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Parallel splitting augmented Lagrangian methods for monotone structured variational inequalities

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Abstract The typical structured variational inequalities can be interpreted as a system of equilibrium problems with a leader and two cooperative followers. Assume that, based on the instruction given by the leader, each follower can solve the individual equilibrium sub-problems in his own way. The responsibility of the leader is to give a more reasonable instruction for the next iteration loop based on the feedback information from the followers. This consideration leads us to present a parallel splitting augmented Lagrangian method (abbreviated to PSALM). The proposed method can be extended to solve the system of equilibrium problems with three separable operators. Finally, it is explained why we cannot use the same technique to develop similar methods for problems with more than three separable operators.

Keywords Augmented Lagrangian method · Separable operators · Structured variational inequalities · Parallel splitting

1 Introduction

Let Ω be a closed convex set in $\mathcal{R}^n$ and F be a continuous mapping from $\mathcal{R}^n$ into itself. Variational inequality is to find a vector $u \in \Omega$, such that

$$(u' - u)^T F(u) \geq 0, \quad \forall u' \in \Omega. \tag{1.1}$$

The variational inequality concerned in this paper has the following typical separable structure:

$$u = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad F(u) = \begin{pmatrix} f(x) \\ g(y) \end{pmatrix} \quad \text{and} \tag{1.2}$$

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$$\Omega = \{(x, y) \mid x \in \mathcal{X},\ y \in \mathcal{Y},\ Ax + By = b\}, \tag{1.3}$$

$\mathcal{X} \subset \mathcal{R}^{n_1}$, $\mathcal{Y} \subset \mathcal{R}^{n_2}$, $A \in \mathcal{R}^{m\times n_1}$, $B \in \mathcal{R}^{m\times n_2}$ are given matrices, and $b \in \mathcal{R}^m$ is a given vector, $f : \mathcal{R}^{n_1} \to \mathcal{R}^{n_1}$, $g : \mathcal{R}^{n_2} \to \mathcal{R}^{n_2}$ are given monotone operators. By attaching a Lagrange multiplier vector $\lambda \in \mathcal{R}^m$ to the linear constraint $Ax + By = b$, Problem (1.1–1.3) can be explained as the following form (see [4, 7, 8]): Find $w = (x, y, \lambda) \in \mathcal{W}$, such that

$$\begin{cases} (x' - x)^T [f(x) - A^T\lambda] \geq 0, \\ (y' - y)^T [g(y) - B^T\lambda] \geq 0, \quad \forall w' \in \mathcal{W}, \\ Ax + By - b = 0, \end{cases} \tag{1.4}$$

where

$$\mathcal{W} = \mathcal{X} \times \mathcal{Y} \times \mathcal{R}^m. \tag{1.5}$$

Problem (1.4, 1.5) is referred as SVI$_2$ (*structured variational inequality with* 2 *separable operators*). Throughout this paper, we assume that the solution set of SVI$_2$ (or SVI$_3$), denoted by $\mathcal{W}^*$, is nonempty. Due to the monotonicity of the problem, $\mathcal{W}^*$ is convex (see Theorem 2.3.5 of [3]). In the following H denotes an $m \times m$ positive definite matrix.

SVI$_2$ can be viewed as a system of equilibrium problem which has a leader and two cooperative followers. Assume that, for given triple (x^k, y^k, λ^k) by the leader, each follower can offer the solution of the individual equilibrium problem

$$x \in \mathcal{X}, \quad (x' - x)^T \big\{ f(x) - A^T[\lambda^k - H(Ax + By^k - b)] \big\} \geq 0, \quad \forall x' \in \mathcal{X}, \tag{1.6}$$

or

$$y \in \mathcal{Y}, \quad (y' - y)^T \big\{ g(y) - B^T[\lambda^k - H(Ax^k + By - b)] \big\} \geq 0, \quad \forall y' \in \mathcal{Y}, \tag{1.7}$$

in his own way, say $\tilde{x}^k$ and $\tilde{y}^k$, respectively. With

$$\tilde{\lambda}^k = \lambda^k - H(A\tilde{x}^k + B\tilde{y}^k - b), \tag{1.8}$$

we get $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k)$ which can be viewed as the feedback information from the followers. In order to obtain the solution of the system of equilibrium problem (1.1–1.3), the question that the leader faced is what kind of $(x^{k+1}, y^{k+1}, \lambda^{k+1})$ will be more beneficial for the next iteration loop.

It is easy to check that $(\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k)$ is a solution of SVI$_2$ if $Ax^k = A\tilde{x}^k$, $By^k = B\tilde{y}^k$ and $\lambda^k = \tilde{\lambda}^k$. If we set $w^{k+1} = \tilde{w}^k$ which is produced by (1.6–1.8), it seems that the convergence cannot be established. One case of our proposed methods in this paper will update the new iterate by

$$w^{k+1} = w^k - \alpha_k(w^k - \tilde{w}^k), \tag{1.9}$$

where $\alpha_k \geq \alpha_{\min} > 0$ is a special step-size. For any $w^* = (x^*, y^*, \lambda^*) \in \mathcal{W}^*$, we will show that the sequence $\{w^k = (x^k, y^k, \lambda^k)\}$ generated by (1.6–1.9) satisfies

$$\|A(x^{k+1} - x^*)\|_H^2 + \|B(y^{k+1} - y^*)\|_H^2 + \|\lambda^{k+1} - \lambda^*\|_{H^{-1}}^2$$

$$\leq \left(\|A(x^k - x^*)\|_H^2 + \|B(y^k - y^*)\|_H^2 + \|\lambda^k - \lambda^*\|_{H^{-1}}^2\right) - \frac{3-2\sqrt{2}}{2}\left(\|A(x^k - \tilde{x}^k)\|_H^2 + \|B(y^k - \tilde{y}^k)\|_H^2 + \|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2\right). \quad (1.10)$$

Throughout this paper, we make the standard assumptions to guarantee that the problem under consideration is solvable and the proposed methods are well defined.

Assumption A

- $f(x)$ and $g(y)$ are monotone, i.e.,

$$(x - x')^T[f(x) - f(x')] \geq 0, \quad \forall x, x' \in \mathcal{X},$$

and

$$(y - y')^T[g(y) - g(y')] \geq 0, \quad \forall y, y' \in \mathcal{Y}.$$

- Problems (1.6) and (1.7) are solvable in each iteration.

For analysis convenience, we denote

$$M = \begin{pmatrix} A^T HA & & \\ & B^T HB & \\ & & H^{-1} \end{pmatrix}. \quad (1.11)$$

Even though M is positive semi-definite, in this paper we use $\|w - \tilde{w}\|_M$ to denote that

$$\|w - \tilde{w}\|_M = \left(\|A(x - \tilde{x})\|_H^2 + \|B(y - \tilde{y})\|_H^2 + \|\lambda - \tilde{\lambda}\|_{H^{-1}}^2\right)^{1/2}.$$

In this way, Inequality (1.10) can be rewritten in a compact form

$$\|w^{k+1} - w^*\|_M^2 \leq \|w^k - w^*\|_M^2 - \frac{3-2\sqrt{2}}{2}\|w^k - \tilde{w}^k\|_M^2. \quad (1.12)$$

2 The proposed method and the relationship to existing methods

This section describes the proposed method and indicates its relationship to the proximal-type methods, the augmented Lagrangian method and the alternating directions method.

