \forall v^* \in \mathcal{V}^*.$$

The above inequality is the Key for convergence analysis !

Set $\alpha = 1$ in (4.2b), the prediction (4.2a) becomes: $w^{k+1} \in \Omega$ such that

$$\theta(u) - \theta(u^{k+1}) + (w - w^{k+1})^T F(w^{k+1}) \ge (v - v^{k+1})^T H(v^k - v^{k+1}), \ \forall w \in \Omega.$$

The generated sequence $\{v^k\}$ satisfies

$$\|v^{k+1} - v^*\|_H^2 \le \|v^k - v^*\|_H^2 - \|v^k - v^{k+1}\|_H^2. \quad \forall v^* \in \mathcal{V}^*.$$

上式是跟 (2.12) 类似的不等式, 是关于核心变量 v 的 PPA 方法.

4.2 Application: Balanced ALM [12]

We consider the linearly constrained convex optimization problem

$$\min\{\theta(x) \mid Ax = b,\ x \in \mathcal{X}\}. \tag{4.3}$$

The Lagrange function of (4.3) is

$$L(x,\lambda) = \theta(x) - \lambda^T(Ax - b), \qquad (x,\lambda) \in \mathcal{X} \times \Re^m. \tag{4.4}$$

The related variational inequality is

$$w^* \in \Omega, \quad \theta(x) - \theta(x^*) + (w - w^*)^T F(w^*) \geq 0, \quad \forall\, w \in \Omega. \tag{4.5a}$$

where

$$w = \begin{pmatrix} u \\ \lambda \end{pmatrix}, \quad F(w) = \begin{pmatrix} -A^T\lambda \\ Ax - b \end{pmatrix} \text{ and } \quad \Omega = \mathcal{U} \times \Re^m. \tag{4.5b}$$

The iterative scheme of Balanced-ALM for the problem (4.3) reads as

$$\begin{cases} x^{k+1} = \arg\min\{L(x,\lambda^k) + \frac{r}{2}\|x - x^k\|^2 \mid x \in \mathcal{X}\}, & \text{(4.6a)} \\ \lambda^{k+1} = \text{argmax}\{L([2x^{k+1} - x^k], \lambda) - \frac{1}{2}\|\lambda - \lambda^k\|^2_{(\frac{1}{r}AA^T + \delta I_m)}\}. & \text{(4.6b)} \end{cases}$$

假如用C-PPA, 就是把 (4.6b) 中的$\frac{1}{2}\|\lambda - \lambda^k\|^2_{(\frac{1}{r}AA^T+\delta I_m)}$ 改成$\frac{1}{2s}\|\lambda - \lambda^k\|^2$

$$\begin{cases} x^{k+1} = \text{argmin}\{\theta(x) + \frac{r}{2}\|x - [x^k + \frac{1}{r}A^T\lambda^k]\|^2 \mid x \in \mathcal{X}\}. & \text{(4.7a)} \\ \lambda^{k+1} = \arg\min\{\frac{1}{2}\|\lambda - \lambda^k\|^2_{(\frac{1}{r}AA^T+\delta I_m)} + \lambda^T(A[2x^{k+1} - x^k] - b)\}. & \text{(4.7b)} \end{cases}$$

where $r > 0$ and $\delta > 0$ are any positive scalars. For example, we can take $\delta = 0.1$. Thus, there is only a positive parameter r is needed to be chosen.

λ^{k+1} from (4.7b) is the solution of the following system of equations:

$$\Big(\frac{1}{r}AA^T + \delta I_m\Big)(\lambda - \lambda^k) + \big(A[2x^{k+1} - x^k] - b\big) = 0.$$

Note that the coefficient matrix is positive definite, and we need to do once the Cholesky decomposition [7].