2.1 The proposed method

The parallel splitting augmented Lagrangian method

Step 0. Let $\varepsilon > 0$, $w^0 = (x^0, y^0, \lambda^0) \in \mathcal{R}^{n_1} \times \mathcal{R}^{n_2} \times \mathcal{R}^m$, $G \in \mathcal{R}^{(n_1+n_2+m)\times(n_1+n_2+m)}$ be symmetric and positive definite, $k = 0$.
Step 1. *Produce* $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k)$ by (1.6–1.8).
Step 2. Convergence verification:

If $\max\{\|A(x^k - \tilde{x}^k)\|_\infty, \|B(y^k - \tilde{y}^k)\|_\infty, \|\lambda^k - \tilde{\lambda}^k\|_\infty\} < \varepsilon$, *then stop.* $(\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k)$ is an acceptable approximate solution.

Step 3. *Update* the new iterate by

$$w^{k+1} = w^k - \alpha_k G^{-1} M(w^k - \tilde{w}^k), \tag{2.1}$$

where

$$\begin{aligned} \alpha_k &= \gamma \alpha_k^*, \quad \gamma \in [1, 2), \\ \alpha_k^* &= \frac{\varphi(w^k, \tilde{w}^k)}{\|G^{-1} M(w^k - \tilde{w}^k)\|_G^2}, \end{aligned} \tag{2.2}$$

and

$$\varphi(w^k, \tilde{w}^k) = \|w^k - \tilde{w}^k\|_M^2 + (\lambda^k - \tilde{\lambda}^k)^T \big[(Ax^k - A\tilde{x}^k) + (By^k - B\tilde{y}^k)\big]. \tag{2.3}$$

Set $k := k + 1$ and *goto* Step 1.

In fact, each iteration of the proposed method consists of two main procedures, namely Step 1 and Step 3. The proposed method can be viewed as the 'leader-followers' procedures. The leader is responsible for designing the algorithm. From the given (x^k, y^k, λ^k), the task of the cooperative followers is to solve their individual subproblems (in Step 1). After receiving the information from the followers, $(\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k)$, the leader should give the new iterate $(x^{k+1}, y^{k+1}, \lambda^{k+1})$ which is accomplished in Step 3.

Remark 2.1 Theoretically, the matrix G is an arbitrary $(n_1 + n_2 + m) \times (n_1 + n_2 + m)$ symmetric and positive definite matrix. For example, we can simply set G to be any positive diagonal matrix. If the matrix M is non-singular, let $G = M$ then (2.1) degenerates to (1.9).

Remark 2.2 Because $\tilde{x}^k \in \mathcal{X}$ and $\tilde{y}^k \in \mathcal{Y}$ are solutions of (1.6) and (1.7), respectively, we have

$$(x' - \tilde{x}^k)^T \big\{ f(\tilde{x}^k) - A^T[\lambda^k - H(A\tilde{x}^k + By^k - b)] \big\} \ge 0, \quad \forall x' \in \mathcal{X} \tag{2.4}$$

and

$$(y' - \tilde{y}^k)^T \big\{ g(\tilde{y}^k) - B^T[\lambda^k - H(Ax^k + B\tilde{y}^k - b)] \big\} \ge 0, \quad \forall y' \in \mathcal{Y}. \tag{2.5}$$

Due to (2.4, 2.5) and using

$$\tilde{\lambda}^k = \lambda^k - H(A\tilde{x}^k + B\tilde{y}^k - b), \tag{2.6}$$

we have

$$\begin{pmatrix} x' - \tilde{x}^k \\ y' - \tilde{y}^k \end{pmatrix}^T \begin{pmatrix} f(\tilde{x}^k) - A^T\tilde{\lambda}^k + A^T H(By^k - B\tilde{y}^k) \\ g(\tilde{y}^k) - B^T\tilde{\lambda}^k + B^T H(Ax^k - A\tilde{x}^k) \end{pmatrix} \ge 0, \quad \forall x' \in \mathcal{X},\ y' \in \mathcal{Y}. \tag{2.7}$$

Remark 2.3 Since $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{R}^m$ is generated by (1.6–1.8) from a given $w^k = (x^k, y^k, \lambda^k)$, according to (2.6) and (2.7), we have

$$\begin{cases} Ax^k = A\tilde{x}^k \\ By^k = B\tilde{y}^k \\ \lambda^k = \tilde{\lambda}^k \end{cases} \Longrightarrow \begin{cases} (x' - \tilde{x}^k)^T[f(\tilde{x}^k) - A^T\tilde{\lambda}^k] \ge 0, \\ (y' - \tilde{y}^k)^T[g(\tilde{y}^k) - B^T\tilde{\lambda}^k] \ge 0, \\ A\tilde{x}^k + B\tilde{y}^k - b = 0, \end{cases} \quad \forall w' \in \mathcal{W}. \tag{2.8}$$

In other words, $\tilde{w}^k$ is a solution of Problem (1.4, 1.5) if $Ax^k = A\tilde{x}^k$, $By^k = B\tilde{y}^k$ and $\lambda^k = \tilde{\lambda}^k$. In fact, this is the base of the stopping criterion in Step 2 of the proposed method.

Remark 2.4 The update form (2.1) in Step 3 of the algorithm is based on the fact that $-G^{-1}M(w^k - \tilde{w}^k)$ is a descent direction of the unknown distance function $\|w - w^*\|_G^2$ at the point w^k. This property will be proved in Sect. 3.1. α_k^* in (2.2) is the 'optimal' step size, which will be shown in Sect. 3.2. We can also use (1.9) to update the new iterate. In that case, instead of (2.2), the 'optimal' step size becomes

$$\alpha_k^* = \frac{\varphi(w^k, \tilde{w}^k)}{\|w^k - \tilde{w}^k\|_M^2}. \tag{2.9}$$

For fast convergence, the practical step size should be multiplied by a relaxed factor $\gamma \in [1, 2)$.

Remark 2.5 Note that $\|w^k - \tilde{w}^k\|_M = 0$ if and only if $Ax^k = A\tilde{x}^k$, $By^k = B\tilde{y}^k$ and $\lambda^k = \tilde{\lambda}^k$. In the case $\|w^k - \tilde{w}^k\|_M \neq 0$, $\varphi(w^k, \tilde{w}^k)$ is positive. We state this fact in the following lemma.

Lemma 2.1 *Let $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k)$ be generated by* (1.6–1.8) *from a given $w^k = (x^k, y^k, \lambda^k)$ and $\varphi(w^k, \tilde{w}^k)$ be defined in* (2.3). *Then we have*

$$\varphi(w^k, \tilde{w}^k) \ge \frac{2 - \sqrt{2}}{2} \|w^k - \tilde{w}^k\|_M^2. \tag{2.10}$$

Proof First, according to the definitions (see (2.3) and (1.11)),

$$\begin{aligned} \varphi(w^k, \tilde{w}^k) &= \|A(x^k - \tilde{x}^k)\|_H^2 + \|B(y^k - \tilde{y}^k)\|_H^2 + \|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2 \\ &\quad + (Ax^k - A\tilde{x}^k)^T(\lambda^k - \tilde{\lambda}^k) + (By^k - B\tilde{y}^k)^T(\lambda^k - \tilde{\lambda}^k). \end{aligned} \tag{2.11}$$

By using the Cauchy–Schwarz Inequality, we have

$$(Ax^k - A\tilde{x}^k)^T(\lambda^k - \tilde{\lambda}^k) \ge -\frac{1}{2}\left(\sqrt{2}\|A(x^k - \tilde{x}^k)\|_H^2 + \frac{1}{\sqrt{2}}\|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2\right) \tag{2.12}$$

and

$$(By^k - B\tilde{y}^k)^T(\lambda^k - \tilde{\lambda}^k) \ge -\frac{1}{2}\left(\sqrt{2}\|B(y^k - \tilde{y}^k)\|_H^2 + \frac{1}{\sqrt{2}}\|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2\right). \tag{2.13}$$

Substituting (2.12, 2.13) in (2.11), we have

$$\varphi(w^k, \tilde{w}^k) \geq \frac{2-\sqrt{2}}{2}\left(\|A(x^k - \tilde{x}^k)\|_H^2 + \|B(y^k - \tilde{y}^k)\|_H^2 + \|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2\right),$$

and the assertion is obtained. □

2.2 Relationships to existing methods

Now, we indicate the relationship of the proposed method to some existing methods. First, the proposed method in this paper is different from the proximal-type methods [1, 2, 9, 11, 12] because sub-problem (1.6) (resp. (1.7)) does not include the proximal term such as $r(x - x^k)$ (resp. $s(y - y^k)$). What is the difference between the proposed method and other similar methods?

The *augmented Lagrangian method* is one of the attractive classical methods in numerical optimization [10] which does not have proximal terms in its sub-problems. When the method is applied to solve SVI_2, the iteration is from a given λ^k to a new λ^{k+1}. It takes the solution of the following auxiliary problem

$$u = (x, y) \in \mathcal{X} \times \mathcal{Y},$$
$$\begin{cases} (x' - x)^T \left\{ f(x) - A^T[\lambda^k - H(Ax + By - b)] \right\} \geq 0, & \forall x' \in \mathcal{X}, \\ (y' - y)^T \left\{ g(y) - B^T[\lambda^k - H(Ax + By - b)] \right\} \geq 0, & \forall y' \in \mathcal{Y}, \end{cases} \tag{2.14}$$

as u^{k+1}. Then the multipliers are updated by

$$\lambda^{k+1} = \lambda^k - \gamma H(Ax^{k+1} + By^{k+1} - b), \tag{2.15}$$

where $\gamma \in (0, 2)$ is a relaxed factor. It is clear that $(x^{k+1}, y^{k+1}, \lambda^{k+1})$ is a solution of SVI_2 if and only if $\lambda^k = \lambda^{k+1}$. It was proved in [8] that, for any $(x^*, y^*, \lambda^*) \in \mathcal{W}^*$, the sequence $\{\lambda^k\}$ generated by the augmented Lagrangian method (2.14, 2.15) for SVI_2 satisfies

$$\|\lambda^{k+1} - \lambda^*\|_{H^{-1}}^2 \leq \|\lambda^k - \lambda^*\|_{H^{-1}}^2 - \gamma(2-\gamma)\|\lambda^k - \lambda^{k+1}\|_{H^{-1}}^2. \tag{2.16}$$

The main disadvantage is that the unknown vectors x and y in (2.14) are overlapped.

If we solve the auxiliary problem (2.14) separately in a determinate order, it leads to the *alternating directions method* [5–8]. Its iteration is from a given duple (y^k, λ^k) to a new duple (y^{k+1}, λ^{k+1}). Namely, it takes the solution of the following problem

$$x \in \mathcal{X}, \quad (x' - x)^T \left\{ f(x) - A^T[\lambda^k - H(Ax + By^k - b)] \right\} \geq 0, \quad \forall x' \in \mathcal{X}, \tag{2.17}$$

as x^{k+1} and then y^{k+1} is produced by solving

$$y \in \mathcal{Y}, \quad (y' - y)^T \left\{ g(y) - B^T[\lambda^k - H(Ax^{k+1} + By - b)] \right\} \geq 0, \quad \forall y' \in \mathcal{Y}. \tag{2.18}$$

Finally, the multipliers are updated by

$$\lambda^{k+1} = \lambda^k - \gamma H(Ax^{k+1} + By^{k+1} - b), \tag{2.19}$$

where $\gamma \in (0, \frac{\sqrt{5}+1}{2})$ is a relaxed factor given by Glowinski [7]. It is clear that $(x^{k+1}, y^{k+1}, \lambda^{k+1})$ is a solution of SVI_2 if $By^k = By^{k+1}$ and $\lambda^k = \lambda^{k+1}$. For any $(x^*, y^*, \lambda^*) \in \mathcal{W}^*$, in the case of $\gamma = 1$, the sequence $\{(y^k, \lambda^k)\}$ generated by alternating directions method (2.17–2.19) for SVI_2 satisfies

$$\begin{aligned} &\|B(y^{k+1} - y^*)\|_H^2 + \|\lambda^{k+1} - \lambda^*\|_{H^{-1}}^2 \\ &\quad \le \big(\|B(y^k - y^*)\|_H^2 + \|\lambda^k - \lambda^*\|_{H^{-1}}^2\big) - \big(\|B(y^k - y^{k+1})\|_H^2 + \|\lambda^k - \lambda^{k+1}\|_{H^{-1}}^2\big). \end{aligned} \tag{2.20}$$

In each iteration, the alternating directions method decomposes the sub-problem (2.14) into (2.17) and (2.18) but not in a parallel wise.

Note that, in our proposed method, Problems (1.6) and (1.7), which produce $\tilde{x}^k$ and $\tilde{y}^k$, are parallel decomposed from (2.14). This is also the reason that we call the proposed method *parallel splitting augmented Lagrangian method*. In addition, instead of taking the solution of the sub-problems, the new iterate in the proposed method is updated by a simple manipulation, e.g., (2.1). Another specialty is that the proposed method can be extended to deal with problems which include three separable operators.

3 Convergence of the proposed PSALM

In the proposed method, the first procedure (accomplished by the followers) offers a descent direction of the unknown distance function, and the second procedure (accomplished by the leader) determines the 'optimal' step size along this direction. This section shows the details.

3.1 The descent direction in the proposed PSALM

For any $w^* \in \mathcal{W}^*$, $G(w^k - w^*)$ is the gradient of the unknown distance function $\frac{1}{2}\|w - w^*\|_G^2$ at the point $w^k \notin \mathcal{W}^*$. A direction d is called a descent direction of $\frac{1}{2}\|w - w^*\|_G^2$ at the point w^k if and only if the inner-product $\langle G(w^k - w^*),\ d\rangle < 0$. Let $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k)$ be generated by (1.6–1.8) from a given $w^k = (x^k, y^k, \lambda^k)$. The objective of this subsection is to show that for any $w^* \in \mathcal{W}^*$,

$$(w^k - w^*)^T M(w^k - \tilde{w}^k) \ge \varphi(w^k, \tilde{w}^k). \tag{3.1}$$

Combining with (2.10), we have

$$(w^k - w^*)^T M(w^k - \tilde{w}^k) \ge \frac{2-\sqrt{2}}{2}\|w^k - \tilde{w}^k\|_M^2. \tag{3.2}$$

It guarantees that $G^{-1}M(\tilde{w}^k - w^k)$ is a descent direction of $\|w - w^*\|_G^2$ at the point $w^k \notin \mathcal{W}^*$. We call (3.2) the key inequality of the proposed methods for problems with two separable operators.