Lemma 3 *For given* w^k*, let* w^{k+1} *be generated by* (4.7)*, then we have*

$$w^{k+1} \in \Omega, \quad \theta(x) - \theta(x^{k+1}) + (w - w^{k+1})^T F(w^{k+1}) \\ \geq (w - w^{k+1})^T H(w^k - w^{k+1}), \quad \forall\, w \in \Omega, \tag{4.8a}$$

where

$$H = \begin{pmatrix} rI_n & A^T \\ A & \frac{1}{r}AA^T + \delta I_m \end{pmatrix} \quad \text{is positive definite.} \tag{4.8b}$$

Proof. According to Lemma 1, x^{k+1} offered by (4.7a) satisfies the variational inequality

$$x^{k+1} \in \mathcal{X}, \quad \theta(x) - \theta(x^{k+1}) + (x - x^{k+1})^T \big\{-A^T\lambda^k + r(x^{k+1} - x^k)\big\} \geq 0, \quad \forall\, x \in \mathcal{X}.$$

Then, for any unknown λ^{k+1}, we have

$$x^{k+1} \in \mathcal{X}, \quad \theta(x) - \theta(x^{k+1}) + (x - x^{k+1})^T (-A^T\lambda^{k+1}) \\ \geq (x - x^{k+1})^T \big\{r(x^k - x^{k+1}) + A^T(\lambda^k - \lambda^{k+1})\big\}, \ \forall x \in \mathcal{X}. \tag{4.9}$$

Similarly, according to Lemma 1, λ^{k+1} offered by (4.7b) is characterized by the variational inequality

$$(\lambda-\lambda^{k+1})^T\Big\{\Big(A[2x^{k+1}-x^k]-b\Big)+\Big(\frac{1}{r}AA^T+\delta I_m\Big)(\lambda^{k+1}-\lambda^k)\Big\}\geq 0, \quad \forall\, \lambda\in\Lambda.$$

It can be rewritten as

$$\begin{aligned}\lambda^{k+1}\in\Lambda,\;\; &(\lambda-\lambda^{k+1})^T(Ax^{k+1}-b)\\ &\geq(\lambda-\lambda^{k+1})^T\big\{(A(x^k-x^{k+1})+\Big(\frac{1}{r}AA^T+\delta I_m\Big)(\lambda^k-\lambda^{k+1})\big\},\end{aligned} \quad (4.10)$$

for all $\lambda\in\Re^m$. Combining (4.9) and (4.10), and using the notation in (4.5), we get

$$\theta(x)-\theta(x^{k+1})+(w-w^{k+1})^T\{F(w^{k+1})+H(w^{k+1}-w^k)\}\geq 0,\;\forall w\in\Omega.$$

where H is given by (4.8b). Notice that the matrix H

$$\begin{aligned}H &= \begin{pmatrix} rI_n & A^T\\ A & \frac{1}{r}AA^T\end{pmatrix}+\begin{pmatrix}0 & 0\\ 0 & \delta I_m\end{pmatrix}\\ &= \begin{pmatrix}\sqrt{r}I_n\\ \sqrt{\frac{1}{r}}A\end{pmatrix}\Big(\sqrt{r}I_n,\sqrt{\tfrac{1}{r}}A^T\Big)+\begin{pmatrix}0 & 0\\ 0 & \delta I_m\end{pmatrix},\end{aligned}$$

for any $u = (x, \lambda) \neq 0$. Thus, we have

$$u^T H u = \left\| \sqrt{r} x + \sqrt{\tfrac{1}{r}} A^T \lambda \right\|^2 + \delta \|\lambda\|^2 > 0,$$

and therefore the matrix H is positive definite. □

事实上, 我们是根据 (4.8) 来设计算法的！ 先写下 (4.8) 的具体形式：

The following is a proximal point algorithm: Find

$$(x^{k+1}, \lambda^{k+1}) \in \Omega, \quad \theta(x) - \theta(x^{k+1}) + \begin{pmatrix} x - x^{k+1} \\ \lambda - \lambda^{k+1} \end{pmatrix}^T \begin{pmatrix} -A^T \lambda^{k+1} \\ A x^{k+1} - b \end{pmatrix}$$

$$\geq \begin{pmatrix} x - x^{k+1} \\ \lambda - \lambda^{k+1} \end{pmatrix}^T \begin{pmatrix} rI & A^T \\ A & \frac{1}{r} A A^T + \delta I_m \end{pmatrix} \begin{pmatrix} x^k - x^{k+1} \\ \lambda^k - \lambda^{k+1} \end{pmatrix}, \tag{4.11}$$

for all $(x, \lambda) \in \Omega$.