Lemma 3.1 *Let $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k)$ be generated by (1.6–1.8) from a given $w^k = (x^k, y^k, \lambda^k)$. Then for any $w^* = (x^*, y^*, \lambda^*) \in \mathcal{W}^*$ we have*

$$(\tilde{\lambda}^k - \lambda^*)^T H^{-1}(\lambda^k - \tilde{\lambda}^k) \\ \geq (A\tilde{x}^k - Ax^*)^T H(By^k - B\tilde{y}^k) + (B\tilde{y}^k - By^*)^T H(Ax^k - A\tilde{x}^k). \quad (3.3)$$

Proof Since $w^* \in \mathcal{W}^*$, $\tilde{x}^k \in \mathcal{X}$ and $\tilde{y}^k \in \mathcal{Y}$, it follows from (1.4) that

$$\begin{pmatrix} \tilde{x}^k - x^* \\ \tilde{y}^k - y^* \end{pmatrix}^T \begin{pmatrix} f(x^*) - A^T\lambda^* \\ g(y^*) - B^T\lambda^* \end{pmatrix} \geq 0, \quad (3.4)$$

and

$$Ax^* + By^* - b = 0.$$

On the other hand, since $x^* \in \mathcal{X}$ and $y^* \in \mathcal{Y}$, from (2.7) we have

$$\begin{pmatrix} x^* - \tilde{x}^k \\ y^* - \tilde{y}^k \end{pmatrix}^T \begin{pmatrix} f(\tilde{x}^k) - A^T\tilde{\lambda}^k + A^T H(By^k - B\tilde{y}^k) \\ g(\tilde{y}^k) - B^T\tilde{\lambda}^k + B^T H(Ax^k - A\tilde{x}^k) \end{pmatrix} \geq 0. \quad (3.5)$$

Adding (3.4) and (3.5), and using the monotonicity of f and g, we obtain

$$\begin{pmatrix} \tilde{x}^k - x^* \\ \tilde{y}^k - y^* \end{pmatrix}^T \begin{pmatrix} A^T(\tilde{\lambda}^k - \lambda^*) - A^T H(By^k - B\tilde{y}^k) \\ B^T(\tilde{\lambda}^k - \lambda^*) - B^T H(Ax^k - A\tilde{x}^k) \end{pmatrix} \geq 0,$$

and it follows that

$$(\tilde{\lambda}^k - \lambda^*)^T \big[(A\tilde{x}^k - Ax^*) + (B\tilde{y}^k - By^*)\big] \\ \geq (A\tilde{x}^k - Ax^*)^T H(By^k - B\tilde{y}^k) + (B\tilde{y}^k - By^*)^T H(Ax^k - A\tilde{x}^k). \quad (3.6)$$

Since $Ax^* + By^* = b$ and $A\tilde{x}^k + B\tilde{y}^k - b = H^{-1}(\lambda^k - \tilde{\lambda}^k)$, from (3.6) we get the result of this lemma directly. □

Lemma 3.2 *Let $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k)$ be generated by (1.6–1.8) from a given $w^k = (x^k, y^k, \lambda^k)$. Then for any $w^* = (x^*, y^*, \lambda^*) \in \mathcal{W}^*$ we have*

$$(\tilde{w}^k - w^*)^T M(w^k - \tilde{w}^k) \geq (\lambda^k - \tilde{\lambda}^k)^T \big[(Ax^k - A\tilde{x}^k) + (By^k - B\tilde{y}^k)\big], \quad (3.7)$$

and consequently

$$(w^k - w^*)^T M(w^k - \tilde{w}^k) \geq \varphi(w^k, \tilde{w}^k), \quad (3.8)$$

where $\varphi(w^k, \tilde{w}^k)$ is defined in (2.3).

Proof First, using matrix M (see (1.11)), we have

$$(\tilde{w}^k - w^*)^T M(w^k - \tilde{w}^k) = (A\tilde{x}^k - Ax^*)^T H(Ax^k - A\tilde{x}^k) + (B\tilde{y}^k - By^*)^T \\ \times H(By^k - B\tilde{y}^k) + (\tilde{\lambda}^k - \lambda^*)^T H^{-1}(\lambda^k - \tilde{\lambda}^k). \quad (3.9)$$

Substituting (3.3) in the last term of the right-hand side of (3.9), we get

$$\begin{aligned}(\tilde{w}^k - w^*)^T M(w^k - \tilde{w}^k) &\\ \geq (A\tilde{x}^k - Ax^*)^T H(Ax^k - A\tilde{x}^k) &+ (B\tilde{y}^k - By^*)^T H(By^k - B\tilde{y}^k)\\ + (B\tilde{y}^k - By^*)^T H(Ax^k - A\tilde{x}^k) &+ (A\tilde{x}^k - Ax^*)^T H(By^k - B\tilde{y}^k).\end{aligned}$$

Using $Ax^* + By^* = b$, from the above inequality we obtain

$$(\tilde{w}^k - w^*)^T M(w^k - \tilde{w}^k) \geq (A\tilde{x}^k + B\tilde{y}^k - b)^T H\big[(Ax^k - A\tilde{x}^k) + (By^k - B\tilde{y}^k)\big]. \tag{3.10}$$

Since $H(A\tilde{x}^k + B\tilde{y}^k - b) = \lambda^k - \tilde{\lambda}^k$, from (3.10) we get assertion (3.7). Using the definition of $\varphi(w^k, \tilde{w}^k)$, (3.8) follows from (3.7) directly. □

From (3.8) and (2.10) we get the key inequality (3.2) and thus we have the following theorem.

Theorem 3.1 *Let $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k)$ be generated by (1.6–1.8) from a given $w^k = (x^k, y^k, \lambda^k)$. If $\|w^k - \tilde{w}^k\|_M \neq 0$, then for any positive definite matrix G and any $w^* \in \mathcal{W}^*$,*

$$-G^{-1}M(w^k - \tilde{w}^k)$$

is a descent direction of the unknown distance function $\frac{1}{2}\|w - w^*\|_G^2$ at the current point w^k.

3.2 The step size and the new iterate

Since $-G^{-1}M(w^k - \tilde{w}^k)$ is a descent direction of $\|w - w^*\|_G^2$ at the point w^k, the new iterate will be determined along this direction by choosing a suitable step size. In order to explain why we have the 'optimal' step α_k^* as defined in (2.2), we let

$$w^{k+1}(\alpha) = w^k - \alpha G^{-1}M(w^k - \tilde{w}^k) \tag{3.11}$$

be the step-size-dependent new iterate and

$$\theta_k(\alpha) = \|w^k - w^*\|_G^2 - \|w^{k+1}(\alpha) - w^*\|_G^2 \tag{3.12}$$

be the profit function of the k-th iteration. Because $\theta_k(\alpha)$ includes the unknown vector w^*, it cannot be maximized directly. The following lemma offers us a lower bound of $\theta_k(\alpha)$ which is a quadratic function of α.