The equivalent form is

$$(x^{k+1},\lambda^{k+1})\in\Omega,\;\;\theta(x)-\theta(x^{k+1})+\begin{pmatrix} x-x^{k+1}\\ \lambda-\lambda^{k+1}\end{pmatrix}^T\begin{pmatrix} -A^T\lambda^{k+1}\\ Ax^{k+1}-b\end{pmatrix}$$

$$\geq\begin{pmatrix} x-x^{k+1}\\ \lambda-\lambda^{k+1}\end{pmatrix}^T\begin{pmatrix} r(x^k-x^{k+1})+A^T(\lambda^k-\lambda^{k+1})\\ A(x^k-x^{k+1})+(\frac{1}{r}AA^T+\delta I_m)(\lambda^k-\lambda^{k+1})\end{pmatrix}, \quad (4.12)$$

for all $(x,\lambda)\in\Omega$.

By a manipulation. the x-part of (4.12) is: $x^{k+1}\in\mathcal{X}$ and

$$\theta(x)-\theta(x^{k+1})+(x-x^{k+1})^T\{-A^T\lambda^k+r(x^{k+1}-x^k)\}\geq 0,\;\;\forall x\in\mathcal{X}.$$

According to Lemma 1, we get x^{k+1} by

$$\begin{aligned} x^{k+1} &= \text{argmin}\{\theta(x)-(\lambda^k)^TAx+\frac{r}{2}\|x-x^k\|^2 \mid x\in\mathcal{X}\}\\ &= \text{argmin}\{\theta(x)+\frac{r}{2}\|x-[x^k+\frac{1}{r}A^T\lambda^k]\|^2 \mid x\in\mathcal{X}\}. \end{aligned}$$

The λ-part of (4.12) is

$$\lambda^{k+1} \in \Re^m, \quad (\lambda - \lambda^{k+1})^T \left\{ \begin{array}{c} (\frac{1}{r}AA^T + \delta I_m)(\lambda^{k+1} - \lambda^k) \\ +[A(2x^{k+1} - x^k) - b] \end{array} \right\} \geq 0, \quad \forall \lambda \in \Re^m.$$

The equivalent form is

$$\lambda^{k+1} = \lambda^k - \Big(\frac{1}{r}AA^T + \delta I_m\Big)^{-1}[A(2x^{k+1} - x^k) - b].$$

以上部分, 主要是讲 H-模下的 PPA 算法:

$$w^{k+1} \in \Omega, \quad \theta(x) - \theta(x^{k+1}) + (w - w^{k+1})^T F(w^{k+1}) \\ \geq (w - w^{k+1})^T H(w^k - w^{k+1}), \quad \forall\, w \in \Omega.$$

从 $H = \begin{pmatrix} rI_n & A^T \\ A & sI_m \end{pmatrix}$ 到 $H = \begin{pmatrix} rI_n & A^T \\ A & \frac{1}{r}AA^T + \delta I_m \end{pmatrix}$.

从需要 $rs > \|A^T A\|$ 到 任意的 $r > 0, \ \delta \approx> 0.$

5 Splitting Methods in a Unified Framework

We study the algorithms using the guidance of variational inequality. The optimal condition of the linearly constrained convex optimization is resulted in a variational inequality:

$$w^* \in \Omega, \quad \theta(u) - \theta(u^*) + (w - w^*)^T F(w^*) \geq 0, \quad \forall\, w \in \Omega. \qquad (5.1)$$

5.1 Algorithms in a unified framework for VI (5.1)

[Prediction Step.] With given v^k, find a vector $\tilde{w}^k \in \Omega$ which satisfying

$$\theta(u) - \theta(\tilde{u}^k) + (w - \tilde{w}^k)^T F(\tilde{w}^k) \geq (v - \tilde{v}^k)^T Q(v^k - \tilde{v}^k), \ \forall w \in \Omega, \qquad (5.2a)$$

where the matrix Q has the property: $Q^T + Q$ is positive definite.