Lemma 3.3 *Let $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k)$ be generated by (1.6–1.8) from a given $w^k = (x^k, y^k, \lambda^k)$ and $\varphi(w^k, \tilde{w}^k)$ be defined in (2.3). Then we have*

$$\theta_k(\alpha) \geq q_k(\alpha), \quad \forall \alpha > 0, \tag{3.13}$$

where

$$q_k(\alpha) = 2\alpha\varphi(w^k, \tilde{w}^k) - \alpha^2\|G^{-1}M(w^k - \tilde{w}^k)\|_G^2. \tag{3.14}$$

Proof It follows from (3.11, 3.12) and (3.8) that

$$\begin{aligned}\theta_k(\alpha) &= \|w^k - w^*\|_G^2 - \|w^k - w^* - \alpha G^{-1}M(w^k - \tilde{w}^k)\|_G^2 \\ &= 2\alpha(w^k - w^*)^T M(w^k - \tilde{w}^k) - \alpha^2\|G^{-1}M(w^k - \tilde{w}^k)\|_G^2 \\ &\geq 2\alpha\varphi(w^k, \tilde{w}^k) - \alpha^2\|G^{-1}M(w^k - \tilde{w}^k)\|_G^2. \end{aligned} \tag{3.15}$$

The right-hand side of (3.15) is $q_k(\alpha)$ and the proof is complete. □

Remark 3.1 Without loss of generality, we assume that $\|w^k - \tilde{w}^k\|_M^2 > 0$, otherwise, as pointed in Remark 2.3, w^k is a solution point of Problem (1.4, 1.5) and the iteration is stopped. Note that $q_k(0) = 0$ and $q_k'(0) = 2\varphi(w^k, \tilde{w}^k) \geq (2-\sqrt{2})\|w^k - \tilde{w}^k\|_M^2 > 0$ (see (2.10)), therefore, there is an $\bar{\alpha}_k > 0$, such that $q_k(\alpha) > 0$ for all $\alpha \in (0, \bar{\alpha}_k)$.

Since $q_k(\alpha)$ is a quadratic function of α, it reaches its maximum at

$$\alpha_k^* = \frac{\varphi(w^k, \tilde{w}^k)}{\|G^{-1}M(w^k - \tilde{w}^k)\|_G^2}, \tag{3.16}$$

this is just the same as defined in (2.2). Because some inequalities are used in proof of (3.8), in practical computation, taking a relaxed factor $\gamma > 1$ is wise for fast convergence. Note that for any $\alpha_k = \gamma\alpha_k^*$, it follows from (3.13, 3.14) and (3.16) that

$$\begin{aligned}\theta_k(\gamma\alpha_k^*) \geq q_k(\gamma\alpha_k^*) &= 2\gamma\alpha_k^*\varphi(w^k, \tilde{w}^k) - \gamma^2\alpha_k^*\big(\alpha_k^*\|G^{-1}M(w^k - \tilde{w}^k)\|_G^2\big) \\ &= \gamma(2-\gamma)\alpha_k^*\varphi(w^k, \tilde{w}^k). \end{aligned} \tag{3.17}$$

In order to guarantee the right-hand side of (3.17) is positive, we often take $\gamma \in [1, 2)$.

Now, we are in the stage to prove the main convergence theorem of this paper.

Theorem 3.2 *For any $w^* = (x^*, y^*, \lambda^*) \in \mathcal{W}^*$, the sequence $\{w^k = (x^k, y^k, \lambda^k)\}$ generated by the proposed method satisfies*

$$\|w^{k+1} - w^*\|_G^2 \leq \|w^k - w^*\|_G^2 - \frac{\gamma(2-\gamma)(3-2\sqrt{2})}{2\|M^{1/2}G^{-1}M^{1/2}\|}\|w^k - \tilde{w}^k\|_M^2. \tag{3.18}$$

Thus we have

$$\lim_{k\to\infty}\|w^k - \tilde{w}^k\|_M = 0, \tag{3.19}$$

and the iterations of the proposed method will be terminated in finite loops.

Proof First, it follows from (3.12) and (3.17) that

$$\|w^{k+1} - w^*\|_G^2 \leq \|w^k - w^*\|_G^2 - \gamma(2-\gamma)\alpha_k^*\varphi(w^k, \tilde{w}^k). \tag{3.20}$$

Since

$$\|G^{-1}M(w^k - \tilde{w}^k)\|_G^2 \leq \|M^{1/2}G^{-1}M^{1/2}\| \cdot \|w^k - \tilde{w}^k\|_M^2, \tag{3.21}$$

it follows from (3.16) and (2.10) that

$$\alpha_k^* \geq \frac{2-\sqrt{2}}{2\|M^{1/2}G^{-1}M^{1/2}\|}. \tag{3.22}$$

Consequently, using (2.10) again, we have

$$\begin{aligned}\alpha_k^*\varphi(w^k,\tilde{w}^k) &\geq \frac{1}{\|M^{1/2}G^{-1}M^{1/2}\|}\left(\frac{2-\sqrt{2}}{2}\right)^2\|w^k-\tilde{w}^k\|_M^2\\ &= \frac{3-2\sqrt{2}}{2\|M^{1/2}G^{-1}M^{1/2}\|}\|w^k-\tilde{w}^k\|_M^2.\end{aligned} \tag{3.23}$$

Substituting (3.23) in (3.20), assertion (3.18) is proved. Therefore, we have

$$\frac{\gamma(2-\gamma)(3-2\sqrt{2})}{2\|M^{1/2}G^{-1}M^{1/2}\|}\sum_{k=0}^{\infty}\|w^k-\tilde{w}^k\|_M^2 \leq \|w^0-w^*\|_G^2, \tag{3.24}$$

and assertion (3.19) follows immediately. Since we use

$$\max\left\{\|A(x^k-\tilde{x}^k)\|_\infty, \|B(y^k-\tilde{y}^k)\|_\infty, \|\lambda^k-\tilde{\lambda}^k\|_\infty\right\} < \varepsilon$$

as the stopping criterion, it follows from (3.19) that the iterations will be terminated in finite loops for any given $\varepsilon > 0$. □

We can use the special update form (1.9) to produce the new iterate. In this case,

$$\begin{aligned}\|w^{k+1}-w^*\|_M^2 &= \|(w^k-w^*)+\alpha_k(\tilde{w}^k-w^k)\|_M^2\\ &= \|w^k-w^*\|_M^2+2\alpha_k(w^k-w^*)^TM(\tilde{w}^k-w^k)+\alpha_k^2\|w^k-\tilde{w}^k\|_M^2.\end{aligned} \tag{3.25}$$

From (3.1) and (3.25), we get

$$\|w^{k+1}-w^*\|_M^2 \leq \|w^k-w^*\|_M^2 - 2\alpha_k\varphi(w^k,\tilde{w}^k)+\alpha_k^2\|w^k-\tilde{w}^k\|_M^2. \tag{3.26}$$

If we set $\alpha_k = \alpha_k^*$ defined in (2.9), it follows from (2.10) that

$$\begin{aligned}\|w^{k+1}-w^*\|_M^2 &\leq \|w^k-w^*\|_M^2 - \alpha_k^*\varphi(w^k,\tilde{w}^k)\\ &\leq \|w^k-w^*\|_M^2 - \left(\frac{2-\sqrt{2}}{2}\right)^2\|w^k-\tilde{w}^k\|_M^2\\ &= \|w^k-w^*\|_M^2 - \left(\frac{3-2\sqrt{2}}{2}\right)\|w^k-\tilde{w}^k\|_M^2.\end{aligned}$$

This is just inequality (1.12) mentioned in the introductory section.

4 Extended method for problems with three separable operators

The parallel splitting augmented Lagrangian method in Sect. 2 can be extended to solve structured variational inequality with three separable operators (SVI_3):

$$\text{Find} \quad u = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \Omega, \quad \text{such that} \quad \begin{cases} (x' - x)^T f(x) \geq 0, \\ (y' - y)^T g(y) \geq 0, \\ (z' - z)^T h(z) \geq 0, \end{cases} \quad \forall u' \in \Omega, \tag{4.1}$$

where

$$\Omega = \{(x, y, z) \mid x \in \mathcal{X}, y \in \mathcal{Y}, z \in \mathcal{Z}, Ax + By + Cz = b\}, \tag{4.2}$$

$\mathcal{X} \subset \mathcal{R}^{n_1}$, $\mathcal{Y} \subset \mathcal{R}^{n_2}$, $\mathcal{Z} \subset \mathcal{R}^{n_3}$, $A \in \mathcal{R}^{m\times n_1}$, $B \in \mathcal{R}^{m\times n_2}$, $C \in \mathcal{R}^{m\times n_3}$ are given matrices, and $b \in \mathcal{R}^m$ is a given vector, $f : \mathcal{R}^{n_1} \to \mathcal{R}^{n_1}$, $g : \mathcal{R}^{n_2} \to \mathcal{R}^{n_2}$, $h : \mathcal{R}^{n_3} \to \mathcal{R}^{n_3}$ are given monotone operators. By attaching a Lagrange multiplier vector $\lambda \in \mathcal{R}^m$ to the linear constraint $Ax + By + Cz = b$, SVI_3 (4.1, 4.2) can be explained as the following form:

$$\text{Find} \quad w = \begin{pmatrix} x \\ y \\ z \\ \lambda \end{pmatrix} \in \mathcal{W}, \quad \text{such that} \quad \begin{cases} (x' - x)^T [f(x) - A^T\lambda] \geq 0, \\ (y' - y)^T [g(y) - B^T\lambda] \geq 0, \\ (z' - z)^T [h(z) - C^T\lambda] \geq 0, \\ Ax + By + Cz - b = 0, \end{cases} \quad \forall w' \in \mathcal{W}, \tag{4.3}$$

where

$$\mathcal{W} = \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \times \mathcal{R}^m. \tag{4.4}$$

For the problems in this section, we denote

$$M = \begin{pmatrix} 2A^T HA & A^T HB & A^T HC & \\ B^T HA & 2B^T HB & B^T HC & \\ C^T HA & C^T HB & 2C^T HC & \\ & & & H^{-1} \end{pmatrix}. \tag{4.5}$$

In general, M is positive semi-definite and thus $\|\cdot\|_M$ is not a norm. Only for analysis convenience, we use the notation $\|w^k - \tilde{w}^k\|_M^2$ to denote $(w^k - \tilde{w}^k)^T M (w^k - \tilde{w}^k)$. Note that for M in (4.5), we have

$$\begin{aligned} \|w^k - \tilde{w}^k\|_M^2 = {} & \big\|A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\big\|_H^2 + \big\|B(y^k - \tilde{y}^k) + C(z^k - \tilde{z}^k)\big\|_H^2 \\ & + \big\|C(z^k - \tilde{z}^k) + A(x^k - \tilde{x}^k)\big\|_H^2 + \|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2. \end{aligned} \tag{4.6}$$

4.1 PSALM for problem with three separable operators

Step 0. Let $\varepsilon > 0$, $\gamma \in [1, 2)$, *G be an* $(n_1 + n_2 + n_3 + m) \times (n_1 + n_2 + n_3 + m)$ positive definite matrix, $w^0 = (x^0, y^0, z^0, \lambda^0) \in \mathcal{R}^{n_1} \times \mathcal{R}^{n_2} \times \mathcal{R}^{n_3} \times \mathcal{R}^m$. *Set* $k = 0$.

Step 1. *Parallel find* $\tilde{x}^k \in \mathcal{X}$, $\tilde{y}^k \in \mathcal{Y}$ and $\tilde{z}^k \in \mathcal{Z}$ which are the solutions of

$$x \in \mathcal{X}, \quad (x'-x)^T\{f(x) - A^T[\lambda^k - H(Ax + By^k + Cz^k - b)]\} \ge 0, \quad \forall x' \in \mathcal{X}, \tag{4.7}$$

$$y \in \mathcal{Y}, \quad (y'-y)^T\{g(y) - B^T[\lambda^k - H(Ax^k + By + Cz^k - b)]\} \ge 0, \quad \forall y' \in \mathcal{Y}, \tag{4.8}$$

and

$$z \in \mathcal{Z}, \quad (z'-z)^T\{h(z) - C^T[\lambda^k - H(Ax^k + By^k + Cz - b)]\} \ge 0, \quad \forall z' \in \mathcal{Z}, \tag{4.9}$$

respectively. *Set*

$$\tilde{\lambda}^k = \lambda^k - H(A\tilde{x}^k + B\tilde{y}^k + C\tilde{z}^k - b). \tag{4.10}$$

Step 2. Convergence verification:
If $\max\big\{\|A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\|_\infty,\ \|B(y^k - \tilde{y}^k) + C(z^k - \tilde{z}^k)\|_\infty,$ $\|C(z^k - \tilde{z}^k) + A(x^k - \tilde{x}^k)\|_\infty,\ \|\lambda^k - \tilde{\lambda}^k\|_\infty\big\} < \varepsilon$, *then stop.*
$(\tilde{x}^k, \tilde{y}^k, \tilde{z}^k, \tilde{\lambda}^k)$ is an acceptable approximate solution.

Step 3. Produce the new iterate by

$$w^{k+1} = w^k - \alpha_k G^{-1} M(w^k - \tilde{w}^k), \tag{4.11}$$

where

$$\alpha_k = \gamma \alpha_k^*, \qquad \alpha_k^* = \frac{\varphi(w^k, \tilde{w}^k)}{\|G^{-1}M(w^k - \tilde{w}^k)\|_G^2} \tag{4.12}$$

and

$$\varphi(w^k, \tilde{w}^k) = \|w^k - \tilde{w}^k\|_M^2 + 2(\lambda^k - \tilde{\lambda}^k)^T \times \big[A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k) + C(z^k - \tilde{z}^k)\big]. \tag{4.13}$$

Set $k := k+1$ and *goto* Step 1.

As in Sect. 2, we list some basic properties of the proposed method for problems with three separable operators in the following remarks.