[Correction Step.] Determine a nonsingular matrix M and a scalar $\alpha > 0$, let

$$v^{k+1} = v^k - \alpha M(v^k - \tilde{v}^k). \qquad (5.2b)$$

v is a part of the elements of the vector w, $\quad v = w$ is also possible.

Convergence Conditions

For the matrices Q and M, there is a positive definite matrix H such that

$$HM = Q. \tag{5.3a}$$

For the matrices H, M and Q satisfied (5.3a), and the step size α in (5.2), the matrix

$$G = Q^T + Q - \alpha M^T HM \succ 0. \tag{5.3b}$$

Methods for $\min - \max$ Problems

In Section 3, the $\min - \max$ problem

$$\min_x \max_y \{\Phi(x, y) = \theta_1(x) - y^T Ax - \theta_2(y) \,|\, x \in \mathcal{X}, y \in \mathcal{Y}\} \tag{5.4}$$

is translated to an equivalent variational inequality:

$$u \in \Omega, \quad \theta(u) - \theta(u^*) + (u - u^*)^T F(u^*) \geq 0, \; \forall\, u \in \Omega, \tag{5.5}$$

where

$$u = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \theta(u) = \theta_1(x) + \theta_2(y), \quad F(u) = \begin{pmatrix} -A^T y \\ Ax \end{pmatrix} \text{ and } \; \Omega = \mathcal{X} \times \mathcal{Y}.$$

For such problem, we have studied in Section 3.

We illustrate how to use the unified framework to modify the methods in Sec. 3.
Taking the $\min - \max$ problem as the example, $w = v = u$ in the framework.

Extended customized PPA

Set the output of (3.11) in §3.2 as the predictor, namely,

$$\begin{cases} \tilde{x}^k = \arg\min\{\Phi(x, y^k) + \frac{r}{2}\|x - x^k\|^2 \mid x \in \mathcal{X}\}, & \text{(5.6a)} \\ \tilde{y}^k = \arg\max\{\Phi\big([2\tilde{x}^k - x^k], y\big) - \frac{s}{2}\|y - y^k\|^2 \mid y \in \mathcal{Y}\} & \text{(5.6b)} \end{cases}$$

In other words, we get the predictor $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k)$ by

$$\begin{cases} \tilde{x}^k = \text{argmin}\{\theta_1(x) + \frac{r}{2}\big\|x - \big[x^k + \frac{1}{r}A^T y^k\big]\big\|^2 \mid x \in \mathcal{X}\}, \\ \tilde{y}^k = \text{argmin}\{\theta_2(y) + \frac{s}{2}\big\|y - \big[y^k + \frac{1}{s}A(2\tilde{x}^k - x^k)\big]\big\|^2 \mid y \in \mathcal{Y}\}. \end{cases}$$

The output $\tilde{w}^k \in \Omega$ of the iteration (5.6) satisfies

$$\theta(u) - \theta(\tilde{u}^k) + (u - \tilde{u}^k)^T F(\tilde{u}^k) \geq (u - \tilde{u}^k)^T Q(u^k - \tilde{u}^k), \ \forall u \in \Omega.$$

It is a form of (5.2a) where

$$Q = \begin{pmatrix} rI & A^T \\ A & sI \end{pmatrix} \quad \text{is symmetric}$$

We take $M = I$ in the correction (5.2b) and the new iterate is updated by

$$w^{k+1} = w^k - \alpha(w^k - \tilde{w}^k), \quad \alpha \in (0, 2).$$

Then, we have and

$$H = QM^{-1} = Q \succ 0$$

and

$$G = Q^T + Q - \alpha M^T H M = (2 - \alpha) H \succ 0.$$

The convergence conditions (5.3) are satisfied. More about customized PPA, please see

♣ G.Y. Gu, B.S. He and X.M. Yuan, Customized Proximal point algorithms for linearly constrained convex minimization and saddle-point problem: a unified Approach, Comput. Optim. Appl., 59(2014), 135-161.