Remark 4.1 First, for any $x \in \mathcal{X}$, $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$, due to (4.7–4.9) and using (4.10), we have

$$\begin{pmatrix} x - \tilde{x}^k \\ y - \tilde{y}^k \\ z - \tilde{z}^k \end{pmatrix}^T \begin{pmatrix} f(\tilde{x}^k) - A^T\tilde{\lambda}^k + A^T H[(By^k - B\tilde{y}^k) + (Cz^k - C\tilde{z}^k)] \\ g(\tilde{y}^k) - B^T\tilde{\lambda}^k + B^T H[(Ax^k - A\tilde{x}^k) + (Cz^k - C\tilde{z}^k)] \\ h(\tilde{z}^k) - C^T\tilde{\lambda}^k + C^T H[(Ax^k - A\tilde{x}^k) + (By^k - B\tilde{y}^k)] \end{pmatrix} \ge 0. \tag{4.14}$$

Remark 4.2 Since $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, z^k, \tilde{\lambda}^k) \in \mathcal{W}$ is generated by (4.7–4.10) from a given $w^k = (x^k, y^k, z^k, \lambda^k)$, according to (4.10) and (4.14), we have if

$$\begin{cases} B(y^k - \tilde{y}^k) + C(z^k - \tilde{z}^k) = 0, \\ A(x^k - \tilde{x}^k) + C(z^k - \tilde{z}^k) = 0, \\ A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k) = 0, \\ \lambda^k - \tilde{\lambda}^k = 0, \end{cases}$$

then

$$\begin{cases} (x' - \tilde{x}^k)^T \{f(\tilde{x}^k) - A^T \tilde{\lambda}^k\} \geq 0, \\ (y' - \tilde{y}^k)^T \{g(\tilde{y}^k) - B^T \tilde{\lambda}^k\} \geq 0, \\ (z' - \tilde{z}^k)^T \{h(\tilde{z}^k) - C^T \tilde{\lambda}^k\} \geq 0, \\ A\tilde{x}^k + B\tilde{y}^k + C\tilde{z}^k - b = 0, \end{cases} \quad \forall w' \in \mathcal{W}$$

and thus $\tilde{w}^k$ is a solution of Problem (4.3). The stopping criterion in Step 2 is based on this statement.

Remark 4.3 Similar as in Sect. 2, for the proposed method for problems with three separable operators, we have the following property of $\|w^k - \tilde{w}^k\|_M$ and $\varphi(w^k, \tilde{w}^k)$.

Lemma 4.1 *Let $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{z}^k, \tilde{\lambda}^k)$ be generated by (4.7–4.10) from a given $w^k = (x^k, y^k, z^k, \lambda^k)$ and $\varphi(w^k, \tilde{w}^k)$ be defined as in (4.13). Then we have*

$$\varphi(w^k, \tilde{w}^k) \geq \frac{2 - \sqrt{3}}{2} \|w^k - \tilde{w}^k\|_M^2. \tag{4.15}$$

Proof We consider the second term of $\varphi(w^k, \tilde{w}^k)$ (see the definition in (4.13)). Note that

$$\begin{aligned} 2(\lambda^k - \tilde{\lambda}^k)^T & [A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k) + C(z^k - \tilde{z}^k)] \\ &= [A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)]^T (\lambda^k - \tilde{\lambda}^k) \\ &\quad + [B(y^k - \tilde{y}^k) + C(z^k - \tilde{z}^k)]^T (\lambda^k - \tilde{\lambda}^k) \\ &\quad + [C(z^k - \tilde{z}^k) + A(x^k - \tilde{x}^k)]^T (\lambda^k - \tilde{\lambda}^k). \end{aligned} \tag{4.16}$$

By using the Cauchy–Schwarz Inequality, we have

$$\begin{aligned} & \left[A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\right]^T (\lambda^k - \tilde{\lambda}^k) \\ & \quad \geq -\frac{\sqrt{3}}{2} \left\|A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\right\|_H^2 - \frac{1}{2\sqrt{3}} \|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2, \\ & \left[B(y^k - \tilde{y}^k) + C(z^k - \tilde{z}^k)\right]^T (\lambda^k - \tilde{\lambda}^k) \\ & \quad \geq -\frac{\sqrt{3}}{2} \left\|B(y^k - \tilde{y}^k) + C(z^k - \tilde{z}^k)\right\|_H^2 - \frac{1}{2\sqrt{3}} \|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2 \quad \text{and} \end{aligned}$$

$$\begin{aligned}&\big[C(z^k - \tilde{z}^k) + A(x^k - \tilde{x}^k)\big]^T(\lambda^k - \tilde{\lambda}^k)\\&\quad \ge -\frac{\sqrt{3}}{2}\|C(z^k - \tilde{z}^k) + A(x^k - \tilde{x}^k)\|_H^2 - \frac{1}{2\sqrt{3}}\|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2.\end{aligned}$$

Substituting them into (4.16) and using (4.6) we get

$$\begin{aligned}&2(\lambda^k - \tilde{\lambda}^k)^T\big[A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k) + C(z^k - \tilde{z}^k)\big]\\&\quad \ge -\frac{\sqrt{3}}{2}\|A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\|_H^2 - \frac{\sqrt{3}}{2}\|B(y^k - \tilde{y}^k) + C(z^k - \tilde{z}^k)\|_H^2\\&\qquad - \frac{\sqrt{3}}{2}\|C(z^k - \tilde{z}^k) + A(x^k - \tilde{x}^k)\|_H^2 - 3\left(\frac{1}{2\sqrt{3}}\|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2\right)\\&\quad = -\frac{\sqrt{3}}{2}\|w^k - \tilde{w}^k\|_M^2.\end{aligned} \tag{4.17}$$

Therefore, it follows from (4.13) and (4.17) that

$$\varphi(w^k, \tilde{w}^k) \ge \frac{2-\sqrt{3}}{2}\|w^k - \tilde{w}^k\|_M^2.$$

The proof is complete. □

4.2 The descent direction of the PSALM for three separable operators

Similarly as in Sect. 3.1, in order to guarantee that $-G^{-1}M(w^k - \tilde{w}^k)$ is a descent direction of $\|w - w^*\|_G^2$ at the point $w^k \notin \mathcal{W}^*$, we need only to show that for any $w^* \in \mathcal{W}^*$,

$$(w^k - w^*)^T M(w^k - \tilde{w}^k) \ge \varphi(w^k, \tilde{w}^k). \tag{4.18}$$

We prove this inequality via following lemmas.

Lemma 4.2 *Let $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{z}^k, \tilde{\lambda}^k)$ be generated by (4.7–4.10) from a given $w^k = (x^k, y^k, z^k, \lambda^k)$. Then for any $w^* = (x^*, y^*, z^*, \lambda^*) \in \mathcal{W}^*$ we have*

$$\begin{aligned}(\tilde{\lambda}^k - \lambda^*)^T H^{-1}(\lambda^k - \tilde{\lambda}^k) &\ge (A\tilde{x}^k - Ax^*)^T H\big[B(y^k - \tilde{y}^k) + C(z^k - \tilde{z}^k)\big]\\&\quad + (B\tilde{y}^k - By^*)^T H\big[A(x^k - \tilde{x}^k) + C(z^k - \tilde{z}^k)\big]\\&\quad + (C\tilde{z}^k - Cz^*)^T H\big[A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\big].\end{aligned} \tag{4.19}$$

Proof Since $w^* \in \mathcal{W}^*$, $\tilde{x}^k \in \mathcal{X}$, $\tilde{y}^k \in \mathcal{Y}$ and $\tilde{z}^k \in \mathcal{Z}$, it follows from (4.3) that

$$\begin{pmatrix}\tilde{x}^k - x^*\\ \tilde{y}^k - y^*\\ \tilde{z}^k - z^*\end{pmatrix}^T \begin{pmatrix} f(x^*) - A^T\lambda^*\\ g(y^*) - B^T\lambda^*\\ h(z^*) - C^T\lambda^*\end{pmatrix} \ge 0, \tag{4.20}$$

and

$$Ax^* + By^* + Cz^* - b = 0.$$

On the other hand, since $x^* \in \mathcal{X}$, $y^* \in \mathcal{Y}$, and $z^* \in \mathcal{Z}$, from (4.14) we have

$$\begin{pmatrix} x^* - \tilde{x}^k \\ y^* - \tilde{y}^k \\ z^* - \tilde{z}^k \end{pmatrix}^T \begin{pmatrix} f(\tilde{x}^k) - A^T\tilde{\lambda}^k + A^T H[(By^k - B\tilde{y}^k) + (Cz^k - C\tilde{z}^k)] \\ g(\tilde{y}^k) - B^T\tilde{\lambda}^k + B^T H[(Ax^k - A\tilde{x}^k) + (Cz^k - C\tilde{z}^k)] \\ h(\tilde{z}^k) - C^T\tilde{\lambda}^k + C^T H[(Ax^k - A\tilde{x}^k) + (By^k - B\tilde{y}^k)] \end{pmatrix} \ge 0. \quad (4.21)$$

Adding (4.20) and (4.21), and using the monotonicity of operator f, g and h, we obtain

$$\begin{pmatrix} \tilde{x}^k - x^* \\ \tilde{y}^k - y^* \\ \tilde{z}^k - z^* \end{pmatrix}^T \begin{pmatrix} A^T(\tilde{\lambda}^k - \lambda^*) - A^T H[(By^k - B\tilde{y}^k) + (Cz^k - C\tilde{z}^k)] \\ B^T(\tilde{\lambda}^k - \lambda^*) - B^T H[(Ax^k - A\tilde{x}^k) + (Cz^k - C\tilde{z}^k)] \\ C^T(\tilde{\lambda}^k - \lambda^*) - C^T H[(Ax^k - A\tilde{x}^k) + (By^k - B\tilde{y}^k)] \end{pmatrix} \ge 0. \quad (4.22)$$

Using $Ax^* + By^* + Cz^* = b$ and $A\tilde{x}^k + B\tilde{y}^k + C\tilde{z}^k - b = H^{-1}(\lambda^k - \tilde{\lambda}^k)$, from (4.22) we get the result of this lemma. □

Lemma 4.3 *Let $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{z}^k, \tilde{\lambda}^k)$ be generated by (4.7–4.10) from a given $w^k = (x^k, y^k, z^k, \lambda^k)$. Then for any $w^* = (x^*, y^*, z^*, \lambda^*) \in \mathcal{W}^*$ we have*

$$\begin{aligned} &(\tilde{w}^k - w^*)^T M(w^k - \tilde{w}^k) \\ &\quad \ge 2(\lambda^k - \tilde{\lambda}^k)^T\big[(Ax^k - A\tilde{x}^k) + (By^k - B\tilde{y}^k) + (Cz^k - C\tilde{z}^k)\big], \end{aligned} \quad (4.23)$$

and consequently

$$(w^k - w^*)^T M(w^k - \tilde{w}^k) \ge \varphi(w^k, \tilde{w}^k), \quad (4.24)$$

where $\varphi(w^k, \tilde{w}^k)$ is defined in (4.13).

Proof First, using the notation M (see (4.5)), we have

$$\begin{aligned} &(\tilde{w}^k - w^*)^T M(w^k - \tilde{w}^k) \\ &= \big[(A\tilde{x}^k - Ax^*) + (B\tilde{y}^k - By^*)\big]^T H\big[A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\big] \\ &\quad + \big[(B\tilde{y}^k - By^*) + (C\tilde{z}^k - Cz^*)\big]^T H\big[B(y^k - \tilde{y}^k) + C(z^k - \tilde{z}^k)\big] \\ &\quad + \big[(C\tilde{z}^k - Cz^*) + (A\tilde{x}^k - Ax^*)\big]^T H\big[C(z^k - \tilde{z}^k) + A(x^k - \tilde{x}^k)\big] \\ &\quad + (\tilde{\lambda}^k - \lambda^*)H^{-1}(\lambda^k - \tilde{\lambda}^k). \end{aligned} \quad (4.25)$$

Substituting (4.19) in the last term of the right-hand side of (4.25) and using $Ax^* + By^* + Cz^* = b$, we get

$$\begin{aligned} &(\tilde{w}^k - w^*)^T M(w^k - \tilde{w}^k) \\ &\quad \ge (A\tilde{x}^k + B\tilde{y}^k + C\tilde{z}^k - b)^T H\big[A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\big] \end{aligned}$$

$$+ (A\tilde{x}^k + B\tilde{y}^k + C\tilde{z}^k - b)^T H\big[B(y^k - \tilde{y}^k) + C(z^k - \tilde{z}^k)\big]$$
$$+ (A\tilde{x}^k + B\tilde{y}^k + C\tilde{z}^k - b)^T H\big[C(z^k - \tilde{z}^k) + A(x^k - \tilde{x}^k)\big]. \quad (4.26)$$

Since $H(A\tilde{x}^k + B\tilde{y}^k + C\tilde{z}^k - b) = \lambda^k - \tilde{\lambda}^k$, from (4.26) we get assertion (4.23). Using the definition of $\varphi(w^k, \tilde{w}^k)$ in this section (see (4.13)), (4.24) follows from (4.23) directly. □

Inequalities (4.24) and (4.15) are bases for constructing parallel splitting augmented Lagrangian method for problems with three separable operators. From them we get

$$(w^k - w^*)^T M(w^k - \tilde{w}^k) \geq \frac{2-\sqrt{3}}{2}\|w^k - \tilde{w}^k\|_M^2, \quad (4.27)$$

which is the key inequality of the proposed method for problems with three separable operators. The remained evidences of convergence analysis are routines and omitted.

5 Conclusions remarks

In this paper, we proposed splitting augmented Lagrangian methods for structured monotone variational inequalities whose operator is composed by two or three separable operators. The main advantage of the proposed method is that the followers can solve their individual sub-problems in a parallel wise. Theoretically, it belongs to the class of modified augmented Lagrangian methods. Unfortunately, the method cannot be extended for problems with more than three separable operators. In general, for problems with p separable operators, we can prove a similar key inequality

$$(w^k - w^*)^T M(w^k - \tilde{w}^k) \geq \frac{2-\sqrt{p}}{2}\|w^k - \tilde{w}^k\|_M^2 \quad (5.1)$$

with proper positive semi-definite matrix M. Note that (3.2) and (4.27) are special cases of (5.1) with $p = 2$ and $p = 3$, respectively. However, for $p \geq 4$, we cannot obtain the assertion that $-G^{-1}M(w^k - \tilde{w}^k)$ is a descent direction of $\|w - w^*\|_G^2$ at the point w^k. Up to now, we cannot use the same technique to develop similar methods for problems with more than three separable operators.

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