Corrected PDHG

Set the output of (3.4) in §3.1 as the predictor, namely,

$$\text{(PDHG)} \quad \begin{cases} \tilde{x}^k = \arg\min\big\{\Phi(x, y^k) + \frac{r}{2}\|x - x^k\|^2 \;\big|\; x \in \mathcal{X}\big\}, & (5.7a) \\ \tilde{y}^k = \arg\max\big\{\Phi(\tilde{x}^k, y) - \frac{s}{2}\|y - y^k\|^2 \,|\, y \in \mathcal{Y}\big\} & (5.7b) \end{cases}$$

Ignoring the constant term in the objective function, the subproblems (3.4) are reduced to

$$\begin{cases} x^{k+1} = \text{argmin}\{\theta_1(x) - x^T A^T y^k + \frac{r}{2}\|x - x^k\|^2 \,|\, x \in \mathcal{X}\}, & (5.8a) \\ y^{k+1} = \text{argmin}\{\theta_2(y) + y^T A x^{k+1} + \frac{s}{2}\|y - y^k\|^2 \,|\, y \in \mathcal{Y}\}. & (5.8b) \end{cases}$$

The output $\tilde{w}^k \in \Omega$ of the iteration (5.7) satisfies

$$\theta(u) - \theta(\tilde{u}^k) + (u - \tilde{u}^k)^T F(\tilde{u}^k) \geq (u - \tilde{u}^k)^T Q(u^k - \tilde{u}^k), \;\; \forall u \in \Omega.$$

It is a form of (5.2a) where

$$Q = \begin{pmatrix} rI & A^T \\ 0 & sI \end{pmatrix} \quad \text{is not symmetric}$$

Correction – I For given v^k and the predictor $\tilde{v}^k$ by (5.7), we use

$$v^{k+1} = v^k - M(v^k - \tilde{v}^k), \tag{5.9}$$

to produce the new iterate, where

$$M = \begin{pmatrix} I_n & \frac{1}{r}A^T \\ 0 & I_m \end{pmatrix}$$

is a upper triangular block matrix whose diagonal part is unit matrix. Note that

$$H = QM^{-1} = \begin{pmatrix} rI_n & A^T \\ 0 & sI_m \end{pmatrix} \begin{pmatrix} I_n & -\frac{1}{r}A^T \\ 0 & I_m \end{pmatrix} = \begin{pmatrix} rI_n & 0 \\ 0 & sI_m \end{pmatrix} \succ 0.$$

In addition,

$$\begin{aligned} G &= Q^T + Q - M^T H M = Q^T + Q - Q^T M \\ &= \begin{pmatrix} rI_n & 0 \\ 0 & sI_m - \frac{1}{r}AA^T \end{pmatrix}. \end{aligned}$$

G is positive definite when $rs > \|A^T A\|$. The convergence conditions (5.3) are satisfied.

Correction – II In the correction step (5.9), the matrix M is a upper-triangular matrix. We can also use the lower-triangular matrix

$$M = \begin{pmatrix} I_n & 0 \\ -\frac{1}{s}A & I_m \end{pmatrix}$$

According to (5.3a), $H = QM^{-1}$, by a simple computation, we have

$$H = \begin{pmatrix} rI_n & A^T \\ 0 & sI_m \end{pmatrix} \begin{pmatrix} I_n & 0 \\ \frac{1}{s}A & I_m \end{pmatrix} = \begin{pmatrix} rI_n + \frac{1}{s}A^TA & A^T \\ A & sI_m \end{pmatrix}.$$

H is positive definite for any $r, s > 0$. In addition,

$$\begin{aligned} G &= Q^T + Q - M^THM = Q^T + Q - Q^TM \\ &= \begin{pmatrix} 2rI_n & A^T \\ A & 2sI_m \end{pmatrix} - \begin{pmatrix} rI_n & 0 \\ 0 & sI_m \end{pmatrix} = \begin{pmatrix} rI_n & A^T \\ A & sI_m \end{pmatrix}. \end{aligned}$$

G is positive definite when $rs > \|A^TA\|$. The convergence conditions (5.3) are satisfied.

5.2 Convergence proof in the unified framework

In this section, assuming the conditions (5.3) in the unified framework are satisfied, we prove some convergence properties

Theorem 1 *Let* $\{v^k\}$ *be the sequence generated by a method for the problem* (5.1) *and* $\tilde{w}^k$ *is obtained in the* k*-th iteration. If* v^k, v^{k+1} *and* $\tilde{w}^k$ *satisfy the conditions in the unified framework, then we have*

$$\|v^{k+1} - v^*\|_H^2 \le \|v^k - v^*\|_H^2 - \alpha\|v^k - \tilde{v}^k\|_G^2, \quad \forall v^* \in \mathcal{V}^*. \tag{5.10}$$

Proof. Using $Q = HM$ (see (5.3a)), the prediction can be written as

$$\theta(u) - \theta(\tilde{u}^k) + (w - \tilde{w}^k)^T F(\tilde{w}^k) \ge (v - \tilde{v}^k)^T HM(v^k - \tilde{v}^k), \ \forall w \in \Omega.$$

By using relation (5.2b), $v^k - v^{k+1} = \alpha M(v^k - \tilde{v}^k)$, we get

$$\alpha\{\theta(u) - \theta(\tilde{u}^k) + (w - \tilde{w}^k)^T F(\tilde{w}^k)\} \ge (v - \tilde{v}^k)^T H(v^k - v^{k+1}), \ \forall w \in \Omega.$$

Setting $w = w^*$ in the above inequality, we get

$$(\tilde{v}^k - v^*)^T H(v^k - v^{k+1}) \ge \alpha\{\theta(\tilde{u}^k) - \theta(u^*) + (\tilde{w}^k - w^*)^T F(\tilde{w}^k)\}, \ \forall w^* \in \Omega^*.$$

By using $(\tilde{w}^k - w^*)^T F(\tilde{w}^k) = (\tilde{w}^k - w^*)^T F(w^*)$ and the optimality of w^*, we have

$$(v^k - v^{k+1})^T(\tilde{v}^k - v^*) \geq 0, \quad \forall v^* \in \mathcal{V}^*. \tag{5.11}$$

Setting $a = v^k$, $b = v^{k+1}, c = \tilde{v}^k$ and $d = v^*$, in the identity

$$2(a-b)^T H(c-d) = \{\|a-d\|_H^2 - \|b-d\|_H^2\} - \{\|a-c\|_H^2 - \|b-c\|_H^2\},$$

it follows from (5.11) that

$$\|v^k - v^*\|_H^2 - \|v^{k+1} - v^*\|_H^2 \geq \|v^k - \tilde{v}^k\|_H^2 - \|v^{k+1} - \tilde{v}^k\|_H^2. \tag{5.12}$$

For the right hand side of the last inequality, we have

$$\begin{aligned}
&\|v^k - \tilde{v}^k\|_H^2 - \|v^{k+1} - \tilde{v}^k\|_H^2 \\
&\quad = \quad \|v^k - \tilde{v}^k\|_H^2 - \|(v^k - \tilde{v}^k) - (v^k - v^{k+1})\|_H^2 \\
&\quad \overset{(5.2b)}{=} \quad \|v^k - \tilde{v}^k\|_H^2 - \|(v^k - \tilde{v}^k) - \alpha M(v^k - \tilde{v}^k)\|_H^2 \\
&\quad = \quad 2\alpha(v^k - \tilde{v}^k)^T HM(v^k - \tilde{v}^k) - \alpha^2(v^k - \tilde{v}^k)^T M^T HM(v^k - \tilde{v}^k) \\
&\quad = \quad \alpha(v^k - \tilde{v}^k)^T(Q^T + Q - \alpha M^T HM)(v^k - \tilde{v}^k) \\
&\quad \overset{(5.3b)}{=} \quad \alpha\|v^k - \tilde{v}^k\|_G^2.
\end{aligned} \tag{5.13}$$

Substituting (5.13) in (5.12), the assertion of this theorem is proved. $\Box$

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Thank you very much for your attention！！